DETERMINATION OF THE ABSORPTION COEFFICIENT OF LAYERS OF A WEAKLY ABSORBING DISPERSED MEDIUM WITH ARBITRARY THICKNESS

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Formulas are derived for calculating the absorption coefficient of a dispersed medium, using measured values of the reflectances and transmittances of layers of finite thickness. Two methods for determining the absorption coefficient in the case of weak absorption are proposed. For the first method to be realized, it is sufficient to have one sample of finite thickness, for the second one two samples are needed: one of finite and the other of infinite thickness. It is shown that the second method permits one to determine the absorption coefficient with an error only slightly greater than the minimal one over a wider range of thicknesses than the first method.

Asymptotic solutions of the direct problem of the transfer equation for thick layers lie at the foundation of many well-known methods for determining the absorption coefficient ε_n of a scattering medium.[1–3] In the case of weak absorption these solutions allow one to obtain simple analytic expressions for estimating the characteristics of the scattering layer from their reflectances and transmittances[1,2,5] or the brightness coefficients at the layer boundaries.[3,4]

In the analysis of the absorption characteristics of different dispersed media, including atmospheric aerosol deposited on a substrate, the layer cannot always be taken to be optically thick and in this case the above methods cease to apply. In this paper we consider a technique for determining the absorption coefficient of a dispersed medium. This technique has none of the limitations associated with the requirement of large optical thickness of the layer. The proposed approach is based on the use of integral equations which relate the brightness at the layer boundaries with the brightness body in the depth mode. The derivation of these equations is based on the fact that under certain conditions of illumination the transfer equation for a planar layer allows separation of variables in the azimuthally averaged brightness coefficients.[6] In this case the radiation intensity at the optical depth $\tau = \varepsilon z$ (ε is the extinction coefficient of the medium; z is the distance from the upper boundary of the layer) is given by $J_\Gamma(\mu, \tau) = \varphi(\mu)\exp(-\Gamma\tau)$, where $\Gamma = \dfrac{1}{\nu_{max}}$; ν_{max} and $\varphi(\mu)$ are the largest eigenvalue and the eigenfunction of the homogeneous transfer equation corresponding to it, respectively:

$$(\nu_{max} - \mu)\varphi(\mu) = \frac{\Lambda\nu_{max}}{2}\int_{-1}^{1} x(\mu, \mu_0)\varphi(\mu_0)d\mu_0, \tag{1}$$

where $x(\mu, \mu_0) = \sum_{l=1}^{N}\frac{2l+1}{2}x_l P_l(\mu_0)$; x_l are the expansion coefficients of the scattering phase function over the Legendre polynomials $P_l(\mu_0)$; N is the number of expansion terms; μ and μ_0 are the cosines of the incident and scattering angles; Λ is the probability of photon survival, which is equal to the ratio of the scattering and extinction coefficients.

Let a plane-parallel layer of the scattering medium with optical depth τ_0 be illuminated from above by radiation whose angular distribution is given by the function $J_\Gamma(\mu, 0)$, and from below by radiation with angular distribution $J_\Gamma(-\mu, \tau_0)$, where $\tau_0 = \varepsilon z_0$; z_0 is the geometric thickness of the layer. Using the separation of variables of the functions $J_\Gamma(\mu, \tau_0)$, we can write the following relations connecting the brightness coefficients at the layer boundaries $\sigma(\mu, \mu_0)$ and $\rho(\mu, \mu_0)$ with the eigenfunction $\varphi(\bar{\mu})$ of the homogeneous transfer function:

$$\int_0^1 \mu_0\sigma(\mu,\mu_0)\varphi(\mu_0)d\mu_0 + e^{-\Gamma\tau_0}\int_0^1 \mu_0\rho(\mu,\mu_0)\varphi(-\mu_0)d\mu_0 = (e^{-\Gamma\tau_0} - e^{-\tau_0/\mu})\varphi(\mu); \tag{2}$$

$$e^{-\Gamma\tau_0}\int_0^1 \mu_0\sigma(\mu,\mu_0)\varphi(-\mu_0)d\mu_0 + \int_0^1 \mu_0\rho(\mu,\mu_0)\varphi(\mu_0)d\mu_0 = (1 - e^{-\Gamma\tau_0}e^{-\tau_0/\mu})\varphi(-\mu) \tag{3}$$

In weakly absorbing layers the brightness body in the depth mode is a linear function of μ_0 (see Ref. 1):

$$\varphi(\mu_0) = 1 + \alpha\mu_0, \tag{4}$$

where the parameter α is defined by the properties of the medium and must be found by measuring the angular distribution of the radiation at the layer boundaries.

