OPTIMAL MEASURER OF THE SCATTERING PHASE MATRIX

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An optimization of a device for measuring 16 components of the scattering phase matrix is proposed. The optimized instrument allows the measurements of the scattering phase matrix to be performed with the maximum possible accuracy with errors in the initial data. The device design provides a very simple control of the polarization components.

The scattering phase matrix D relates the Stokes vector of the radiation from a source $S_s = [I_s\, Q_s U_s V_s]^T$ and the Stokes vector of the scattered radiation $S_r = [I_r\, Q_r U_r V_r]^T$ coming to the receiver by the relation

$$S_r = DS_s\ . \qquad (1)$$

In the general case when the medium is anisotropic, it is impossible to indicate in advance the equal and zero elements, and so it is necessary to determine all the 16 elements of the matrix D.

Importance of the problem in measuring D have been noted in Ref. 1, and its urgency is kept up to date due to the methodological and technical difficulties.

A well–known block diagram of the device for measuring D involves an optical generator the radiation from which passes successively the polarizer and the phase element, which shifts the phase of the perpendicular components by the angle δ_s. Light scattered by the medium comes to the receiver, in which it successively passes through the phase element, which shifts the phase of the perpendicular components by the angle δ_r, the polarizer, and the interference filter. The polarization block of the radiation source transforms the Stokes vector of the output radiation of the optical generator $S_0 = [I_0 Q_0 U_0 V_0]^T$. This transformation is determined[2] by the Muller matrices of the polarizer P_s and the phase element F_s.

Let us determine S_s taking into account that the transformation has the form $S_s = F_s P_s S_0$

$$S_s = [I_s Q_s U_s V_s]^T = \frac{1}{2}(I_0 + Q_0 \cos 2\theta + U_0 \sin 2\theta) \times$$
$$\times [1,\ \cos 2\varphi \cos 2(\varphi - \theta) + \cos \delta_s \sin 2\varphi \sin 2(\varphi - \theta),$$
$$\sin 2\varphi \cos 2(\varphi - \theta) - \cos \delta_s \cos 2\varphi \sin 2(\varphi - \theta),$$
$$\sin \delta_s \sin 2(\varphi - \theta)]^T\ , \qquad (2)$$

where θ is the orientation angle of the transmission plane of the polarizer relative to the x axis of the coordinate system of the radiation source and φ is the angle of orientation of the fast axis of the phase element.

In order to determine 16 elements of the scattering phase matrix 16 independent equations are needed, and taking into account that every type of radiation polarization contains 4 Stokes parameters it suffices to generate 4 polarizations for the source. Let us write these equations taking into account Eqs. (1) and (2)

$$DS_{si} = S_{ri}\ ,\quad i = 1, 2, 3, 4\ . \qquad (3)$$

System (3) should to be well–posed, i.e., its solution should to be low–sensitive to the errors or uncertainties in the initial data.

The optimum measuring instrument gives such coefficients to system (3) that their inaccuracy imposes the least effect on the accuracy of its solution, and, simultaneously, the instrument is easy in performance of the control of polarization blocks of the source and the receiver.

Let us introduce the matrix W, whose lines are the parameters of the Stokes vector of the radiation from the source

$$W = [S_{s1}\, S_{s2}\, S_{s3}\, S_{s4}]^T. \qquad (4)$$

Let us group system (3) in four systems, each determining a line D

$$WD_1 = I_{iw}\ ;\ \ WD_2 = Q_{iw}\ ;\ \ WD_3 = U_{iw}\ ;\ \ WD_4 = V_{iw}\ , \qquad (5)$$

where

$D_1 = [D_{1n}]^T$; $D_2 = [D_{2n}]^T$; $D_3 = [D_{3n}]^T$; $D_4 = [D_{4n}]^T$; $n = 1, 2, 3, 4$; $I_{iw} = [I_{ri}]^T$; $Q_{iw} = [Q_{ri}]^T$; $U_{iw} = [U_{ri}]^T$; $V_{iw} = [V_{ri}]^T$; $i = 1, 2, 3, 4$. Let ΔW be a disturbance of the matrix W due to the inaccuracy in determining θ, φ, and δ_s, and ΔI_r, ΔQ_r, ΔU_r, and ΔV_r be the errors in determining the Stokes parameters I_r, Q_r, U_r, and V_r.

