INTERCOMPARISON OF MODELS OF THE ATMOSPHERIC TURBULENCE SPECTRUM

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This paper deals with the theoretical and experimental studies of the parameters of optical waves aimed at comparing the different models of the atmospheric turbulence spectrum. For inhomogeneous optical paths in the atmosphere the notion of the turbulence spectrum averaged over the path is introduced.

In Ref. 1 several models of the spectral power density of the fluctuations of the refractive index of the atmosphere $\Phi_n(\kappa, \xi)$ were analyzed. It is well known that the following models, describing the behavior of the spectral power density of the fluctuations of the refractive index of the atmosphere in the region adjacent to the energy range, are most widely used:

– von Karman model[2]

$$\Phi_n(\kappa, \xi) = 0.033\, C_n^2(\xi)\, L_{0K}^{11/3}(1 + \kappa^2 L_{0K})^{-11/6}, \quad (1)$$

– exponential model[2]

$$\Phi_n(\kappa, \xi) = 0.033\, C_n^2(\xi)\, \kappa^{-11/3}\{1 - \exp[-\kappa^2/\kappa_{0H}^2]\}, \quad (2)$$

– Greenwood–Tarazano model[3]

$$\Phi_n(\kappa, \xi) = 0.033\, C_n^2(\xi)\, (\kappa^2 L_{0G}^2 + \kappa L_{0G})^{-11/6}, \quad (3)$$

Naturally, in these models described by formulas (1)–(3) the outer scales L_{0K}, κ_{0H}^{-1}, and L_{0G} somewhat differ. Here $C_n^2(\xi)$ is the intensity of the turbulent fluctuations.

Let us perform an intercomparison of models (1)–(3) on the basis of the calculation of the variance of the phase fluctuations of optical wave propagating though the atmospheric layer. For the starting approximation, let us use the relation, describing the phase fluctuations $S(\rho)$ of the optical wave propagating through the turbulent atmosphere in the smooth perturbation approximation[4]:

$$S(\rho) = \kappa \int_0^L dx \int\int d^2 n(\boldsymbol{\kappa}, x) \cos \frac{k^2(L - x)\,\gamma}{2\kappa} \exp(i\kappa\rho\gamma), \quad (4)$$

where κ is the radiation wave number, $\gamma = 1$ for plane wave and $\gamma = x/L$ for spherical wave, and L is the length of the optical path. It is not difficult to show that for the fluctuations[5]

$$\langle d^2 n(\boldsymbol{\kappa}_1, x_1)\, d^2 n(\boldsymbol{\kappa}_2, x_2)\rangle =$$

$$= 2\pi\delta(x_1 - x_2)\, \delta(\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2)\, \Phi_n(\boldsymbol{\kappa}_1, x_1)\, d^2\kappa_1\, d^2\kappa\, dx_1\, dx_2 \quad (5)$$

the variance of the phase fluctuations of a plane wave is

$$\langle S^2\rangle = 2\pi^2 \kappa^2 \int_0^L d\xi \int_0^\infty d\kappa\, \kappa\, \Phi_n(\kappa, \xi) \left[1 + \cos\frac{k^2(L - \xi)}{2\kappa}\right]. \quad (6)$$

For $\kappa_0^2 L/\kappa \ll 1$, where κ_0^{-1} is the value of the outer scale of turbulence, Eq. (6) transforms into

$$\langle S^2\rangle = 4\pi^2 \kappa^2 \int_0^L d\xi \int_0^\infty d\kappa\, \kappa\, \Phi_n(\kappa, \xi). \quad (7)$$

Let us compare the variances of the phase fluctuations calculated for models (1)–(3) on a homogeneous path. For models (1), (2), and (3) we have

$$\langle S^2\rangle_1 = \frac{12}{5} \pi^2\, 0.033 \kappa^2\, C_n^2\, L\, L_{0K}^{5/3} \quad (8)$$

(for $L_{0K}^{-1} \ll \kappa_m$, where κ_m is the wave number for the inner scale of turbulence),

$$\langle S^2\rangle_2 = \frac{12}{5} \pi^2\, 0.033 \Gamma\left(\frac{1}{6}\right) \kappa^2\, C_n^2\, L\, \kappa_{0H}^{-5/3}, \quad (9)$$

$$\langle S^2\rangle_3 = 4\pi^2\, 0.033 \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(\frac{11}{6}\right)} \kappa^2\, C_n^2\, L\, L_{0G}^{5/3}. \quad (10)$$

From the condition of equal variances $\langle S^2\rangle_1 = \langle S^2\rangle_2 = \langle S^2\rangle_3$ for models (1)–(3), we obtain the relationships of the scales

$$L_{0G} \simeq 0.27 L_{0K}, \quad \kappa_{0H}^{-1} \simeq 0.36 L_{0K}, \quad \kappa_{0H}^{-1} \simeq 1.33 L_{0G}. \quad (11)$$

Thus the calculations of the optical characteristics performed for one model of the spectrum can be generalized to the other model using relationships (11).

