Polarization of light field generated by a point unidirectional source in a turbid anisotropically scattering medium

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The small-angle modification of the method of spherical harmonics is generalized for the case of a point unidirectional (PU) light source in an infinite anisotropically scattering medium. Within the framework of this generalization, the vector radiative transfer equation is solved for an unpolarized PU source. Equations describing the state of polarization of the scattered radiation are reduced to the form convenient for use in engineering practice. The equations obtained are analyzed, and it is shown that the minimum of polarization coincides with the direction of sighting at the maximum of radiance, that corresponds to the Umov law.

The problem of describing polarized radiation propagation through a turbid medium can be reduced to determination of the Green's function of the vector radiative transfer equation (VRTE).[1,2] Physically, this means irradiation of the medium with a point unidirectional (PU) source. This paper is devoted to solution of this problem. It is proposed to solve it based on the generalization, for the case of polarization, of the existing small-angle solution of the scalar radiative transfer equation. As a solution, we took the small-angle modification of the method of spherical harmonics (MSH)[3,4] because it is most general among other small-angle methods.[3]

To describe polarization, we will use the CP representation[5,6]:

$$\mathbf{L} = \begin{bmatrix} L_{+2} \\ L_{+0} \\ L_{-0} \\ L_{-2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} Q - iU \\ I - V \\ I + V \\ Q + iU \end{bmatrix}, \qquad (1)$$

where (I, Q, U, V) are the Stokes parameters.[1]

Let a PU source of unpolarized radiation emitted along the direction $\hat{\mathbf{q}}$ be located at some point of an infinite homogeneous medium (the cap ∧ indicates the unit vector), and it is required to determine the column vector of polarization parameters $\mathbf{L}(r, \hat{\mathbf{q}}, \hat{\mathbf{l}})$ in the direction $\hat{\mathbf{l}}$ at the point spaced by r from the source. Then the VRTE boundary-value problem can be written in the form[7,8]:

$$\begin{cases} -\dfrac{\sqrt{1-\mu^2}}{r}\left(\sqrt{1-\eta^2}\,\dfrac{\partial \mathbf{L}(r,\mu,\eta,\varphi)}{\partial \eta}\cos\varphi + \right. \\ \left. + \dfrac{\eta}{\sqrt{1-\eta^2}}\,\dfrac{\partial \mathbf{L}(r,\mu,\eta,\varphi)}{\partial \varphi}\sin\varphi\right) + \varepsilon \mathbf{L}(r,\mu,\eta,\varphi) + \\ + \mu\,\dfrac{\partial \mathbf{L}(r,\mu,\eta,\varphi)}{\partial r} + \dfrac{1-\mu^2}{r}\,\dfrac{\partial \mathbf{L}(r,\mu,\eta,\varphi)}{\partial \mu} = \\ = \dfrac{\sigma}{4\pi}\displaystyle\int_0^{2\pi}\int_{-1}^{1} \overleftrightarrow{R}(-\chi)\,\overleftrightarrow{x}(\mu,\varphi,\mu',\varphi') \times \\ \times \overleftrightarrow{R}(\chi')\,\mathbf{L}(r,\eta,\mu',\varphi')\,d\mu' d\varphi', \\ \mathbf{L}(r,\mu,\eta,\varphi)\big|_{r\to 0} = \dfrac{\mathbf{L}_0}{(2\pi r)^2}\,\delta(\mu-1)\delta(\eta-1), \end{cases} \qquad (2)$$