Let us consider the system of equations (5) which determines D_1. By superposing the measurement errors we obtain a solution of the system

$$(W + \Delta W)[D^*_{1n}]^T = [I_{ri} + \Delta I_{ri}]^T, \qquad n, i = 1, 2, 3, 4\ , \qquad (6)$$

where $D^*_{1n} = D_{1n} + \Delta D_{1n}$, and ΔD_{1n} are the errors of the solution. The rest disturbances are related by the expression from Ref. 3

$$\delta D_1 \le \frac{\text{cond}\ W}{1 - \text{cond}\ W \delta W}(\delta W + \delta I_{iw})\ , \qquad (7)$$

where $\text{cond}\ W = \frac{1}{n}\sqrt{\text{Tr} W^* W}\ \sqrt{\text{Tr}(W^{-1})^* W^{-1}}$ is the number of W conditionality, n is the matrix order, W^* and $(W^{-1})^*$ are the transpose and complex conjugate of W and W^{-1} matrices, respectively,

$$\delta D_1 = \left(\sum_{n=1}^{4} \Delta D^2_{1n} \Big/ \sum_{n=1}^{4} D^2_{1n}\right)^{1/2},$$

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$$\delta W = \left(\sum_{m=1}^{4}\sum_{n=1}^{4} \Delta W_{mn}^2 \Big/ \sum_{m=1}^{4}\sum_{n=1}^{4} W_{mn}^2 \right)^{1/2}$$

$$\delta I_{\rm iw} = \left(\sum_{i=1}^{4} \Delta I_{{\rm r}i}^2 \Big/ \sum_{i=1}^{4} I_{{\rm r}i}^2 \right)^{1/2}$$

are the relative disturbances of D_1, W, and $I_{\rm iw}$, respectively.

It follows from Eq. (6) that relative disturbances δW and $\delta I_{\rm iw}$ are added linearly, and, therefore, minimum of δD_1 is provided at the minimum of δW, $\delta I_{\rm iw}$, and cond W. Then one can conclude that the optimum algorithm of measuring D should have minimum cond W and measure $S_{{\rm r}i}$ with maximum accuracy.

The values δD_2, δD_3, and δD_4, which are relative disturbances of the vectors formed by the second, third, and fourth lines of D with δW and cond W and the relative disturbances $\delta Q_{\rm iw}$, $\delta U_{\rm iw}$, and $\delta V_{\rm iw}$, respectively, are related to each other by expressions analogous to Eq. (7).

The Stokes vector $S = [IQUV]^{\rm T}$ of the radiation at the output of the polarization block of the receiver is determined by the formula

$$S = {\rm P_r} \Phi_{\rm r} S_{\rm r} , \tag{8}$$

where $\rm P_r$ and $\Phi_{\rm r}$ are the Muller matrices of the polarizer and the phase element, respectively.

Since the sensitive element records only the intensity of radiation, it suffices to measure its intensity at the output of the polarization block of the receiver at four values of its Muller matrix and to solve the obtained system of equations in order to find four Stokes parameters of the radiation. Taking into account Eq. (8) we obtain that the radiation with the Stokes vector $S_{\rm r1}$ for one value of the Muller matrix of the polarization block gives the intensity at the output of the block

$$I_{11} = \frac{1}{2}[I_{\rm r1} + Q_{\rm r1} [\cos 2\alpha \cos 2(\alpha - \beta) +$$

$$+ \cos \delta_{\rm r} \sin 2\alpha \sin 2(\alpha - \beta)] + U_{\rm r1}[\sin 2\alpha \cos 2(\alpha - \beta) -$$

$$- \cos \delta_{\rm r} \cos 2\alpha \sin 2(\alpha - \beta)]] - V_{\rm r1} \sin \delta_{\rm r} \sin 2(\alpha - \beta)\} , \tag{9}$$

where β is the angle of orientation of the transmission plane of the polarizer relative to the x_1 axis of the receiver coordinate system and α is the angle of orientation of the phase element.

The expressions for the intensities I_{12}, I_{13}, and I_{14} obtained for the rest three values of the Muller matrix of the polarization block of the receiver differ from Eq. (9) by variations of the parameters α, β, and δ_r.

