PHASE PROBLEM, WAVE–FRONT DISLOCATIONS AND EQUATION FOR THE TWO–DIMENSIONAL OPTICAL FIELD INTENSITY

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Received February 20, 1995

Based on analytical relations obtained for reconstructing the wave phase from the distribution of wave intensity, the peculiarities of the two-dimensional wave-front structure formation have been investigated, which are connected with interference zeros of the field. The presence of field zeros is shown to be insufficient for the existence of dislocations. The transition from the parabolic wave equation to the nonlinear integro-differential equation for the field intensity has been made for the first time.

The phase-front dislocations, occurring in the interference optical fields,[1,2] became recently a matter for scientific research and indicator of a series of wave phenomena and processes in nonlinear optics and laser physics.[3-5] In addition, line of scientific research is formed connected with the use of dislocations for remote diagnostics of natural media.[6] However, it should be noted that practically in all the theoretical papers devoted to these problems it is assumed that the presence of wave field zeros is sufficient condition for dislocation formation. When analyzing the dislocation statistics, one suggests that it is completely determined by zero-carrier statistics.[7,8] To check the uniqueness of such correspondence, we consider the process of dislocation formation, solving simultaneously the problem of the wave-phase reconstruction from measurements of wave intensity distribution (a so-called phase problem in optics).[9,10] When solving this problem, we obtain a set of consequences being useful, in our opinion, in the wave propagation theory as a whole.

We consider the case of a two-dimensional wave propagating in the half-space $z > 0$. To describe the propagation process, we use the parabolic wave equation[11]

$$2\,i\,k\,\frac{\partial}{\partial z}\,U(x, z) + \frac{\partial^2}{\partial x^2}\,U(x, z) + k^2\,\varepsilon'(x, z)\,U(x, z) = 0\ , \qquad (1)$$

where $\varepsilon'(x, z) = [\varepsilon(x, z) - \bar{\varepsilon}]/\bar{\varepsilon}$, $k = \frac{2\,\pi}{\lambda}\sqrt{\bar{\varepsilon}}$, and $\bar{\varepsilon}$ is the constant mean value of ε. This equation describes the evolution of the slowly varying complex amplitude of a monochromatic beam $U(x, z)\exp(-\,i\omega t + ikz)$ propagating along the z axis in an inhomogeneous refractive medium with the dielectric constant $\varepsilon(x, z)$.

By substituting $U(x, z) = I^{1/2}(x, z)\,\exp\{is(x, z)\}$, where $I(x, z)$ is the intensity, and $s(x, z)$ is the wave phase, Eq. (1) is transformed into the set of radiative transfer equations

$$\frac{\partial}{\partial x}\left\{I(x, z)\,\frac{\partial}{\partial x}\,s(x, z)\right\} + k\,\frac{\partial}{\partial z}\,I(x, z) = 0 \qquad (2)$$

and eikonal

$$-\,2\,k\left\{I(x, z)\,\frac{\partial}{\partial z}\,s(x, z)\right\} - \left\{I(x, z)\,\frac{\partial}{\partial x}\,s(x, z)\right\}^2 / I(x, z) - \frac{1}{4}\left(\frac{\partial}{\partial x}\,I(x, z)\right)^2 / I(x, z) + \frac{1}{2}\,\frac{\partial^2 I(x, z)}{\partial x^2} + k^2\,\varepsilon'(x, z)\,I(x, z) = 0\ . \qquad (3)$$

Equation (2) expresses the energy conservation law in differential form. It can be easy solved for the expression in braces:

$$I(x, z)\,\frac{\partial}{\partial x}\,s(x, z) = \left\{I(x, z)\,\frac{\partial}{\partial x}\,s(x, z)\right\}\Bigg|_{x\,=\,x_0} - k\int\limits_{x_0}^{x} \mathrm{d}x'\,\frac{\partial}{\partial z}\,I(x', z)\ . \qquad (4)$$