where $\eta = (\hat{\mathbf{q}}, \hat{\mathbf{r}})$; $\mu = (\hat{\mathbf{l}}, \hat{\mathbf{r}})$; $\varphi = \pi - \psi$, ψ is the dihedral angle between the planes $\hat{\mathbf{q}} \times \hat{\mathbf{r}}$ and $\hat{\mathbf{l}} \times \hat{\mathbf{r}}$; $\hat{\mathbf{r}}$ determines the direction from the source to the observation point; $\mathbf{L}_0 = (0;\, 0.5;\, 0.5;\, 0)^{\mathrm{T}}$; ε and σ are the extinction and scattering coefficients; $\overleftrightarrow{R}(\hat{\mathbf{l}} \times \hat{\mathbf{l}}' \to \hat{\mathbf{r}} \times \hat{\mathbf{l}})$ is the matrix of transformation of the vector parameter at rotation of the reference plane; χ and χ' are the dihedral angles between the planes $\hat{\mathbf{l}} \times \hat{\mathbf{l}}'$, $\hat{\mathbf{r}} \times \hat{\mathbf{l}}$ and $\hat{\mathbf{r}} \times \hat{\mathbf{l}}'$, $\hat{\mathbf{l}} \times \hat{\mathbf{l}}'$, respectively; $\overleftrightarrow{x}()$ is the scattering phase matrix. The plane $\hat{\mathbf{r}} \times \hat{\mathbf{l}}$ is taken as a reference plane.

The problem is solved based on the method of spherical harmonics. Therefore, let us present the solution as the following series:

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$$\mathbf{L}(r, \mu, \eta, \varphi) = \frac{1}{2\pi} \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=-\infty}^{\infty} i^m \frac{2p+1}{2} \frac{2k+1}{2} \times$$

$$\times \overset{\leftrightarrow}{Y}{}_k^m(\mu)\, Q_p^m(\eta)\, \mathrm{e}^{im\varphi}\, C_{kp}^m(r), \quad (3)$$

where

$$\overset{\leftrightarrow}{Y}{}_p^m(\mu) = \mathrm{Diag}(P^p_{m,+2}(\mu),\, P^p_{m,0}(\mu),\, P^p_{m,0}(\mu),\, P^p_{m,-2}(\mu)),$$

$$Q_p^m(\mu) = \sqrt{\frac{(p-m)!}{(p+m)!}}\, P_p^m(\mu), \quad P^k_{mn}(\mu), \quad P_p^m(\mu)$$

are, respectively, the generalized Legendre functions and the associated Legendre functions.[9] The boundary conditions can be reduced to the form

$$\lim_{r \to 0} \mathbf{C}_{kp}^m(r) = \mathbf{L}_0/(2\pi r^2). \quad (4)$$

Resolve the scattering matrix as

$$[\overset{\leftrightarrow}{x}(\gamma)]_{rs} = \sum_{k=0}^{\infty} (2k+1)\, [\overset{\leftrightarrow}{x}_k]_{rs}\, P^k_{rs}(\gamma), \quad (5)$$

where $\gamma = (\hat{\mathbf{l}}\hat{\mathbf{l}}')$, and the indices r and s take the values -2, -0, $+0$, $+2$. Multiply the equation by $\overline{\overset{\leftrightarrow}{Y}{}_k^m(\mu)}$, $Q_p^m(\eta)$, $\mathrm{e}^{-im\varphi}$ and integrate over the entire range of variability of the arguments. Using equations for the generalized Legendre functions[9,10] similarly to the scalar case,[11] we obtain the infinite system of connected differential MSH equations for the coefficients $\mathbf{C}_{kp}^m(r)$:

$$\frac{\partial}{\partial r}\left(\sum_{j=\pm 1} \overset{\leftrightarrow}{A}{}^m_{k+\delta_{1j}}\, \mathbf{C}^m_{p,k+j}(r) + \overset{\leftrightarrow}{B}_k\, \mathbf{C}^m_{kp}(r)\right) +$$

$$+ \frac{1}{r}\left(\sum_{j=\pm 1} j(k+j+\delta_{1j}) t_{k+\delta_{1j}} \overset{\leftrightarrow}{C}_{k+\delta_{1j}}\, \mathbf{C}^m_{p,k+j} + m\overset{\leftrightarrow}{B}_k\, \mathbf{C}^m_{pk}\right) +$$