Substituting Eq. (4) into Eqs. (2) and (3) and setting $\mu = 1$, we obtain the following relations, which are valid for layers of arbitrary thickness:

$$\sigma_1 + \alpha\sigma_2 + t(\rho_1 - \alpha\rho_2) = (t-T)(1+\alpha); \tag{5}$$

$$t(\sigma_1 - \alpha\sigma_2) + (\rho_1 + \alpha\rho_2) = (1-tT)(1-\alpha), \tag{6}$$

where

$$\sigma_n = \int_0^1 \mu_0^n \sigma(\mu=1,\mu_0)d\mu_0; \quad \rho_n = \int_0^1 \mu_0^n \rho(\mu=1,\mu_0)d\mu_0;$$

$$t = e^{-\Gamma\tau_0}; \quad T = e^{-\tau_0}.$$

For the sake of convenience, we introduce the coefficients $k_\sigma = \sigma_2/\sigma_1$ and $k_\rho = \rho_2/\rho_1$, which depend on the medium properties and the layer thickness. Formulas (5) and (6) then have the following forms:

$$\sigma_1(1+\alpha k_\sigma) + t\rho_1(1-\alpha k_\rho) = (t-T)(1+\alpha); \tag{7}$$

$$t\sigma_1(1-\alpha k_\sigma) + \rho_1(1+\alpha k_\rho) = (1-tT)(1-\alpha). \tag{8}$$

Let us now turn our attention to the fact that by virtue of the symmetry of $\sigma(\mu, \mu_0)$ and $\rho(\mu, \mu_0)$ in the variables μ and μ_0 $\sigma(\mu_0) = \sigma(\mu = 1, \mu_0) = \sigma(\mu_0, \mu = 1)$ and $\rho(\mu_0) = \rho(\mu = 1, \mu_0) = \rho(\mu_0, \mu = 1)$ can be determined by measuring the angular dependence of the brightness coefficients of the radiation exiting the layer for the case of normal incidence of the beam on the layer. From Eqs. (5)–(8) it follows that

$$\alpha = \sqrt{\frac{(1-\rho_1^2)-(\sigma_1+T)^2}{(1+k_\rho\rho_1)^2-(k_\sigma\sigma_1+T)^2}}; \tag{9}$$

$$t = \frac{(\sigma_1+T)+\alpha(k_\sigma\sigma_1+T)}{(1-\rho_1)+\alpha(1+k_\rho\rho_1)}; \tag{10}$$

$$\Gamma = -\frac{1}{\tau_0}\ln t. \tag{11}$$

A relation between the parameters α, Λ, and Γ can be found using Eqs. (1) and (4). Substituting Eq. (4) into Eq. (1), multiplying both sides of Eq. (1) by μ, and taking the integral over μ within the limits $[-1, 1]$, we obtain

$$\alpha = 3(1-\Lambda)/\Gamma. \tag{12}$$

Using Eqs. (11) and (12), it is possible to determine the absorption coefficient of the layer:

$$\varepsilon_n = \varepsilon(1-\Lambda) = \frac{1}{3}\frac{1}{z_0}\alpha(-\ln t), \tag{13}$$

i.e., ε_n is determined from data on the geometrical thickness of the layer and the moments of the brightness coefficients at the layer boundaries.

