Moving spatial filtering method in atmospheric turbulence simulation

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Received May 15, 2006

Numerical phase-screen method is described for computer simulation of atmospheric turbulence with predetermined spatial spectrum in the case of transversal wind. The method is based on the spatial smoothing of the initial uncorrelated random field using a spatial filter obtained by the inverse Fourier transform of the original spectrum. The method allows one to simulate phase screens of virtually arbitrary length and is free of some restrictions of the spectral method. Examples of Kolmogorov spectrum field simulation are presented.

Adaptive optical systems provide real-time enhancement of the quality of object images observed though perturbed atmosphere. In developing and analyzing such systems, computer-simulation techniques are used for modeling the atmospheric turbulence. One of the most frequently used approaches is the phase-screen method[1] which is based on the assumption that random influence of the turbulent medium layer on light beam manifests itself at distances larger than the length of the diffraction conversion of phase perturbations of a light field to amplitude ones. This allows one to replace a layer of a turbulent medium of finite thickness by a set of thin two-dimensional phase screens with preset spatial distortion spectra $\Phi_T(\boldsymbol{\kappa})$, $\boldsymbol{\kappa} = \{k_x, k_y\}$, and to consider these screens δ-correlated along the direction of beam propagation. Varying screen parameters both homogeneous and inhomogeneous (along the propagation path) distribution of atmospheric turbulence can be simulated.

As a rule, screens $\mathbf{S}(x, y)$ are synthesized using the spectral method,[1] i.e., the two-dimensional spectrum $|\Phi_T(\boldsymbol{\kappa})|$ is formed with a preset amplitude and random, from point to point, phases $\varphi(\boldsymbol{\kappa})$, then the obtained random spectrum is inverse Fourier transformed:

$$\mathbf{S}(\mathbf{r}) \leftarrow FT^{-1}\left\{|\Phi_T(\boldsymbol{\kappa})|\exp[i\varphi(\boldsymbol{\kappa})]\right\}. \quad (1)$$

First applications of the method were based on the use of independent static screens, which allowed calculating averaged statistical parameters of the model under study. However, needs in simulation of real optical systems stimulated development of the phase-screen methods for screens changing sufficiently smoothly ("continuously") between two discrete moments in time.

One of the early studies,[2] devoted to random two-dimensional surface formation to study fluctuation processes in plasma, describes, in addition to the spectral method, the method based on the synthesis of an optimal two-dimensional statistical predictor, which is an analog of the time approach. Both amplitude and phase of the formed spectrum were random in the spectral approach.[2] In Ref. 1, it was theoretically justified that only phases of the formed spectrum are to be random, while its amplitude is to be kept deterministic.

The difficulty in synthesizing two-dimensional surfaces was noted in Ref. 2 concerned with the finiteness of the computational grid, at which the inequalities

$$\delta \ll \Lambda \ll L_c \ll N\delta \quad (2)$$

are to be met. Here δ is the distance between pixels' centers; Λ is the variable called gradient length (characteristic size of "small" fluctuations); L_c is the length of correlation of the generated field; N is the generated realization's size along one of the sides. If one demands that each of the right-hand quantity in Eq. (2) to be at least one order of magnitude greater than the left-hand one, then the minimum size N is about 10^3; this has become attainable relatively recently. However, even that size of the computational grid is often insufficient for adequate simulation of the atmospheric turbulence. The point is that the ratio between characteristic sizes of turbulent fluctuations, i.e., inner ($\sim 10^{-3}$ m) and outer ($\sim 10^1$–10^3 m) scales, is 10^4–10^6, which requires special actions to reproduce low-frequency harmonics of the spatial fluctuation spectrum of the atmospheric refractive index.

Large-scale fluctuations mostly contribute to the phase dispersion of a light beam propagated through the atmosphere. Therefore, many authors proposed different techniques for their reproduction while generating phase screens. References 1, 3, and 4 describe techniques based on superimposing of two phase screens; one of them includes high and medium spatial frequencies of phase fluctuations and is formed by the spectral method, another one reproduces low frequencies and is formed by superposition of Zernike polynomials or the Karhunen–Loeve function.