$$+ \frac{1}{2r} \sum_{j=\pm 1} f_p^{-jm} \left(\sum_{\nu=\pm 1} \nu T^{\nu jm}_{k+\nu}\, \overset{\leftrightarrow}{C}_{k+\delta_{1j}}\, \mathbf{C}^{m+j}_{p,k+\nu} - j f^{jm}_k\, \overset{\leftrightarrow}{B}_k\, \mathbf{C}^{m+j}_{pk}\right) =$$

$$= -(2k+1)\, \varepsilon(\overset{\leftrightarrow}{1} - \Lambda \overset{\leftrightarrow}{x}_k)\, \mathbf{C}^m_{kp}, \quad (6)$$

where

$$f_p^m = \sqrt{(p+m)(p-m+1)};\ [\overset{\leftrightarrow}{B}_k]_{rs} = \frac{(2k+1)s}{k(k+1)}\, \delta_{rs};$$

$$[\overset{\leftrightarrow}{C}_k]_{rs} = \frac{\sqrt{k^2 - s^2}}{k}\, \delta_{rs}; \quad \Lambda = \frac{\sigma}{\varepsilon};$$

$$[\overset{\leftrightarrow}{A}{}_k^m]_{rs} = \frac{\sqrt{(k^2 - m^2)(k^2 - s^2)}}{k}\, \delta_{rs}; \quad t_k = \sqrt{k^2 - m^2}; \quad (7)$$

$$T_k^m = \sqrt{(k+m)(k+m+1)},$$

and the successively repeating indices mean their product.

Radiation scattering in actual media occurs for particles much larger than the radiation wavelength, thus leading to the strongly anisotropic scattering phase function. This circumstance causes strong angular dependence of the brightness body. Therefore, the brightness spectrum weakly depends on the number k; it is a smooth, monotonically decreasing function. The angular dependence of polarization parameters is not that strong, but the spectrum here contains a large number of terms,[12] and this defines the dominant effect of the terms with $p, k \gg 1$ in the series (3). These properties of the spectrum allow us to redefine the connection between $\mathbf{C}^m_{kp}(r)$ and $\mathbf{C}^m_{k\pm 1,p}(r)$ and to break the obtained system of equations. For this purpose, the expansion coefficients are assumed continuously depending on their indices and the resolution[3,4]

$$\mathbf{C}^m(r, k \pm 1, p) \approx \mathbf{C}^m(r, k, p) \pm \partial \mathbf{C}^m(r, k, p)/(\partial k) \quad (8)$$

is conducted.

However, for a PU source these assumptions do not lead to an analytically solvable equation. Therefore, assume additionally that the dependence of polarization parameters on the zenith angle is stronger than on the azimuth angle: $p, k \gg |m|$, that is, the brightness body is more anisotropic than asymmetric. This allows us to assume the following:

$$\overset{\leftrightarrow}{A}{}_k^m \approx k\overset{\leftrightarrow}{1}, \quad t_k \approx k, \quad \overset{\leftrightarrow}{C}_k \approx \overset{\leftrightarrow}{1},$$
$$\overset{\leftrightarrow}{B}_k \approx \overset{\leftrightarrow}{0}, \quad f_p^m \approx p, \quad T_k^m \approx k + m. \quad (9)$$

Introduce the function

$$\mathbf{C}^m(k, p) = \mathbf{Y}^m(k, p)/r^2. \quad (10)$$

Substituting these equations into Eq. (6), we obtain the equation

$$\frac{\partial \mathbf{Y}^m}{\partial r} + \frac{k}{r}\frac{\partial \mathbf{Y}^m}{\partial k} + \frac{p}{2r}\left[\frac{\partial \mathbf{Y}^{m+1}}{\partial k} + \frac{\partial \mathbf{Y}^{m-1}}{\partial k} + \right.$$
$$\left. + \frac{1}{k}\left((m+1)\mathbf{Y}^{m+1} - (m-1)\mathbf{Y}^{m-1}\right)\right] = -\varepsilon(1 - \Lambda \overset{\leftrightarrow}{x}_k)\, \mathbf{Y}^m. \quad (11)$$