It is easy to see that relation (11) is transformed into the well-known formula

$$\varepsilon_n = q\Gamma\cdot\Gamma\varepsilon, \tag{13a}$$

where $q = (3 - x_1)^{-1}$. The parameters $q\Gamma$ and $\Gamma\varepsilon$ which enter into Eq. (13a) can be found from measurements of the values of the reflectance of an infinitely thick layer and of the transmittance of optically thick layers.[3–5,10] Note that with such an approach, contrary to the approach proposed in this work, it is necessary to deal with not one but with several samples.

Let us estimate the region of applicability of the obtained relations (9) and (10), using known calculational data on $\sigma(\mu)$ and $\rho(\mu)$. Calculations made with the help of data from Ref. 7 show that at $\Lambda \geq 0.99$ the error in the determination of t, α, and ε_n using formulas (9), (10), and (13) for the layer with optical depth $\tau_0 = 1$ did not exceed 3.5% and decreased as $\Lambda \to 1$. Calculations made with the help of the formulas used in Ref. 3 and 5 for such an optically thin layer at $\Lambda \geq 0.99$ indicate an error of about 50%. Numerical checks with the formulas used to determine the brightness coefficients obtained in Ref. 3 showed that the difference in values parameters Λ and t used to calculate the brightness coefficients $\sigma(\mu)$ and $\rho(\mu)$ in Ref. 3, and in the values of these parameters given by formulas (10)–(11) did not exceed 10% for $\Gamma = 0 \div 0.25$, $\tau_0 = 5 \div 30$, and $x = -\frac{2}{3}$ to $\frac{2}{3}$ (x_1 value used in our work is related with $\bar{x}_1$ used in Ref. 3, by the relation $\bar{x}_1 = \frac{3}{2}x_1$.

Let us now turn our attention to the fact that the parameters k_σ and k_ρ enter into Eqs. (7) and (8) as the coefficients of the parameter α, the value of which is small. Therefore, one can expect that if the coefficients k_σ and k_ρ are changed slightly by varying the optical parameters of the medium and the layer thickness, their influence will be small and, to determine the parameters α and t of the medium, one must only find the first moments of the brightness coefficients σ_1 and ρ_1. To determine the character of the variation of the coefficients k_σ and k_ρ, the brightness coefficients $\sigma(\mu)$ and $\rho(\mu)$ were calculated for layers of optical thickness $\tau_0 = 1,2,3$ and, for an infinitely thick layer from the data listed in the tables in Refs. 6 and 7. The results obtained indicate that for the medium with the phase function $x(\mu) = 1$ at $\tau_0 \leq 3$ and for the infinitely thick layer in which $x(\mu) = 1 + \frac{3}{2}x_1\mu$ at $x_1 = 0$ and 2/3, the values of k_σ change from 0.64 to 0.68. The experimental results for several weakly absorbing objects are as follows: one layer of filter paper has $k_\sigma = 0.71$ and $k_\rho = 0.68$; four layers of filter paper have $k_\sigma = 0.71$ and $k_\rho = 0.69$; a $BaSO_4$ layer with $z_0 = 0.5$–6 mm has $k_\sigma = 0.72$.

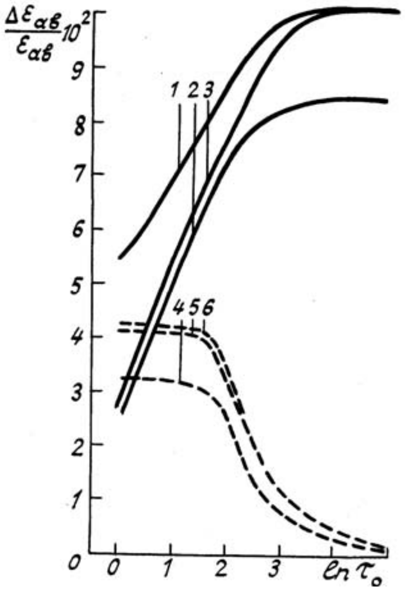

FIG. 1. Dependence of $\Delta\varepsilon_n/\varepsilon_n$ on τ_0 at $\Delta k_\rho = 0.2$ and $\Delta k_\sigma = 0$ (solid curves), Δk_ρ and $\Delta k_\sigma = 0.2$ (dashed curves) for $x_1 = 0$, $1 - \Lambda = 10^{-4}$ (curves 1, 4); $x_1 = 2/3$, $1 - \Lambda = 10^{-2}$ (curves 2, 5); $x_1 = 2/3$, $1 - \Lambda = 10^{-4}$ (curves 3, 6).