An extension of the spectral method to the problem of synthesis of the screens $\mathbf{S}(x, y, t)$, moving at a speed $\bar{\mathbf{V}}$, within the hypothesis of frozen

0235-6880/06/10 818-05 $02.00

turbulence (Taylor hypothesis) is described in Ref. 1. The hypothesis is formulated as

$$\mathbf{S}(\mathbf{r},t) = \mathbf{S}(\mathbf{r} - \mathbf{V}t, 0), \quad \mathbf{r} = \{x, y\}. \tag{3}$$

The extension makes use of the well-known property of the Fourier transform, i.e., shift of an original corresponds to adding a linear component to the phase of the Fourier transform:

$$\mathbf{S}(\mathbf{r} - \mathbf{V}t) \leftarrow FT^{-1}\left\{\left|\Phi(\mathbf{\kappa})\right|\exp[i\varphi(\mathbf{\kappa})]\exp[i\mathbf{\kappa}\mathbf{V}t]\right\}. \tag{4}$$

This method is sufficiently simply realizable and allows moving the screen along an arbitrary direction; however, the cyclic shift of the same realization is performed in this case. The period of an exact replica is determined by the screen size N and the direction of motion.[1]

Another synthesis method for "smoothly" time-changing screens has been proposed in Ref. 5. The method consists in modifying the random phase factor $\exp[\varphi(k)]$ in each spectrum pixel (i, j) at every time step t_n according to the following equation:

$$\exp[\varphi(i,j,t_{n+!})] \to \left\{p\exp[\varphi_1(i,j,t_n)] + \right.$$
$$\left. + \sqrt{1-p^2}\exp[\varphi_2(i,j,t_n)]\right\}\exp[i\mathbf{\kappa}\mathbf{V}(\Delta t)], \tag{5}$$

where $0 \le p \le 1$, $\Delta t = t_{n+1} - t_n$, and new random phase addition $\varphi_2(i, j)$, like the main phase $\varphi_1(i, j)$, is uniformly distributed over the $[0, 2\pi]$ interval. Such an approach in combination with the cyclic shift allows the additional time transformation of the turbulent inhomogeneities distribution due to small-scale (less or close to a beam size) fluctuations of wind speed to be accounted for along with the transport of the turbulent inhomogeneities by mean wind. The coefficient $p = \exp(-\Delta t/\tau)$ allows one to regulate the relative contribution of random wind fluctuations to the time transformation of inhomogeneities in comparison with their mean transport. In particular, $p = 1$ agrees with the Taylor hypothesis (i.e., the contribution of random wind fluctuations is neglected), while $p = 0$ corresponds to the situation when mean wind is absent or has same direction as the beam propagation. In the latter case, inhomogeneities vary only due to random wind speed fluctuations with the characteristic time τ.

This paper describes the method of generating the phase screens based on moving spatial filtering. Screens synthesized by the method have much longer cyclicity, which is determined only by the capacity of a used randomizer. This allows much more reliable analysis of statistical parameters of the adaptive systems to be performed.

Description of the method

Consider the spectral method of random screen synthesis as a typical digital filtering problem. Let the "noise" screen $\mathbf{a} = \{a_{ij}\}$ has the sizes $N \times N$, where a_{ij} is the uncorrelated Gaussian "white noise" of unit intensity. The spectrum $\mathbf{A} = \{A_{ij}\}$ of such noise[6] is an ensemble of random complex variables with unity amplitude and phases uniformly distributed over the $[0, 2\pi]$ interval. Superimpose the filter $\mathbf{\Phi} = \{\Phi_{ij}\}$ with the preset amplitude dependence $|\Phi(i, j)|$ and zero phase $\mathrm{Arg}(\Phi_{ij})$ on this noise (in the general case the spectrum phase should meet the antisymmetry condition[7]; in the spectral method, phase is considered equal to zero[1]): multiply the noise spectrum by the filter spectral function and perform the inverse Fourier transform. As a result, one obtains the complex random field $\mathbf{S} = \{S_{ij}\}$:

$$\mathbf{S} = FT^{-1}[\mathbf{A} \cdot \mathbf{\Phi}]. \tag{6}$$

Real, $\mathrm{Re}(\mathbf{S})$, and imaginary, $\mathrm{Im}(\mathbf{S})$, parts of the field $\mathbf{S}$ are mutually orthogonal[1] random fields with the preset spectrum amplitude $|\Phi(i, j)|$. Note, that the initial noise screen $\mathbf{a}$ as well as the spectral filtering function $\mathbf{\Phi}$ and the resulting field $\mathbf{S}$ have the same size $N \times N$, where N usually equals to integer power of the 2 in using the fast Fourier transform algorithm. The initial noise field $\mathbf{a}$ does not explicitly present in the standard spectral method and according to Eq. (1) the random spectrum $\mathbf{A} \cdot \mathbf{\Phi}$ is formed.