Define the new function $\mathbf{f}(r, \mathbf{k}, \mathbf{p})$ so that the sought functions $\mathbf{Y}^m$ are the coefficients of its expansion into the Fourier series:

$$\mathbf{f}(r, \mathbf{k}, \mathbf{p}) = \sum_{m=-\infty}^{\infty} \mathbf{Y}^m(r, k, p)\, \mathrm{e}^{im\psi}, \quad (12)$$

the vectors $\mathbf{k}$ and $\mathbf{p}$ are equal to the indices k and p in the absolute values and lie in the same plane, and ψ is the angle between them.

Multiply Eq. (11) by $\mathrm{e}^{im\psi}$ and sum it up over m from $-\infty$ to $+\infty$. We obtain

$$\frac{\partial \mathbf{f}(r, \mathbf{k}, \mathbf{p})}{\partial r} + \frac{1}{r}(\mathbf{k} + \mathbf{p}, \nabla_k)\, \mathbf{f}(r, \mathbf{k}, \mathbf{p}) =$$

$$= -\varepsilon(1 - \Lambda \overleftrightarrow{x}_k)\, \mathbf{f}(r, \mathbf{k}, \mathbf{p}). \qquad (13)$$

Solving this equation similarly to the scalar case[11] by the method of integration along a characteristic, we have

$$\mathbf{f}(r, \mathbf{k}, \mathbf{p}) =$$
$$= \frac{1}{2\pi} \exp\left[-\varepsilon r + \sigma r \int_0^1 \overleftrightarrow{x}(\,|(\mathbf{k} + \mathbf{p})\xi - \mathbf{k}|\,)\, \mathrm{d}\xi\right] \mathbf{L}_0, \qquad (14)$$

that leads to the equation

$$\mathbf{C}^m_{kp}(r) = \frac{e^{-\varepsilon r}}{4\pi^2 r^2} \int_0^{2\pi} \exp\left\{-im\psi + \sigma r \int_0^1 \overleftrightarrow{x} \times \right.$$
$$\left. \times \left(\left|\sqrt{p^2(1-\zeta)^2 + k^2\zeta^2 - 2\zeta(1-\zeta)pk\cos\psi}\right|\right) \mathrm{d}\zeta \right\} \mathrm{d}\psi\, \mathbf{L}_0. \qquad (15)$$

Equations (3) and (15) determine the polarization parameters of the light field generated by the unpolarized PU source in the infinite medium. At small sighting and emission angles, the obtained equations lead to the solution, which can be obtained from the development of the approach from Refs. 13–15.

These equations imply calculation of the double integral and summation over three variables. Simplify the solution obtained by accepting some assumptions.

Consider relatively large optical depths, at which the angular dependences of the polarization parameters are the functions smoother than the scattering phase matrix. Correspondingly, the spectrum of the scattering phase matrix is smoother than that of the vector parameter. Then, within the range, where the spectrum of the vector parameter is nonzero, the spectrum of the scattering matrix varies only slightly, and the consideration can be restricted to a small number of terms when expanding it into the Taylor series. Consider the class of the scattering phase functions with the zero first derivative in the spectrum:

$$\overleftrightarrow{x}(k) \approx \overleftrightarrow{1} - \frac{k^2}{a^2} \overleftrightarrow{B}, \quad \overleftrightarrow{B} = \begin{pmatrix} 1 & b & \bar{b} & 0 \\ b & 1 & 0 & \bar{b} \\ \bar{b} & 0 & 1 & b \\ 0 & \bar{b} & b & 1 \end{pmatrix},$$

$$b = -\frac{1}{2}(P + iQ), \qquad (16)$$

where $0 \le P,\ Q \le 1$ determine the appearance of the linear and circular polarizations in a scattering event; a is a parameter of the scattering phase function. We can show that the spectrum (5) of the scattering phase matrix in the form[4,12]