The quantity in the left-hand side of the equation represents the transverse component of the Umov–Poynting vector[11] $L_x(x, z)$. Because the beam energy is localized along the direction of preferred propagation (along the z axis), it is evident that $L_x(x_0, z) \to 0$ as $x_0 \to \pm\,\infty$. Then instead of Eq. (4) we can write

$$L_x(x, z) = I(x, z) \frac{\partial}{\partial x} s(x, z) = -k \int_{-\infty}^{x} \mathrm{d}x' \frac{\partial}{\partial z} I(x', z) =$$

$$= -\frac{k}{2} \int_{-\infty}^{\infty} \mathrm{d}x' \,\mathrm{sign}\,(x - x') \frac{\partial}{\partial z} I(x', z) \ , \tag{5}$$

where sign x is the signum function. By means of Eq. (5) and eikonal (3) we also can derive the longitudinal component of the Poynting vector in terms of the intensity

$$L_z(x, z) = I(x, z) \left[k + \frac{\partial}{\partial z} s(x, z) \right] =$$

$$= \left[k + k \frac{\varepsilon'(x, z)}{2} \right] I(x, z) + \frac{1}{4k} \frac{\partial^2}{\partial x^2} I(x, z) -$$

$$- \frac{1}{8k} \left\{ \frac{\partial I(x, z)}{\partial x} \right\}^2 / I(x, z) - \frac{1}{2k} \left(\frac{k}{2} \right)^2 \times$$

$$\times \left\{ \int_{-\infty}^{\infty} \mathrm{d}x' \,\mathrm{sgn}\,(x - x') \frac{\partial}{\partial z} I(x', z) \right\}^2 / I(x, z) \ . \tag{6}$$

For the parabolic equation, we know the consequence of the energy conservation law

$$\int_{-\infty}^{\infty} I(x, z) \,\mathrm{d}x = \mathrm{const} \ .$$

From Eq. (5), we obtain one more consequence

$$\int_{-\infty}^{\infty} L_x(x, z) \,\mathrm{d}x = \int_{-\infty}^{\infty} I(x, z) \frac{\partial}{\partial x} s(x, z) \,\mathrm{d}x = 0 \ .$$

Equations (5) and (6) make it possible to reconstruct the wave phase. In fact, if the field intensity does not vanish in space $\{x, z\}$, we can write

$$\frac{\partial}{\partial x} s(x, z) =$$

$$= \left\{ -\frac{k}{2} \int_{-\infty}^{\infty} \mathrm{d}x' \,\mathrm{sign}(x - x') \frac{\partial}{\partial z} I(x', z) \right\} / I(x, z) \ , \tag{7}$$

$$\frac{\partial}{\partial z} s(x, z) = \frac{k}{2} \varepsilon'(x, z) + \frac{1}{I(x, z)} \frac{1}{4k} \frac{\partial^2}{\partial x^2} I(x, z) -$$

$$- \frac{1}{8k} \left\{ \frac{\partial I(x, z)}{\partial x} \right\}^2 / I^2(x, z) - \frac{1}{2k} \left(\frac{k}{2} \right)^2 \times$$

$$\times \left\{ \int_{-\infty}^{\infty} \mathrm{d}x' \,\mathrm{sgn}\,(x - x') \frac{\partial}{\partial z} I(x', z) \right\}^2 / I^2(x, z) \ . \tag{8}$$

Solutions of these equations are of the form

$$s(x, z) = s(x_0, z) -$$

$$- \frac{k}{2} \int_{x_0}^{x} \mathrm{d}x' \frac{\int_{-\infty}^{\infty} \mathrm{d}x'' \,\mathrm{sign}(x' - x'') \frac{\partial}{\partial z} I(x'', z)}{I(x', z)} \ , \tag{9}$$

$$s(x_0, z) = s(x_0, 0) + \int_0^z \mathrm{d}z' \left\{ \frac{k}{2} \varepsilon'(x_0, z') + \frac{1}{4k} \frac{1}{I(x_0, z')} \times \right.$$