Consider the solution of the two-dimensional field filtering problem in space. In this case, the resulting field is presented by the convolution of the initial noise field $\mathbf{a}$ and the pulsed response function of the filter, $\mathbf{H} = \{H_{ij}\}$:

$$\mathbf{S} = \mathbf{a} * \mathbf{H}, \tag{7}$$

where $\mathbf{H}$ is calculated by means of the Fourier transform of the spectral filter function $\mathbf{\Phi}$:

$$\mathbf{H} = \mathrm{Re}(FT^{-1}[\mathbf{\Phi}]). \tag{8}$$

In the case of infinite-sized fields, Eqs. (6) and (7) are completely equivalent and reflect one of the feature of the Fourier transform, namely, the product of spectra corresponds to the convolution of originals. In the case of finite-sized fields, which is of practical interest, there is an essential distinction that if the size of the pulsed response function $\mathbf{H}$ is $N \times N$, then the "initial" noise field must have $(N + M - 1) \times$ $\times (N + K - 1)$ size to obtain the resulting field $\mathbf{S}$ of $M \times K$ size. In particular, if $K = M = N$ then the size of the initial noise field should be $(2N - 1)^2$, i.e., virtually 4 times (by number of pixels – field elements) larger then the size of $\mathbf{S}$. At the same time, memory required for calculating the convolution (7) is only twice as larger as those for calculations by Eq. (6), since the variables $\mathbf{A}$ and $\mathbf{\Phi}$ in spectral representation are complex, while the variables in Eq. (7) are real. Emphasize, that in contrast to the spectral method the size of pulsed response function is not related to the size of the screen generated.

At time varying of the initial noise field $\mathbf{a}$, the resulting field $\mathbf{S}$ also varies (Eq. (7)). In the particular case of time variation – moving along some direction

$$\mathbf{a}(x, y, t) = \mathbf{a}(\mathbf{r} - \mathbf{V}t), \quad \mathbf{r} = \{x, y\}, \tag{9}$$

the resulting field "moves" along the same direction corresponding to the speed $\mathbf{V}$.

Moving along the direction of one of the coordinates is the most easily realizable. Consider, step by step, the algorithm of forming the random rectangular field $M \times K$, moving along the x-axis, by the filter $\mathbf{H}$ of $N \times N$ size.

1. Calculate the spectral filter function $\mathbf{\Phi} = \{\Phi_{ij}\}$ of $N \times N$ size with the preset amplitude dependence $|\Phi(i, j)|$ and zero phase $\mathrm{Arg}(\Phi_{ij}) \equiv 0$.

2. Calculate the pulsed response function $\mathbf{H} = \mathrm{Re}(FT^{-1}[\mathbf{\Phi}])$ using the Fourier transform. If the procedure has been fulfilled correctly, then $\mathrm{Im}(FT^{-1}[\mathbf{\Phi}]) \approx 0$. The size of the obtained function is $N \times N$.

3. Generate the field $\mathbf{a} = \{a_{ij}\}$ of Gaussian random variables with the zero average and unit variance, which consists of $(M + N - 1)$ line and N columns.

4. Calculate the convolution of the field $\mathbf{a}$ and function $\mathbf{H}$. As a result, one obtains the column $\mathbf{b}$ of M elements:

$$b_i = \sum_{j=1}^{N} \sum_{k=1}^{N} a_{(i+k-1),j} H_{kj}. \tag{10}$$

Place the column $\mathbf{b}$ as the first column of the resulting field $\mathbf{S}$:

$$S_{i1} \leftarrow b_i \tag{11}$$

and move all rows of $\mathbf{S}$ one element to the right.

5. Move every row of the initial array $\mathbf{a}$ one element to the right and put new random numbers a_{i1} at the first place in every row.

6. Repeat steps 4 and 5 K times; the resulting field $\mathbf{S}$ is completely full of smoothed values S_{ij}, $i = \overline{1...M}$, $j = \overline{1...K}$.