$$\overleftrightarrow{x}(\mu) = x(\mu) \begin{pmatrix} \nu & \omega & \bar{\omega} & 0 \\ \omega & 1 & 0 & \bar{\omega} \\ \bar{\omega} & 0 & 1 & \omega \\ 0 & \bar{\omega} & \omega & \nu \end{pmatrix},$$

$$\nu = \frac{(1 + \mu)^2}{4}, \quad \omega = b(\mu^2 - 1), \qquad (17)$$

roughly satisfies this expansion, if the spectrum x_k of the scattering phase function $x(\mu)$ at expansion into a Taylor series over the index has no first derivative. Assume that the parameter a has a rather large value, at which Eq. (16), as well as the approximations (8) and (9) are acceptable.

The well-known equation for the matrix exponent is[16]

$$\exp(z\overleftrightarrow{K}) = \sum_{j=1}^{4} \frac{\exp(z\zeta_j)(\zeta_j \overleftrightarrow{1} - \overleftrightarrow{K})^{\mathrm{V}}}{\left.\dfrac{\mathrm{d}}{\mathrm{d}u} \det(u\overleftrightarrow{1} - \overleftrightarrow{K})\right|_{u = \zeta_j}}, \qquad (18)$$

where $(\ )^{\mathrm{V}}$ stands for the matrix of algebraic adjuncts, and ζ_j are solutions of the equation $\det(\zeta \overleftrightarrow{1} - \overleftrightarrow{K}) = 0$. Our task is to compile the series (3) with the coefficients (15). The spectrum of the sought solution is smooth, and we are interested in large k and p and small observation angles, therefore the sums can be replaced by integrals, and the generalized Legendre functions can be replaced by Bessel functions by analogy with the well-known relation between the Hankel transform and expansion into a series over spherical functions.[9] Then the series (3) corresponds to the integral transformation

$$\mathbf{L}(r, \mathbf{q}, \mathbf{I}) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}^2 k \times$$
$$\times \int_0^{\infty}\int_0^{2\pi} p\mathrm{d}p\mathrm{d}\xi \exp(i\mathbf{kq} + i\mathbf{pI})\, \mathrm{O}(\xi)\, \mathbf{L}(r, \mathbf{k}, \mathbf{p}), \qquad (19)$$

$$\mathbf{L}(r, \mathbf{k}, \mathbf{p}) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \mathbf{C}^m(r, k, p) \exp(-im\zeta), \qquad (20)$$

where $\cos\xi = (\hat{\mathbf{p}}\,\hat{\mathbf{I}})$; $\cos\zeta = (\hat{\mathbf{k}}\,\hat{\mathbf{p}})$; $\mathrm{O}(\xi) = \mathrm{Diag}\,(\exp(2i\xi);\ 1;\ 1;\ \exp(-2i\xi))$; $\mathbf{q} = \hat{\mathbf{r}} - \hat{\mathbf{q}}$; $\mathbf{I} = \hat{\mathbf{r}} - \hat{\mathbf{I}}$.

Substitute Eqs. (16), (18), and (20) into Eq. (19). The integral (19) can be calculated exactly. The final equations have the form

$$\mathbf{L}(r, \mathbf{q}, \mathbf{I}) = \frac{\exp[-\varepsilon r(1 - \Lambda)]}{4r^2} \frac{3a^4}{(2\pi\sigma r)^2} \times$$
$$\times \begin{pmatrix} M(\lambda_2) - M(\lambda_1) \\ L(\lambda_2) + L(\lambda_1) \\ L(\lambda_2) + L(\lambda_1) \\ \overline{M(\lambda_2)} - \overline{M(\lambda_1)} \end{pmatrix}, \qquad (21)$$

