# ONE–DIMENSIONAL IMAGE FILTERING, OPTIMAL WITH RESPECT TO RESOLUTION

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*Received February 8, 1991*

*The problem of optimal image filtering is solved in accordance with the criterion of maximum resolution in the one–dimensional version. It is shown that the obtained solution can be employed to improve the images of the observed targets in an optical system operating in a turbulent atmosphere under conditions of random refraction.*

## INTRODUCTION

It is well known[1–3] that the quality of optical systems, as well as of any other image constructing systems like TV, x–ray, and photographic systems is characterized by its resolution according to Foucault. The problems of analyzing the implicit capabilities of the IS's, which would make it possible to improve (adjust or control) their resolutions, are the permanent investigation subject in the USSR[4–6] and abroad.[7–9] Meanwhile, as far as we know, from the literature there are no data on the theoretically maximum attainable values of the resolution for the IS of one or another type and on the choice of the operational characteristics of the IS, which would correspond to that theoretical maximum. Since in the general form this problem is extremely complicated, in this paper we present its solution only in a one–dimensional version which is applicable to the linear IS's with the following model structure: initial image, distorting filter, additive noise, correcting filter, and reconstructed image.

## MATHEMATICAL FORMULATION OF THE PROBLEM

Let the resolution of an image constructing system be defined as follows:[10]

$$R = \mu\left\{\bar{\nu} \ge 0 \,|\, k_0 G(\nu) \ge K(\nu),\ G(\nu) > 0,\ 0 \le \nu \le \bar{\nu}\right\},$$

where $\mu$ is the Lebesgue measure, $\nu$ and $\bar{\nu}$ are the spatial frequencies, $k_0$ is the initial contrast (contrast of resolvable elements at the input of the IS), $G$ is the resultant modulation transfer function (MTF) of the IS, and $K$ is the threshold contrast.

In what follows, by analogy to Ref. 11 we will assume that the threshold contrast is solely determined by the noise at the IS output, i.e.,

$$K(\nu) \equiv K = M_{\text{thr}}\delta,$$

where $M_{\text{thr}}$ is the threshold signal–to–noise ratio and $\delta$ is the relative rms value of the noise at the IS output.

After simple transformations the problem under consideration is reduced to maximizing of the functional

$$R(\Phi) = \mu\left\{\bar{\nu} \ge 0 \,|\, H(\nu)\Phi(\nu) \ge c\int\limits_{-\infty}^{+\infty} S(\nu)\Phi(\nu)\,d\nu,\right.$$
$$\left. H(\nu)\Phi(\nu) > 0,\ 0 \le \nu \le \bar{\nu}\right\} \qquad (1)$$

over all $\Phi$ under conditions that 1) $c \ge 0$, 2) $H$, $\Phi$, and $S$ be nonnegative even functions over the entire axis, 3) $S$ and $H$ be continuous at the zeroth point, 4) $S(0) \ge S(\nu)$, 5) $S(0) < \infty$, and 6) $H(0) = \Phi(0) = 1$. Here $H = |\tilde{h}|^2$ and $\Phi = |\tilde{\varphi}|^2$ are the squared MTF's of the distorting and correcting filters, respectively; $\tilde{h}$ and $\tilde{\varphi}$ denote the optical transfer function (OTF) of the distorting and correcting filters, respectively; $c = [M_{\text{thr}}/(k_0 B)]^2$, $B$ is the brightness of the background of the input image, and $S$ is the spectral density of the noise.

Prior to solving the problem itself let us introduce a number of additional designations and remarks:

$P_Q = \{\bar{\nu} \ge 0 \,|\, Q(\nu) > 0,\ 0 \le \nu \le \bar{\nu}\}$, and $A_\Phi$ is the set in braces of the right side of Eq. (1), $D = \left\{\Phi \,|\, c\int\limits_{-\infty}^{+\infty} S(\nu)\,\Phi(\nu)\,d\nu \le 1\right\}$,

$\eta(\bar{\nu}) = 2c\int\limits_0^{\bar{\nu}} \dfrac{S(\nu)}{H(\nu)}d\nu$, $\bar{B} = \{\bar{\nu} \ge 0 \,|\, \eta(\bar{\nu}) \le 1\}$, and $\varnothing$ is the empty set.