In the calculation of the variance of the jitter in the image of a star in the focal plane of a telescope, we use the following simplifying assumptions: the amplitude fluctuations of an optical wave are weak and the telescope aperture is Gaussian. Then (see Ref. 2) the variance of the jitter in the image will be

$$\sigma_a^2(R) = 16\pi^2\, 0.033 F^2 \int_0^L d\xi\, C_n^2(\xi) \times$$

$$\times \int_0^\infty d\kappa\, \kappa^{-2/3} \{1 - \exp(-\kappa^2/\kappa_{0H}^2)\} \exp(-\kappa^2 R^2/2) \quad (12)$$

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for model 2. Here R is the effective radius of Gaussian aperture. Simple calculations based on formula (12) yield

$$\sigma_a^2(R) = 8\pi^2\, 0.033\Gamma\left(\frac{1}{6}\right) 2^{1/6} F^2 R^{-1/3} \times$$

$$\times \int_0^L d\xi\, C_n^2(\xi) \left\{1 - (1 + 2\kappa_{0H}^{-2} R^{-2})^{-1/6}\right\} . \quad (13)$$

Further calculations require the real profiles of the model parameters $C_n^2(\xi)$ and $\kappa_{0H}^{-1}(x)$.

For von Karman model (1) of turbulence spectrum in Ref. 5 the variance of the jitter in the image for circular telescope aperture was calculated

$$\sigma_a^2(R) = (\pi\, \kappa^2 R^2)^{-1} \int_0^{2R} \rho\, d\rho \left[D_s''(\rho) + \frac{D_s'(\rho)}{\rho}\right] \times$$

$$\times \left\{\arccos(\rho/2R) - (\rho/2R)\sqrt{1 - (r/2R)^2}\right\} , \quad (14)$$

where $D_s(\rho)$ is the structure phase function and $2R$ is the diameter of a receiving aperture. In the region of applicability of the smooth perturbation method the structure phase function was substituted by the structure function of complex phase, in what follows

$$D_s''(\rho) + \frac{D_s'(\rho)}{\rho} = 8\pi^2 \kappa^2 \int_0^L d\xi \int_0^\infty d\kappa\, \kappa^3 J_0(\kappa, \rho)\, \Phi_n(\kappa, \xi) . \quad (15)$$

Moreover, the function characterizing the averaging effect of the circular aperture in Eq. (14) was approximated by the power–law function[6]

$$\left(\arccos x - x\sqrt{1 - x^2}\right) \approx \frac{\pi}{2} [1 - 1.25x + 0.25x^4] .$$

The results of calculation of the angles of arrival by Eq. (14) for homogeneous path are shown by curve *2* in Fig. 1. In addition, the $R^{-1/3}$ dependence for Kolmogorov's turbulence spectrum (curve *1*)

$$\sigma_a^2(2R) \simeq 2.84\, C_n^2\, L(2R)^{-1/3} \quad (16)$$

is also shown here. Comparing curves *1* and *2* one can find that the finite value of the outer scale of turbulence leads to faster rate of decrease of the variance of the jitter in the image with the increase of receiving aperture diameter (when $2R > 0.1\,\kappa_0^{-1}$) as compared with the results calculated by Eq. (16) applicable only in the case $2R\,\kappa_0 \ll 1$ (in which curves *1* and *2* coincide).

Equation (13) can be used to calculate the variance of the jitter in the image on inhomogeneous path, when the corresponding vertical profiles of $C_n^2(\xi)$ and $\kappa_0^{-1}(x)$ are used.

The aperture dependence of the variance of the jitter in the image of a star formed by the telescope was experimentally tested in Ref. 7 as part of the integrated measurements of astroclimate on the site of the BTA largest Russian telescope. The jitter in the image was experimentally measured using a photoelectric attachment positioned in the focal plane of the telescope with main mirror 605 mm in diameter. The aperture was changed by mounting of opaque circular diaphragms of different diameters.