$$M(\lambda) = \frac{2\sigma r}{a^2\lambda}\frac{[2l - q\exp(i\varphi)]^2}{(2\mathbf{I}+\mathbf{q})^4}\exp\left(-\frac{3a^2\,\mathbf{q}^2}{4\sigma r\lambda}\right)\times$$

$$\times\left\{1-\exp\left[-\frac{a^2(2\mathbf{I}+\mathbf{q})^2}{4\sigma r\lambda}\right]\left[1+\frac{a^2(2\mathbf{I}+\mathbf{q})^2}{4\sigma r\lambda}\right]\right\}\xrightarrow[\mathbf{I},\,\mathbf{q}\to 0]{}$$

$$\xrightarrow[\mathbf{I},\,\mathbf{q}\to 0]{}\frac{a^2}{8\lambda^3\sigma r}[2l - q\exp(i\varphi)]^2, \qquad (22)$$

$$L(\lambda) = \frac{1}{\lambda^2}\exp\left[-\frac{a^2}{\lambda\sigma r}(\mathbf{I}^2+\mathbf{q}^2+\mathbf{Iq})\right],$$

$$\lambda_1 = 1+P,\quad \lambda_2 = 1-P,\quad |\mathbf{q}| = q,\quad |\mathbf{I}| = l.$$

Equations (21) and (22) are VRTE solutions. Taking into account that

$$\delta(x) = \lim_{c\to 0}\frac{1}{c\sqrt{\pi}}\exp\left(-\frac{x^2}{c^2}\right), \qquad (23)$$

we can show that the solution obtained satisfies the boundary conditions (2) written in the form

$$\mathbf{L}(r,\hat{\mathbf{q}},\hat{\mathbf{I}})\big|_{r\to 0} = \frac{\mathbf{L}_0}{r^2}\,\delta(\hat{\mathbf{r}}-\hat{\mathbf{I}})\,\delta(\hat{\mathbf{r}}-\hat{\mathbf{q}}). \qquad (24)$$

Expansion of the spectrum of scattering phase function into the series similar to Eq. (16) for the scalar radiative transfer equation was proposed in Ref. 15.

Let us analyze the equation obtained. It falls in the class of diffusion approximations (DA), what is confirmed by the impossibility of expanding it into a series over the number of scattering events (powers of Λ). From the equality $L_{+0} = L_{-0}$ it follows that $V = 0$ at small emission and sighting angles and the processes connected with the appearance of circular polarization are weak. In the case of sighting at the source, $\mathbf{I} = \mathbf{q} = 0$ and $M(\lambda) = 0$ along the emission axis, and when polarization is absent, what corresponds to the symmetry of the problem. In this case, $M(\lambda)$ can be determined accurate to the factor of $\exp(2i\varphi)$ describing rotation of the reference plane $\hat{\mathbf{I}}\times\hat{\mathbf{r}}$ in the CP representation,[7] since its position in this case is not determined.

The brightness from solution (21) is maximum, when

$$\mathbf{I}^2+\mathbf{q}^2+\mathbf{Iq} = \min \Leftrightarrow 5 - 3\hat{\mathbf{r}}\hat{\mathbf{q}} - \hat{\mathbf{I}}(3\hat{\mathbf{r}}-\hat{\mathbf{q}}) =$$

$$= \min \Leftrightarrow \hat{\mathbf{I}} \uparrow\uparrow 3\hat{\mathbf{r}} - \hat{\mathbf{q}}. \qquad (25)$$

This result is already known.[15] The appearance of maximum under condition (25) is the consequence of multiple scattering, since both the first and the second orders of scattering have maximum in the case of sighting at the source.