7. At every repetition of the steps 4 and 5 the resulting array $\mathbf{S}$ moves by one element and is filled up from the left with the column of new values.

Maximum cyclicity length of the generated random field is determined by the capacity of the randomizer used. Thus, in using a 32-bit randomizer, the maximum length of the random sequence $N_{\max} = 2^{32}$; to obtain a "band" of the width N, width of the initial noise field is to be equal $(2N - 1)$ for the same size of the pulsed filter response function; maximum length Q of the smoothed field correspondingly equals to $(N_{\max} - 2N^2 + 3N - 1)/(2N - 1)$, whence an assessment

$$Q \approx \frac{N_{\max}}{2N} - N \tag{12}$$

is obtained for $N_{\max}$, $N^2 \gg N \gg 1$. According to Eq. (12) Q equals approximately $2^{22} \approx 4 \cdot 10^6$ at $N = 2^{10}$.

Generating phase screens with Kolmogorov turbulence spectrum

The method of sliding smoothing was developed for synthesis of random phase screens to be used for simulating operation of adaptive optical phase conjugation systems under the anisoplanatic conditions.[8,9] In simulating such a system it was necessary to synthesize screens as "test" phase fields, use of which allows one to obtain results testable by other techniques, e.g. analytical calculations. Such calculations are carried out easier if using Kolmogorov statistics of the refractive index fluctuations in the turbulent atmosphere. The spatial spectrum of the Kolmogorov turbulence is defined by the equation[1]

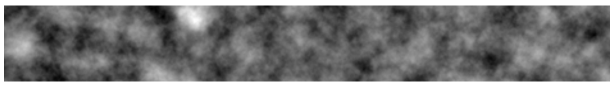
$$\Phi_n(\mathbf{k}) = 0.033 C_n^2 \mathbf{k}^{-11/3}, \tag{13}$$

where the structure constant C_n^2 characterizes the intensity of turbulent fluctuations of the atmospheric refractive index. For a plane wave passing through a layer of a turbulent medium of thickness L, the two-dimensional geometrical-optics spectral density $\Phi_\varphi(k)$ of phase fluctuations has the following form[1]

$$\Phi_\varphi(k) = \frac{8\pi^2}{\lambda^2} L \Phi_n(k), \tag{14}$$

where λ is the radiation wavelength. This equation was derived assuming the fluctuations of the atmospheric refractive index to be delta-correlated along the direction of radiation propagation. The peculiarity of Kolmogorov spectrum is the absence of typical scales of spatial frequencies: at high spatial frequencies, the spectrum continuously falls-off down to zero and increases without limit at low ones, having non-integrable singularity at zero. It is impossible to simulate the Kolmogorov spectrum correctly; so, either its modification by introducing explicitly the outer scale, "cutting off" the spectral density at low frequencies, or use screens with *a fortiori* larger size than the light beam aperture. In so doing, the amplitude of zero spatial frequency of a discrete spectrum is set equal to zero, as it determines only the average value of the synthesized field.

Figure 1 shows two fragments of the phase screens $\mathbf{S}_1$ and $\mathbf{S}_2$ of 128×1024 size with the spectrum (14) generated using the method of sliding smoothing by applying pulsed response functions $\mathbf{H}$ of different widths.



$\mathbf{S}_1$

a

$\mathbf{S}_2$

b

Fig. 1. Phase screens of 128×1024 pixels size generated using pulsed response function $\mathbf{H}$ of different widths: $\mathbf{S}_1$ screen was obtained with $\mathbf{H}$ of the width equal to 128 (*a*) and screen $\mathbf{S}_2$ using $\mathbf{H}$ of the width equal to 512 (*b*).

In generating $\mathbf{S}_1$ the $\mathbf{H}$ function used had width equal to 128 (Fig. 2) while the $\mathbf{S}_2$ screen was generated using the width equal to 512. As is seen from the comparison of the fragments, the 4-fold increase of the $\mathbf{H}$ function's width allows one to generate a screen with significantly larger relative weight of the low-frequency spatial harmonics. To quantitatively compare the generated screens we used their expansion over Zernike polynomials on a round aperture[10]:

$$\mathbf{S} = \sum_{n,m} \beta_{nm} \mathbf{Z}_{nm}, \qquad (15)$$

where β_{nm} are the expansion coefficients, n is the maximum power of the radial polynomial, m is the azimuth part of the Zernike mode. Similar approach was used in Refs. 1, 3, and 4 as well. Theoretical average values $\langle \beta_{nm}^2 \rangle$ for Kolmogorov turbulence spectrum were obtained in Ref. 11.