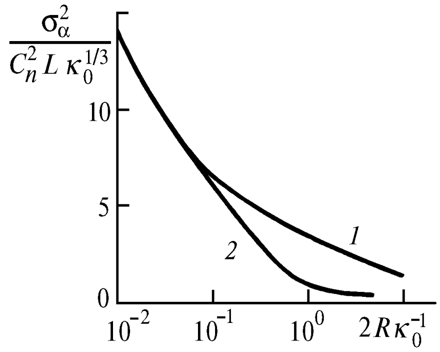

FIG. 1. Aperture dependence of the variance of the fluctuations in the angles of arrival of optical wave on a homogeneous path: 1) calculation for Kolmogorov's turbulence spectrum and 2) calculation for model (1) at $\kappa_0 = L_{0K}^{-1}$.

The constructional feature of the employed telescope was shadowing of the central part of the mirror. In practice we deal with the telescope having the annular aperture with the diameter of the shadowed zone $D_1 = 115$ mm. Previously in Ref. 8 the variance of the jitter in the image of a star was calculated for telescope having annular aperture with the external diameter D_1 and internal diameter D_2. It was shown that in the power–law portion of aperture dependence of the variance of the jitter in the image we have

$$\sigma_a^2(R) = 5.69\, f(n)\, D_2^{-1/3} \int_0^\infty d\xi\, C_n^2(x) , \quad (17)$$

where $n = D_1/D_2$,

$$f(n) = (1 - n^2)^{-2}\left\{1 + n^{11/3} - 2.15 \frac{n^2}{(1+n)^{1/3}}\, {}_2F_1\left(\frac{1}{6}, \frac{3}{2}; 3; \frac{4n}{(1+n)^2}\right)\right\} . \quad (18)$$

Here the function $f(n)$ describes the influence of shadowing of the central part of the telescope and gives the quantitative overestimation of the variance of the jitter in the image for the telescope with annular aperture as compared with the circular telescope. Some values of the function $f(n)$ are presented in Table I. It is not difficult to see that the function $f(n)$ considerably differs from unity only for large shadowing zone (more than 50% of the aperture). These results describing the influence of shadowing of the central part of the telescope were used in Ref. 7 to correct the results of measurements.

TABLE I.

n	0	0.5	0.75	0.8	0.9	0.95
$f(n)$	1	1.08	1.37	1.67	4.33	15.17

Two runs of observations (performed for different levels of atmospheric turbulence) of aperture dependence $\sigma_a^2(R)$ with external diameters of the telescope of 152, 215, 313, 492, and 605 mm are shown in Fig. 2. The internal diameter of the annular aperture of the telescope was constant and equal to

115 mm. The measured data are in relative units. The $R^{-1/3}$ power–law dependence is shown here too for comparison. As was previously shown theoretically, the data of measurements of the variance of the jitter considerably deviate from the $R^{-1/3}$ dependence. This confirms the conclusion (made from comparison of Figs. 1 and 2) about the influence of the finiteness of the outer turbulence scale.

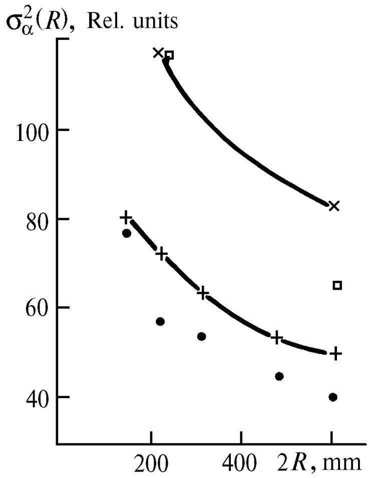

FIG. 2. Experimental check of the aperture dependence of the variance of the jitter in the image of a star in the focal plane of the telescope with a diameter of 605 mm: • *and* □ *denote the data of measurements for different levels of turbulence and* × *and* + *denote the* $R^{-1/3}$ *dependence corresponding to the turbulence spectrum with infinite outer scale.*