Determine the degree of polarization as $R = \sqrt{Q^2+U^2+V^2}/I$. For simplicity, consider the case $P = 1$, $\varphi = 0$. Then

$$M(\lambda_2) = L(\lambda_2) = 0, \qquad Q = V = 0,$$

$$R = \frac{|Q|}{I} = \frac{L_{+2}+L_{-2}}{L_{+0}+L_{-0}} = \frac{M(\lambda_1)}{L(\lambda_1)} = \frac{e^y - 1 - y}{2y},$$

$$y = \frac{a^2(2l-q)^2}{4\sigma r(1+P)}. \qquad (26)$$

The minimum of the function (26) equal to zero takes place at $2l = q$, what is equivalent to the condition (25) at small angles. The appearance of the point with zero polarization coinciding with the point of maximum brightness is the result of multiple scattering that corresponds to the Umov law[17]: the degree of polarization of the radiation reflected by a plane layer of a turbid medium decreases with the increase of the reflection coefficient.

Analysis of the function (26) shows that it has only one extreme. Physically, this seems incorrect: at large angles the degree of polarization should decrease again, so one maximum more should exist. Consequently, the obtained solutions are valid at small observation and emission angles.

Now consider how the obtained VRTE solution is connected with the known solutions of the same order of accuracy. For this purpose, express the field generated by a plane unidirectional source of unpolarized radiation incident normally onto an infinite homogeneous layer of turbid medium as:

$$\mathbf{L}(z,\hat{\mathbf{I}}\hat{\mathbf{z}}) = \int \overleftrightarrow{R}(\chi)\,\mathbf{L}(z,\hat{\mathbf{z}}\hat{\mathbf{r}},\hat{\mathbf{I}}\hat{\mathbf{r}},\varphi)\,d^2\rho,$$

$$\mathbf{r} = \{\rho, z\}, \qquad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}, \qquad (27)$$

where the azimuth angle φ is measured as rotation relative the vector $\mathbf{r}$; χ is the angle of rotation of the reference plane $\hat{\mathbf{r}}\times\hat{\mathbf{I}}$ to the plane $\hat{\mathbf{z}}\times\hat{\mathbf{I}}$.

Let us integrate over the solid angle

$$\mathbf{L}(z,\hat{\mathbf{I}}\hat{\mathbf{z}}) = \frac{1}{2\pi}\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=-\infty}^{\infty} i^m\,\frac{2p+1}{2}\,\frac{2k+1}{2}\sqrt{\frac{(p-m)!}{(p+m)!}}\times$$

$$\times\int_{(2\pi)}\overleftrightarrow{R}(\chi)\,\overleftrightarrow{Y}_k^m(\mu)\,P_p^m(\eta)\,e^{im\varphi}\,\mathbf{C}_{kp}^m(r)\,\frac{r^2\,d\hat{\mathbf{r}}}{(\hat{\mathbf{r}}\hat{\mathbf{z}})}. \qquad (28)$$

For further transformations, accept the following small-angle assumptions

$$r \approx z,\quad (\hat{\mathbf{r}},\hat{\mathbf{z}}) \approx 1,\quad \mathbf{C}_{kl}^m(r)\,r^2\,d\hat{\mathbf{r}}/(\hat{\mathbf{r}},\hat{\mathbf{z}}) \approx z^2\,\mathbf{C}_{kl}^m(z)\,d\hat{\mathbf{r}},$$

and, taking into account that backscattering is small, pass from the integral over only the hemisphere to the integral over the full solid angle. This transforms Eq. (28) into the form

$$\mathbf{L}(z, \hat{\mathbf{l}}\hat{\mathbf{z}}) = \frac{1}{2\pi} \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=-\infty}^{\infty} i^m \frac{2p+1}{2} \frac{2k+1}{2} \sqrt{\frac{(p-m)!}{(p+m)!}} \times$$

$$\times\, \mathbf{C}_{kp}^{m}(z)\, z^2 \int \overleftrightarrow{R}(\chi)\, \overleftrightarrow{Y}_k^m(\mu)\, P_p^m(\eta)\, e^{im\varphi}\, d\hat{\mathbf{r}}. \quad (29)$$