It is obvious that the sets $A_\Phi$ and $P_Q$ are convex for $\forall\Phi$ and $Q$ and it follows from conditions 1 and 2 that the set $\bar{B}$ is also convex. For this reason, $\forall A \in \{\bar{B}, P_Q, A_\Phi \,|\, Q, \Phi$ are arbitrary$\}$ is convex (if not empty, then it consists of nonnegative numbers), and in accordance with Refs. 12 and 13 it can only be the set

$$A = [0, p] \text{ or } A = [0, q),$$

where $p$ and $q$ are some nonnegative numbers (possibly infinite). If $A = \varnothing \Rightarrow q = 0$. If $A \ne \varnothing$, it is obvious that

$$A = [0, a] \text{ or } A = [0, a),\ a = \mu(A) = \sup(A).$$

From the foregoing and conditions 3 and 6 it follows that $P_H \ne \varnothing$ and $\alpha = \mu(P_H) = \sup(P_H) > 0$. One can easily see that $R(\Phi) \le \mu(P_{H\Phi}) \le \mu(P_H)$ and $\mu(P_\Phi)$. From conditions 1–6 it follows that $D \ne \varnothing$, and from conditions 3, 5, and 6 one has that $S/H$ is continuous within some closed vicinity of zero $W(0)$ and $\eta$ is continuous on $W(0)$ and hence $\eta(0) = 0$ and $\bar{B} \ne \varnothing$, and according to Ref. 14 $\exists!\ 0 < \bar{\nu} \in W(0)$ is such that $\eta(\bar{\nu}) \le 1$ that results in $\beta = \mu(\bar{B}) = \sup(\bar{B}) > 0$.

Let us proceed to solving the above–stated problem.

0235-6880/91/10 720-03 $02.00 

**Case 1.** $S = 0$ almost everywhere. Then $S\Phi = 0$ almost everywhere for $\forall\Phi$. Therefore, as it follows from the well–known theorem[12]

$$\int_{-\infty}^{+\infty} S(\nu)\Phi(\nu)\, d\nu = 0 \text{ for } \forall\Phi \Rightarrow R(\Phi) = \mu(P_{H\Phi}).$$

From this we obtain that $R_{\max} = \alpha$ and it is reached on $\forall\Phi$, for which $\mu(P_\Phi) \geq \alpha$, in other words, on such an $\forall\Phi$, which is positive on the set $(-\alpha, \alpha)$ and which is arbitrary outside it, and $\Phi(0) = 1$.

**Case 2.** $S \neq 0$ almost everywhere.

**Case 2.1.** $\int_{-\infty}^{+\infty} S(\nu)\Phi(\nu)\, d\nu = 0$. In this case from condition 2 and in accordance with the well–known theorem[15] $S\Phi = 0$ almost everywhere. Since $S \neq 0$ almost everywhere $S(0) > 0$ according to conditions 2 and 4. And from condition 3 it follows that there exists a vicinity $\Omega(0)$ around zero point where $S > 0$. Hence we obtain that $\Phi = 0$ almost everywhere on $\Omega(0) \Rightarrow \mu(P_\Phi) = 0 \Rightarrow R(\Phi) = 0$.

**Case 2.2.** $\int_{-\infty}^{+\infty} S(\nu)\Phi(\nu)d\nu > 0$. From condition 6 it follows that if $c\int_{-\infty}^{+\infty} S(\nu)\Phi(\nu)d\nu > 1$ then $R(\Phi) = 0$. Thus in the case 2 the maximum of functional (1) can be reached only on $\Phi \in D$. From the condition 6 and taking into account a property of the set $D$ it follows that $A_\Phi \neq \varnothing$ for $\forall\Phi \in D \Rightarrow R(\Phi) = \mu(A_\Phi) = \sup(A_\Phi)$.

Let now $\Phi \in D$ and $\bar{\nu} \in A_\Phi$ be chosen arbitrarily. For $\forall\bar{\nu} \in A$ in accordance with the definition of this set the inequality

$$H(\nu)\Phi(\nu) \geq c\int_{-\infty}^{+\infty} S(\nu)\Phi(\nu)\, d\nu,\ 0 \leq \nu \leq \bar{\nu} \tag{2}$$

is valid. From the monotony of a definite integral and from condition 2, applying inequality (2) for the second time, we obtain

$$\int_{-\infty}^{+\infty} S(\nu)\Phi(\nu)\, d\nu \geq 2c \int_0^{\bar{\nu}} S(\nu)\Phi(\nu)\, d\nu \geq \eta(\bar{\nu}) \int_{-\infty}^{+\infty} S(\nu)\Phi(\nu)\, d\nu.$$

From this and the above said taking condition 1 into account it follows that for all $\forall\bar{\nu} \in A_\Phi$ and $\forall\Phi \in D$ the inequality $\eta(\nu) \leq 1 \Rightarrow A_\Phi \subset \bar{B}$ holds, and for $\forall\Phi \in D \Rightarrow R(\Phi) \leq \beta$, while for $\forall F \in D \Rightarrow R_{\max} \leq \beta$.