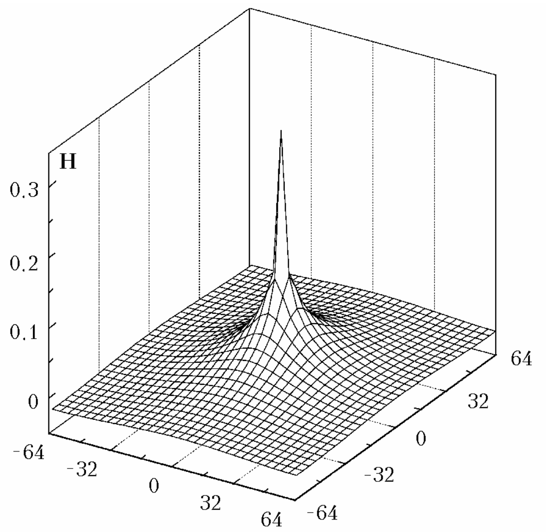


Fig. 2. Pulsed response function of 128 × 128 size.

These values determine the contribution of corresponding modes to the total variance of phase fluctuations on the round aperture and are independent of the azimuth index m:

$$\langle \beta_{nm}^2 \rangle = \alpha_n \gamma, \quad \gamma = \left(\frac{D}{r_0} \right)^{5/3}. \qquad (16)$$

Here D is the diameter of the round aperture, r_0 is the correlation radius of the light field propagated through the turbulent layer of the atmosphere[1,11] (Fried's radius). The coefficients α_n were used for comparison of generated screens. Thirty-five polynomials with the radial index n from 1 to 7 were used in the calculations. The round aperture was shifted along the screen with the step equal to its size D and average values $\langle \beta_{nm}^2 \rangle$ of the squared coefficients of the screen expansion over Zernike polynomials were calculated. These values were then averaged over the azimuth number m:

$$\langle \alpha_n \rangle = \gamma^{-1} (n+1)^{-1} \sum_{m=1}^{n+1} \langle \beta_{nm}^2 \rangle. \qquad (17)$$

To estimate the average Fried's radius r_0 of the generated screens and calculate the normalizing parameter γ, the averaged sums of the squared coefficients of the screen expansion over polynomials of the order higher than one (σ_1^2) and higher than two (σ_2^2) were used, whose theoretical values were calculated according to Ref. 11:

$$\sigma_1^2 = \sum_{n=2}^{7} (n+1)\alpha_n = 0.122,$$
$$\sigma_2^2 = \sum_{n=3}^{7} (n+1)\alpha_n = 0.0524. \qquad (18)$$

The coefficients $\langle \alpha_n \rangle$ calculated by Eqs. (16) and (17) as well as their theoretical values for the $\mathbf{S}_1$ and $\mathbf{S}_2$ screens, shown in Fig. 1, are presented in Fig. 3.