Using the addition formula for the generalized spherical functions, their orthogonality, and connection with the associated Legendre functions, the integral over the solid angle can be calculated and Eq. (29) takes the form

$$\mathbf{L}(z, \hat{\mathbf{l}}\hat{\mathbf{z}}) = \sum_{k=0}^{\infty} \frac{2k+1}{4\pi} \overleftrightarrow{Y}_k^0(\hat{\mathbf{l}}\hat{\mathbf{z}}) \sum_{m=-\infty}^{\infty} i^m\, \mathbf{C}_{kk}^m(z)\, z^2. \quad (30)$$

Substitution of solution (15) allows us to calculate the internal sum over m in Eq. (30):

$$\sum_{m=-\infty}^{\infty} i^m\, \mathbf{C}_{kk}^m(z)\, z^2 = \exp\{-\varepsilon r(1 - \Lambda \overleftrightarrow{x}_k)\}, \quad (31)$$

what completely corresponds to the results obtained in Ref. 4.

Figures 1–2 compare the solutions obtained in MSH and DA [Eqs. (16)–(22)] as applied to a plane unidirectional source with the results of statistical modeling. It is worth noting the high accuracy of MSH, while DA is unable to describe even qualitatively the brightness body at any depth, but describes the degree of polarization with practically acceptable accuracy at the optical depth $\tau > 10$.

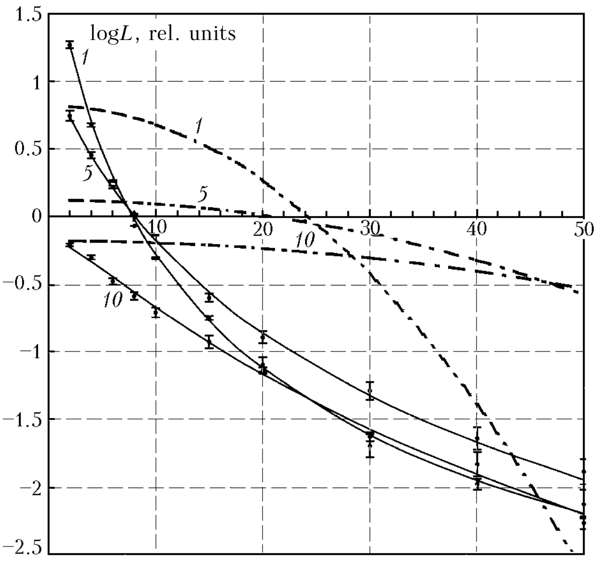

Fig. 1. Distribution of brightness L inside a layer of turbid medium; figures at the curves mean the optical depth. Medium parameters: $\Lambda = 0.8$, Heney–Greenstein scattering phase function with $g = 0.97$.

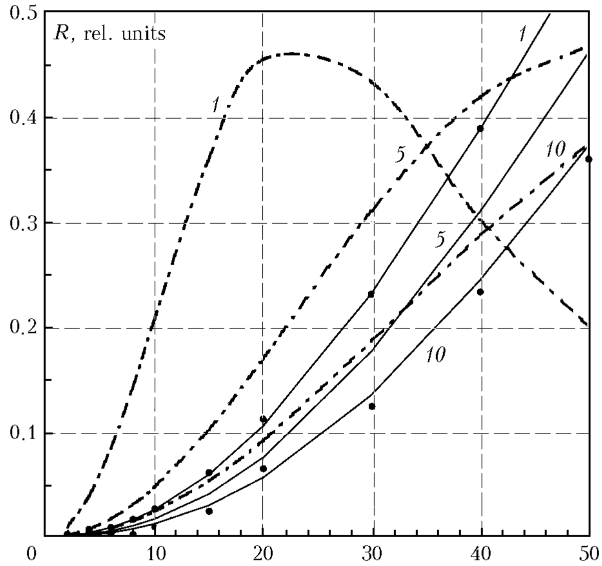

Fig. 2. Degree of polarization R inside a layer of a turbid medium; figures at the curves mean the optical depth. Medium parameters: $\Lambda = 0.8$, Heney–Greenstein scattering phase function with $g = 0.97$.

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