On the other hand, we have already obtained that $R_{\max} \leq \alpha$. Thus, finally we have that $R_{\max} \leq \gamma = \min(\alpha, \beta)$. Let us now show that $R_{\max} = \gamma$ and it is reached on the function

$$\Phi_{\text{opt}}(\nu) = \begin{cases} 1/H(\nu), & |\nu| < \gamma\ ; \\ 0\ , & |\nu| \geq \gamma\ . \end{cases} \tag{3}$$

To this end, obviously, it is quite sufficient to show that

$$2c \int_0^{\gamma} S(\nu)\Phi_{\text{opt}}(\nu)\, d\nu \leq 1\ .$$

From Eq. (3) and conditions 1, 2, 4, and 6 it follows that the function

$$\xi(\bar{\nu}) = 2c\ (\int_0^{\bar{\nu}} S(\nu)\ \Phi_{\text{opt}}(\nu)\, d\nu$$

is continuous everywhere, and that it monotonically increases and coincides with the function $\eta$, at least, on the set $[0, \gamma) \subset \bar{B}$. For this reason,

$$\sup_{0<\nu<\gamma} \xi(\nu) = \sup_{0<\nu<\gamma} \eta(\nu) \leq \sup_{\nu\in\bar{B}} \eta(\nu) \leq 1\ . \tag{4}$$

From the fact that the quantity $\xi$ is continuous everywhere and is monotonically increasing it follows that[12]

$$\sup_{0<\nu<\gamma} \xi(\nu) = \xi(\gamma)\ . \tag{5}$$

Therefore, combining Eqs. (4) and (5) we will obtain the sought–after result.

## SOME RESULTS OBTAINED FROM THE PROBLEM SOLUTION

Thus, under sufficiently general restrictions the theoretically achievable maximum resolution of the linear IS's is

$$R_{\max} = \min(\alpha, \beta) \tag{6}$$

and it is reached at

$$|\tilde{\varphi}_{\text{opt}}(\nu)| = \begin{cases} \dfrac{1}{|\tilde{h}(\nu)|}\ , & |\nu| < R_{\max}\ ; \\ 0\ , & |\nu| \geq R_{\max}\ , \end{cases} \tag{7}$$

where

$$\alpha = \sup\{\ \bar{\nu} \geq 0\, |\, |\tilde{h}(\nu)| > 0,\ 0 \leq \nu \leq \bar{\nu}\ \}\ ;$$

$$\beta = \sup\left\{ \bar{\nu} \geq 0\, \middle|\, 2c \int_0^{\bar{\nu}} \frac{S(\nu)}{|\tilde{h}(\nu)|^2}\, d\nu \leq 1 \right\}\ . \tag{8}$$

Note that the quantity $\alpha$ in the case of a continuous optical transfer function $\tilde{h}$ is its first positive zero, and therefore the "sup" sign in the given relations for $\alpha$ and $\beta$ can be replaced by "$\mu$".

## DISCUSSION OF THE OBTAINED RESULTS

Expression (7) for the OTF of an optimal, with respect to the resolution, correcting filter coincides with that for the OTF of an inversion filter[17] which is used to reconstruct optical images. The coincidence is correct to the parameter $R_{\max}$. The difference is that in this paper we have chosen $R_{\max}$ from the condition of maximum resolution in the Foucault sense and then it is analytically calculated, based on the use of formulas (6) and (8), while in Ref. 17 this parameter is chosen from the condition of minimum rms error for the initial–image reconstruction and it is defined analytically as a solution of the equation[17]

$$|\tilde{h}(\nu)|^2 = \frac{S(\nu)}{S_0(\nu)} \tag{9}$$

with respect to $\nu$, where $S_0$ is the spectral density of the initial image.

One can readily see that in Eqs. (6) and (8), on the one hand, and in Eq. (9), on the other, different *a priori* information about the initial image is used. In the first case the contrast $k_0$ and the background brightness $B$ are used, while in the second case – only the spectral density $S_0$ of the initial image.

The results which have been obtained in this paper, i.e., formulas (6) and (8), can be applied, in particular, to signal filtering in scanning optical systems operating in the turbulent atmosphere under conditions of random refraction, if, in accordance with Refs. 1, 16, and 18, it is assumed that

$$\tilde{\Psi}_{\text{opt}}(\omega) = \tilde{\varphi}_{\text{opt}}(\omega/v)$$

is the transfer function of a temporal filter, $v$ is the velocity of an image scanning (for airborne lidars this is the velocity of flight), $\omega$ is the temporal frequency, and

$$\tilde{h} = L(\nu, 0) ,$$

where $L(\nu_x, \nu_y)$ is the product of the OTF's of the turbulent atmosphere and of the receiving aperture of the optical system, $\nu_x$ and $\nu_y$ are the spatial frequencies along the image–scanning direction (aircraft flight direction) and along the direction which is perpendicular to that one, respectively;

$$S(\nu) = \int_{-\infty}^{\infty} F(\nu, \nu_y)\, d\nu_y ,$$

where $F$ is the spectral density of fluctuation of the refractive index of the medium (air), which represents a quantitative model of the random refraction.

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