$$\times \frac{\partial^2}{\partial x^2} I(x_0, z') - \frac{1}{8k} \frac{1}{I^2(x_0, z')} \left[\frac{\partial}{\partial x} I(x_0, z') \right]^2 -$$

$$\left. - \frac{k}{8} \frac{1}{I^2(x_0, z')} \left[\int_{-\infty}^{\infty} \mathrm{d}x' \,\mathrm{sgn}(x_0 - x') \frac{\partial}{\partial z'} I(x', z') \right]^2 \right\} \tag{10}$$

and enable one to calculate the value of phase over all space $z > 0$, if the phase constant $s(x_0, 0)$ in the plane $z = 0$ is known. Such definiteness is achieved at the sacrifice of the intensity measurement over all space from an initial plane to an observation point. Let us assume that at some points of space $x = x_d$, $z = z_d$ the intensity of wave field vanishes. To describe such a situation, we turn to Eqs. (5) and (6). Since for many optical problems the relative phase distribution is necessary, we consider only Eq. (5) with z as a parameter. It can be shown that in the vicinity of points where $I(x_{di}, z_d) = 0$ differential equation (5) admits the representation

$$\frac{1}{2} (x - x_{di})^2 \frac{\partial^2}{\partial x^2} I(x_{di}, z_d) \frac{\partial s(x, z)}{\partial x} =$$

$$= -\frac{k}{2} (x - x_{di})^2 \frac{\partial^2}{\partial z \,\partial x} I(x_{di}, z_d) \ ,$$

and at the points $x = x_{di}$ becomes identical. At these points, the phase can take arbitrary values, and integral lines passing through them represent the straight lines parallel to the ordinate. In the regions from one such point to the other, solution (5) coincides with classical solution (9). Since the initial phase value $s(x_0, z)$ can be determined only to within the constant $2\pi n$ ($n = \pm 1, \pm 2, \ldots, \pm N$), the particular solution in the regions between the points of singularity x_{di} may be any curve obtained from Eq. (9) by $2\pi n$ shift. Integral curves along the entire x axis may consist of branches of solution (9) and vertical line segments of length $2\pi n_i$ (i is the serial number of point) extended to the transition to the next branch. Such a solution with localized discontinuities at points x_{di} and continuous segments extending from x_{di} to x_{di+1} will be a generalized solution[12] of differential equation (5). A generalized function is the derivative of such a solution:

$$\frac{\partial}{\partial x} s(x, z) = \left\{\frac{\partial}{\partial x} s(x, z)\right\} + \sum_i [s(x_{di}, z_d)]\, \delta(x - x_{di}) ,$$

where $\left\{\frac{\partial}{\partial x} s(x, z)\right\}$ is a piecewise-continuous function on x (Ref. 12), and $[s(x_{di}, z_d)] = s(x_{di} + 0) - s(x_{di} - 0)$ is the discontinuity at point x_{di}. This function satisfies the energy conservation law (5) including the singularity points $x = x_d$, $z = z_d$ because for $I(x_{di}, z_d) = 0$ the equality of theory of generalized functions $I(x_d, z_d)\, \delta(x - x_d) = 0$ (Ref. 12) is satisfied. It seems likely that within the framework of the theory being studied it is impossible to specify the rule for selection (control parameter) of any specific piecewise-continuous solution. Therefore, among the infinite set of possible solutions a solution without discontinuity appears, i.e., without dislocations. Strictly speaking, zero intensity at any point of space enables one only to assume the availability of wave-front dislocations at this point.