Let us consider the influence of the errors Δk_ρ and Δk_σ of assigning the coefficients k_ρ and k_σ on the relative error

$$\Delta\varepsilon_n / \varepsilon_n = \frac{1}{\varepsilon_n(k_\rho, k_\sigma, \tau_0)}[\varepsilon_n(k_\rho + \Delta k_\rho, k_0 + \Delta k_\sigma, \tau_0) - $$

$$-\varepsilon_n(k_\rho, k_\sigma, \tau_0)].$$

Figure 1 shows the dependence of the error $\Delta\varepsilon_n/\varepsilon_n$ on τ_0 for the scattering phase function $x(\mu) = 1 + \frac{2}{3}x_1\mu$. It is clear from this figure that the relative error in determining ε_n due to the inaccuracy in determining k_ρ increases with the thickness of the optical layer, remaining less than 11% even for very weak absorption when $1 - \Lambda = 10^{-4}$. Calculations show that for $\Delta k_\rho = 0.02$ the value of $\Delta\varepsilon_n/\varepsilon_n$ does not exceed 1.2%. The error $\Delta\varepsilon_n/\varepsilon_n$ (due to inaccurately set K_σ) decreases with τ_0 increase and does not exceed 5%, if $\Delta k_\sigma = 0.2$; and, if $\Delta k_\sigma = 0.02$, then $\Delta\varepsilon_n/\varepsilon_n$ does not exceed 0.5%. According to the above data, k_σ and k_ρ vary only slightly for actual weakly absorbing media: $k_\sigma \cong 0.71$ and $k_\rho \cong 0.68$ with errors $\Delta k_\sigma = \Delta k_\rho \cong 0.01$. For such media, ε_n can therefore be determined by measuring only the transmittance σ_1 and reflectance ρ_1.

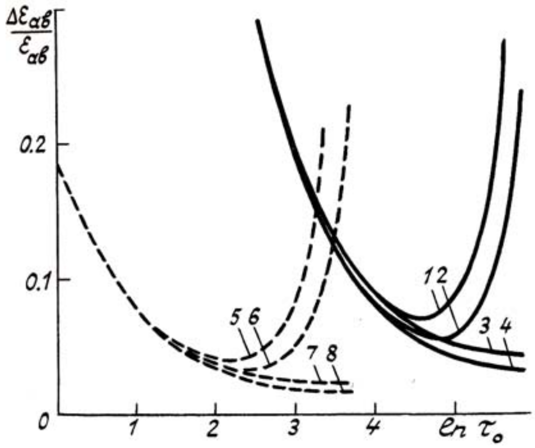

FIG. 2. Dependence of $\Delta\varepsilon_n/\varepsilon_n$ on τ_0 for the first procedure of determining ε_n at $\Delta k_\sigma = \Delta k_\rho = 0$: 1) $\Delta\sigma_1 = 0.001$, $\Delta\rho_1 = 0$, $x_1 = 0$; 2) $\Delta\sigma_1 = 0.001$, $\Delta\rho_1 = 0$, $x_1 = 2/3$; 3) $\Delta\sigma_1 = 0$, $\Delta\rho_1 = 0.001$, $x_1 = 0$; 4) $\Delta\sigma_1 = 0$, $\Delta\rho_1 = 0.001$, $x_1 = 2/3$; 5) $\Delta\sigma_1 = 0.005$, $\Delta\rho_1 = 0$, $x_1 = 0$; 6) $\Delta\sigma_1 = 0.005$, $\Delta\rho_1 = 0$, $x_1 = 2/3$; 7) $\Delta\sigma_1 = 0$, $\Delta\rho_1 = 0.005$, $x_1 = 0$; 8) $\Delta\sigma_1 = 0$, $\Delta\rho_1 = 0.005$, $x_1 = 2/3$. $1 - \Lambda = 10^{-4}$ (solid curves); $1 - \Lambda = 10^{-2}$ (dashed curves).