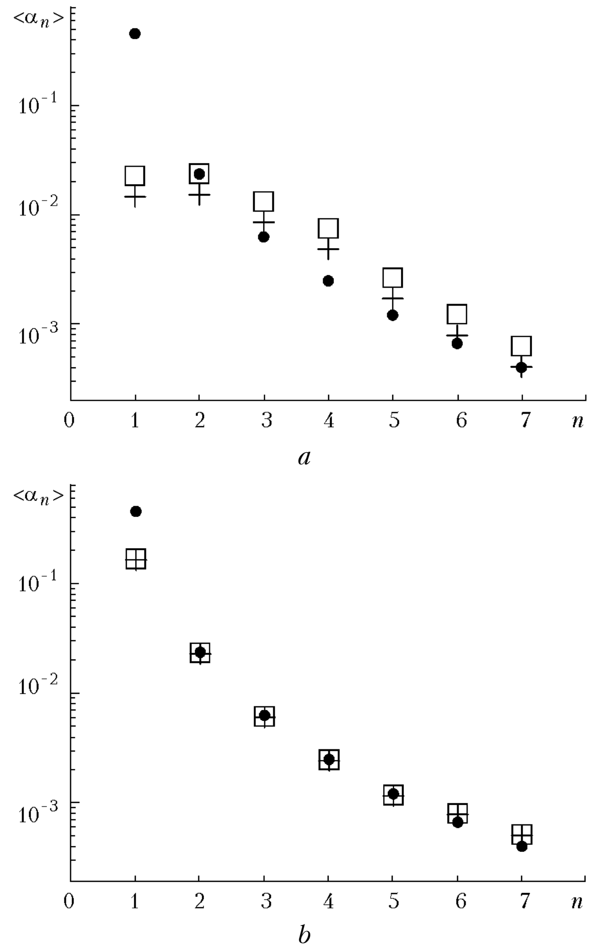


Fig. 3. Normalized coefficients $\langle \alpha_n \rangle$ of screens expansion over Zernike polynomials: normalizing to σ_1^2 sum ( ); normalizing to σ_2^2 sum (+); theoretical values (•).

For the $\mathbf{S}_1$ screen generated with the pulsed response function of 128 width, the coefficients $<\alpha_1>$ are significantly less while the coefficients $<\alpha_3>$ to $<\alpha_7>$ are noticeably larger than the theoretical values that reflects insufficient fraction of lower harmonics in its spectrum. For the $\mathbf{S}_2$ screen generated with the pulsed response function of 512 width, the coefficients $<\alpha_2>$ to $<\alpha_7>$ are very close to their theoretical values and only the coefficient $<\alpha_1>$, corresponding to aperture-average wave tilts, is a little less then its theoretical values. Similar result, i.e. increase of the fraction of low-frequency spatial harmonics in the generated phase screen was obtained in Refs. 12 and 13 owing to the use of a modified spectral method according to which the density of the computational grid nodes was iteratively increased in the spectral space in the neighborhood of the zero harmonic.

In accordance with the theoretical model, the distribution of the probability density of phase distortions in the turbulent atmosphere must be Gaussian.[1] Phase screens are to have the same distribution. Therefore, a randomizer with the normal distribution and zero mean and unity variance was used for filling the initial uncorrelated field **a**. Besides, the fields were generated using random numbers with the uniform distribution (with the same mean and variance). As the comparison showed, use of the initial fields with different statistics results in virtually similar distributions of the resulting screen. The Table below gives comparative characteristics of the distributions obtained for fields of 128×8192 size; here μ_1 is the mean value, σ^2 is the variance, coefficients of skewness μ_3/σ^3 and kurtosis $\mu_4/\sigma^4 - 3$, where μ_i is the central moment of the ith order. The results show that when needed, a more simple and fast randomizer with the uniform distribution can be used instead of a randomizer with the normal distribution.

Statistical characteristics of the screen generated for normal and uniform distributions of the initial noise field a

Distribution of **a**	μ_1	σ^2	Skewness	Kurtosis
normal	−0.11	6.49	0.12	−0.18
uniform	−0.1	6.36	$-2\cdot10^{-4}$	0.12

Calculating convolution in the method of sliding smoothing requires more operations than in the fast Fourier transform. Therefore, the method of generating moving screens proposed takes longer computation time than the spectral one. Nevertheless, this disadvantage is made up by the possibility of simulating fields of virtually any size. Moreover, in a random field simulated with the spectral method, a spurious correlation occurs on the scales larger than half of the screen's width due to spatial periodicity of the Fourier harmonics summed. Therefore only a quarter of the screen obtained have to be used for such fields.[12] The technique of sliding smoothing allows one to increase the usable fraction of the screen's width by means of increasing the size of the pulsed response function.

A modification of the spectral method, described in Ref. 14, allows approximate simulation of the rectangular phase screens as well as their "stitching" to obtain screens of required length and eliminate periodicity. In contrast, the sliding smoothing allows one to obtain screens of arbitrary large length within one regular procedure. Besides, in order to obtain any required transformation of the resulting screen it is sufficient to perform the transformation of the initial uncorrelated field and then apply sliding smoothing to it. To obtain such a result within the spectral method, the required spectrum transformation can be very complicated.

Generation of moving screens by the method of sliding smoothing can be easily supplemented with the above-mentioned technique to account for wind speed fluctuations,[5] that allow a better simulation of the light beam propagation through the turbulent atmosphere to be performed.

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