In our opinion, expressions (7)–(8) for phase derivatives in terms of the intensity make it possible to obtain an interesting relationship. Acting on Eqs. (7) and (8) by the operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial x}$, respectively, and equating the results of differentiation, we have

$$\frac{\partial}{\partial z}\left\{\frac{1}{I(x, z)} \frac{\partial}{\partial z}\left[\int_{-\infty}^{\infty} \mathrm{d}x' \,\mathrm{sign}(x' - x)\, I(x', z)\right]\right\} =$$

$$= \frac{\partial}{\partial x}\, \varepsilon'(x, z) + \frac{1}{2k^2} \frac{\partial}{\partial x}\left\{\frac{1}{I(x, z)} \frac{\partial^2 I(x, z)}{\partial x^2}\right\} -$$

$$- \frac{1}{4k^2} \frac{\partial}{\partial x}\left\{\frac{1}{I^2(x, z)}\left[\frac{\partial I(x, z)}{\partial x}\right]^2\right\} -$$

$$- \frac{1}{4} \frac{\partial}{\partial x}\left\{\frac{1}{I^2(x, z)}\left[\frac{\partial}{\partial z}\int_{-\infty}^{\infty} \mathrm{d}x' \,\mathrm{sgn}(x' - x)\, I(x', z)\right]^2\right\} \quad (11)$$

– the nonlinear integro-differential equation for the intensity. From Eq. (11) with $k \to \infty$, the extraordinary geometric-optical approximation

$$\frac{\partial}{\partial z}\left\{\frac{1}{I(x, z)} \frac{\partial}{\partial z}\left[\int_{-\infty}^{\infty} \mathrm{d}x' \,\mathrm{sign}(x' - x)\, I(x', z)\right]\right\} =$$

$$= \frac{\partial}{\partial x}\, \varepsilon'(x, z)\, \frac{1}{4} \frac{\partial}{\partial x} \times$$

$$\times \left\{\frac{1}{I^2(x, z)}\left[\frac{\partial}{\partial z}\int_{-\infty}^{\infty} \mathrm{d}x' \mathrm{sign}\,(x' - x)\, I(x', z)\right]^2\right\} \quad (12)$$

follows containing neither eikonal nor phase. Owing to the complexity of Eqs. (11)–(12), it is hardly probable that they can be a serious alternative to traditional equations of wave and geometric optics. However, in our opinion, their analysis and investigation of possible ways of their solution will be useful.

ACKNOWLEDGMENTS

The paper was supported in part by the Russian Fundamental Research Foundation, Project No. 94–02–04995–a.

REFERENCES

1. M.V. Berry, in: *Singularities in Waves and Rays (*Les Houches Summer School, Amsterdam, 1980), p. 453.
2. N.B. Baranova and B. Ya. Zel'dovich, Zh. Eksp. Teor. Fiz. **80**, 1789 (1981).
3. F.T. Arecchi, G. Giacomelli, P.L. Ramazza, and S. Residori, Phys. Rev. Lett. **67**, 3749 (1991).
4. R. Neubecker, M. Kreuzer, and T. Tschudi, Opt. Commun. **96**, 117 (1993).
5. N.N. Rozanov, Opt. Spektrosk. **75**, 861 (1993).
6. V.A. Zhuravlev, I.K. Kobozev, Yu.A. Kravtsov, et al., Akust. Zh. **39**, 764 (1993).
7. N.B. Baranova, B. Ya. Zel'dovich, A.V. Mamaev, et al., Pis'ma Zh. Eksp. Teor. Fiz. **33**, 206 (1981).
8. V.A. Zhuravlev, I.K. Kobozev, and Yu.A. Kravtsov, Zh. Eksp. Teor. Fiz. **102**, 483 (1992).
9. H.A. Ferwerda, in: *Inverse Source Problem in Optics,* ed. by N.P. Baltes (Berlin–Heidelberg–New York, 1978).
10. T.I. Kuznetsova, Usp. Fiz. Nauk **1954**, 677 (1988).
11. S.M. Rytov, Yu.A. Kravtsov, and V.I. Tatarskii, *Introduction into Statistical Radio Physics* (Nauka, Moscow, 1980).
12. V.S. Vladimirov, *Generalized Functions in Mathematical Physics* (Nauka, Moscow, 1979).