Let us analyze the influence on the accuracy of determining ε_n of errors in the measurements of σ_1 and ρ_1. We calculate the relative error

$$\Delta\varepsilon_n / \varepsilon_n = \frac{1}{\varepsilon_n(\sigma_1, \rho, k_\rho, k_\sigma)}[\varepsilon_n(\sigma_1 + \Delta\sigma_1\rho_1 + \Delta\rho_1, k_\rho, k_\sigma) - $$

$$-\varepsilon_n(\sigma_1, \rho_1, k_\rho, k_\sigma)]$$

as a function of the optical thickness τ_0, at $k_\sigma = 0.71$ and $k_\rho = 0.68$.

Figure 2 presents calculational results for a medium with a spherical scattering phase function (curves 1, 3, and 7) and with the elongate scattering phase function $x = 1 + \mu$, (curves 2, 4, 6, and 8). The dependences $\Delta\varepsilon_n/\varepsilon_n(\tau_0)$ at $\Delta\rho_1 = 0$ have a minimum. As Λ increases, its position shifts toward larger optical depths, and the minimal error grows. With increase in the elongateness of the scattering phase function $x(\mu)$, there occurs a small shift of minimum toward larger τ_0 values and a decrease in the value of the minimal possible error in the determination of ε_n. At $1 - \Lambda = 10^{-4}$ and $x_1 = \frac{2}{3}$ the minimal error in the determination of ε_n due to the measurement error $\Delta\sigma_1 = 0.001$ is equal to $\delta_0\varepsilon_n \cong 5.6\%$ and attained at the optical depth $\tau_0 \cong 1.2\%$. The dependence $\Delta\varepsilon_n/\varepsilon_n(\tau_0)$ at $\sigma_1 = 0$ is monotonically decreasing and coincides with $\left.\frac{\Delta\varepsilon_n}{\varepsilon_n}\right|_{\Delta\rho_1=0}$ at small optical depths, as was confirmed by calculations of ε_n using the experimental data from Refs. 8 and 9 for MS-14 glass. The calculations performed show that with decreasing thickness of the glass layer the random deviations of ε_n from its mean value increase. This result is explained by the fact that according to the estimates made in Ref. 9, for MS–14 glass $1 - \Lambda \cong 2 \cdot 10^{-5}$ and $\varepsilon_n \cong 120$ mm . Hence, layers of 0.3–6 mm thickness correspond to the left-hand part of the curve

$$\left.\frac{\Delta\varepsilon_n}{\varepsilon_n}(\tau_0)\right|_{\Delta\rho_1=0}.$$

Let us consider yet another way of determining ε_n. It consists in measuring not only the brightness coefficients $\sigma(\mu)$ and $\rho(\mu)$ for the finite-thickness layers but also the brightness coefficient of the infinitely thick layer $\rho_\infty(\mu)$. And indeed, in the limit $t \to 0$ (as follows from Eq. (8))

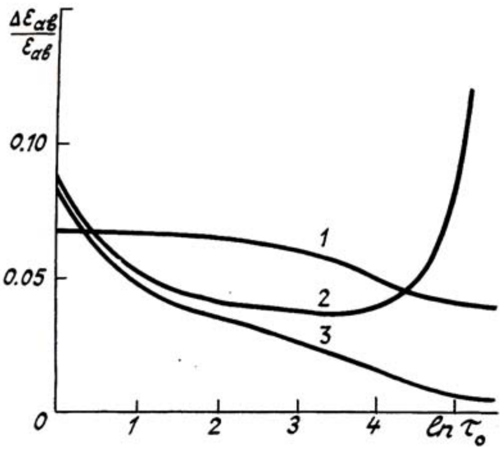

FIG. 3. Dependence of $\Delta\varepsilon_n/\varepsilon_n$ on τ_0 for the second procedure of determining ε_n at $1 - \Lambda = 10^{-4}$.

$$\alpha = \frac{1-\rho_{1\infty}}{1+k_\rho\rho_{1\infty}}, \tag{14}$$

where $\rho_{1\infty} = \int_0^1 \rho_\infty(\mu)\mu d\mu$.