It seems expedient to me to compare the data of optical measurements with the results of model calculations. At present a large number of model altitude dependences of the structure parameter of refractive index $C_n^2(h)$ and of the outer scale of turbulence $L_0(h)$ is known. The altitude models of $C_n^2(h)$ are rather numerous (see Refs. 2, 4, 9, and 10). Here we consider only two of them. The first is the model proposed in Ref. 10 describing the so–called best, intermediate, and worst conditions for optical observations. It is of interest to us the model[11] describing the propagation of visible radiation at night

$$C_n^2(h)\,[\mathrm{m}^{-2/3}] = \begin{cases} 3.4\cdot10^{-15}\,, & 0 \le h \le 18.5\ \mathrm{m}\,,\\ 2.87\cdot10^{-12}h^{-2}\,, & 18.5\ \mathrm{m} \le h \le 110\ \mathrm{m}\,,\\ 3.4\cdot10^{-15}\,, & 110\mathrm{m} \le h \le 1500\ \mathrm{m}\,,\\ 3.87\cdot10^{-7}h^{-3}\,, & 1500\ \mathrm{m} \le h \le 7200\ \mathrm{m}\,,\\ 2.00\cdot10^{-16}h^{-1/2}\,, & 7200 \le h \le 20000\ \mathrm{m}\,. \end{cases}$$

As for the altitude models of the outer scale of the atmospheric turbulence, they are less in number. It makes sense to point out that the linear growth of the outer scale was reported in Ref. 12 at low altitudes h, and according to Ref. 4, the value of L_0 became comparable to the altitude above the underlying surface. Recently the following models have been reported in Refs. 11 and 13:

$$\kappa_0^{-1}(h) = 5 \Bigg/ \left[1 + \left(\frac{h - 7500}{2000}\right)^2\right], \qquad (19)$$

$$\kappa_0^{-1}(h) = 4 \Bigg/ \left[1 + \left(\frac{h - 8500}{2500}\right)^2\right]. \qquad (20)$$

Using these models of L_0 and C_n^2, one can calculate the fluctuations of the parameters of optical waves propagating along inhomogeneous paths. Thus, the jitter in the image of a star is calculated on the basis of Eq. (13) for propagation in the zenith direction. It is seen from the analysis of Eq. (13) that all the salient features of the inhomogeneous optical path will be described by the behavior of the function

$$f(R, H_0, H) = \frac{\int\limits_{H_0}^{H} d\xi\, C_n^2(\xi)\,[1 + 2k_0^{-2} R^{-2}]^{-1/6}}{\int\limits_{H_0}^{H} d\xi\, C_n^2(\xi)}\,.$$

Thus, for homogeneous path

$$f(R, H_0, H) = f(R) \equiv (1 + 2\kappa_0^{-2} R^{-2})^{-1/6}\,.$$

If the aperture of the telescope $R \ll \kappa_0^{-1}$, then

$$f(R, H_0, H) \equiv 1\,.$$

To introduce the notion of the outer scale $\kappa^{-1}_{\mathrm{o.\,av.}}$ averaged over the optical path, an attempt must be made to approximate it by the trial–and–error method

$$f(R, H_0, H) = (1 + 2\kappa^{-2}_{\mathrm{o.\,av.}} R^{-2})^{-1/6}\,.$$

Our calculations show that the outer scale averaged over the entire vertical column of the atmosphere $\kappa^{-1}_{\mathrm{o.\,av.}} \approx 0.5$ m at $H_0 = 0$ and $H = 20000$ m for the best conditions in the atmosphere, according to Gurvich (Ref. 10), and $\kappa^{-1}_{\mathrm{o.\,av.}} = 1.0$ m for the worst conditions in the atmosphere. In calculation the model (19) of the outer scale of turbulence was used. It is of interest to note that the results obtained for model proposed in Ref. 11 are the same, as a whole.

It should be noted that the results of this paper agree fairly well with the data of Ref. 11. This demonstrates that the behavior of the turbulence spectrum in the region of low frequencies is adequately described by models (1)–(3). The parameters C_n^2 and κ_0^{-1} of these models, in their turn, are described on the basis of empirical altitude dependences. To calculate the characteristics of optical waves propagating through the atmosphere, one can introduce the average integral spectrum

$$\int\limits_0^{\infty} d\xi\, \Phi_n(\kappa, \xi) = 0.025\kappa^{-2}\, r_0^{-5/3}\, \kappa^{-11/3} \left\{1 - \exp(-\,\kappa^2/\,\kappa^2_{\mathrm{o.\,av.}})\right\},$$

where r_0 is Fried's radius.

Because of the considerable growth[4] of the inner scale of the turbulence l_0 with increase of the altitude ($l_0 \sim \sqrt[3]{h}$) and the finite outer scale of the turbulence, one can expect essential narrowing of inertial range of spectrum of atmospheric turbulent inhomogeneities at high altitudes $h(h > 5000$ m). In its turn, this can alter some trends in the description of optical wave fluctuations. In particular, this is of importance when estimating the operating efficiency of adaptive systems in the atmosphere.[15]

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