Using this value of α, we determine, first, t using formula (10) and, then, ε_n from Eq. (13).

Let us consider the error in determining ε_n using this calculational procedure. It is evident that in this case it is determined by the errors incurred in measuring $\rho_{1\infty}$, σ_1, and ρ_1. The contribution to the error due to the inaccuracy in measuring $\rho_{1\infty}$ at $1 - \Lambda = 10^{-4}$ and $x(\mu) = 1$ is represented by curve 1 in Fig. 3. Curves 2 and 3 in this figure correspond to the contributions of the σ_1 and ρ_1 measurement errors to the error in determining ε_n.

Let us compare the errors in determining ε_n in both procedures. As can be seen from Fig. 3, the errors caused by the ρ_1 and $\rho_{1\infty}$ measurement errors decrease with increase in the layer thickness; moreover, for thin layers, the error caused by inaccuracies in the measurement of ρ_1 is substantially smaller than that incurred in the first procedure (see Fig. 2). The error due to error in the measurement of σ_1 for $\tau_0 \le 115$ is smaller than that incurred in the first procedure. As τ_0 increases, this error grows in both methods and contributes greatly to the total error in the determination of ε_n. As the calculations show, the dependence of the error components on the absorption and elongateness of the scattering phase function $x(\mu)$ have the same character as in the first method. As A decreases, all of the components of the relative error decrease, and the minimum in the dependence of $\left.\frac{\Delta\varepsilon_n}{\varepsilon_n}\right|_{\Delta\rho_1\neq 0}$ on τ_0 shifts toward smaller values of τ_0. As the elongateness of the scattering phase function increases, all components of the relative error also decrease, and the minimum of the dependence $\left.\frac{\Delta\varepsilon_n}{\varepsilon_n}\right|_{\Delta\rho_1=0}$ on τ_0 shifts towards larger values of τ_0.

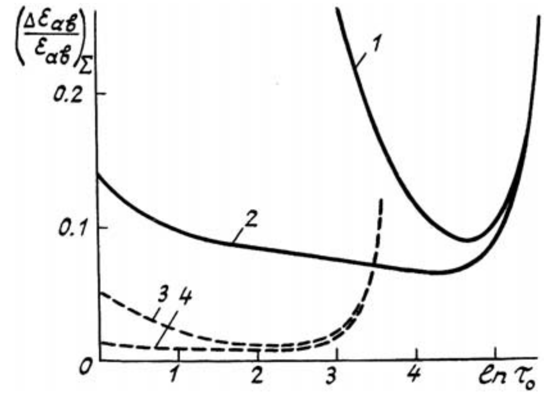

FIG. 4. Dependence of $(\Delta\varepsilon_n/\varepsilon_n)_\Sigma$ on τ_0 for the first and second procedures of determining ε_n at $\Delta\rho_1 = \Delta\sigma_1 = \Delta\rho_{1\infty} = 0.001$: the first procedure (curves 1 and 3); the second procedure (curves 2 and 4). $1 - \Lambda = 10^{-4}$ (solid curves); $1 - \Lambda = 10^{-2}$ (dashed curves).

In conclusion we present data on the total relative error in determining the absorption coefficient using both methods. To determine the total error, we use the well-known rule of addition of variances of random values. The dependence of the total error $\left(\frac{\Delta\varepsilon_n}{\varepsilon_n}\right)_\Sigma$ on $\ln\tau_0$ for both methods for $\Delta\rho_1 = \Delta\sigma_1 = \rho_{1\infty} = 10^{-3}$ is shown in Fig. 4. It can be seen from the figure that the minimal errors attained in the first and the second methods are similar, but the second method makes it possible to determine ε_n with an error only slightly different from the minimal one over a wider interval of layer thicknesses than is the case in the first method.

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