Let us write down the system of equations which determine $S_{\rm r1}$, in the matrix form

$$\frac{1}{2} M[I_{\rm r1} Q_{\rm r1} U_{\rm r1} V_{\rm r1}]^{\rm T} = [I_{11} I_{12} I_{13} I_{14}]^{\rm T} . \tag{10}$$

By introducing the matrix K inverse to M, we write down Eq. (10) in the form

$$[I_{\rm r1} Q_{\rm r1} U_{\rm r1} V_{\rm r1}]^{\rm T} = 2K [I_{11} I_{12} I_{13} I_{14}]^{\rm T} . \tag{11}$$

Taking into account that I_{21}, I_{22}, I_{23}, and I_{24} are the intensities obtained from measurements of $S_{\rm r2}$; I_{31}, I_{32}, I_{33}, and I_{34} from measurements of $S_{\rm r3}$; and, I_{41}, I_{42}, I_{43}, and I_{44} from measurements of $S_{\rm r4}$, one can write down

$$[S_{{\rm r}i}] = 2K [I_{ip}]^{\rm T} , \; p = 1, 2, 3, 4 . \tag{12}$$

Let us introduce the matrix $G = [I_{ip}]$, i, p = 1, 2, 3, 4 and taking into account Eqs. (5), (10), (11), and (12), we obtain the lines

$$D_1 = 2W^{-1}{\rm G}\,[k_{1n}]^{\rm T} ; \; D_2 = 2W^{-1}{\rm G}\,[k_{2n}]^{\rm T} ;$$

$$D_3 = 2W^{-1}{\rm G}\,[k_{3n}]^{\rm T} ; \; D_4 = 2W^{-1}{\rm G}\,[k_{4n}]^{\rm T} ;$$

$$n = 1, 2, 3, 4 . \tag{13}$$

Relative disturbances for system (10) are related as follows:

$$\delta S_{\rm r1} \le \frac{{\rm cond}\, M}{1 - {\rm cond}\, M \delta M} (\delta M + \delta I_1) , \tag{14}$$

where

$$\delta M = \left(\sum_{m=1}^{4}\sum_{n=1}^{4} \Delta M_{mn}^2 \Big/ \sum_{m=1}^{4}\sum_{n=1}^{4} M_{mn}^2 \right)^{1/2}$$

are the relative disturbances of the matrix M,

$$\delta S_{\rm r1} = \left[\frac{\Delta I_{\rm r1}^2 + \Delta Q_{\rm r1}^2 + \Delta U_{\rm r1}^2 + \Delta V_{\rm r1}^2}{I_{\rm r1}^2 + Q_{\rm r1}^2 + U_{\rm r1}^2 + V_{\rm r1}^2} \right]^{1/2} ,$$

are the relative disturbance of S_{n1}, and

$$\delta I_1 = \left(\sum_{p=1}^{4} \; I_{1p}^2 \Big/ \sum_{p=1}^{4} I_{1p}^2 \right)^{1/2}$$

is the relative disturbance of the measured intensities caused by the errors of measurement ΔI_{1p}.

It follows from Eq. (14) that relative disturbances δM and δI_1 are added linearly, and, therefore, the minimum of $\delta S_{\rm r1}$ is provided when δM and δI_1 reach their minima. Then one can conclude that when measuring $S_{\rm r1}$ it is necessary to reach minimum cond M and to measure I_{11}, I_{12}, I_{13}, and I_{14} with maximum accuracy. It is evident that at more accurately measured vectors $S_{{\rm r}i}$, the less are $\delta I_{\rm iw}$, $\delta Q_{\rm iw}$, $\delta_{\rm iw}$, and $\delta V_{\rm iw}$. M differs from W by the opposite sign of elements in the fourth column. One can show that cond M = cond W, therefore let us analyze W.

Thus the matrix W is always degenerated if only the angle of the polarizer orientation varies, or if the angles θ and φ vary simultaneously at $\varphi - \theta$ = const. If the parameter $\delta_{\rm r}$ changes, it is necessary to change at least once the orientation of the phase element or of the polarizer in order to make W nondegenerate. On the other hand the matrix W is always nondegenerate if the orientation of the phase element is changed, therefore the control of the orientation of the phase element is most convenient. The numerical analysis[4] has shown that cond M is minimum at some orientation of the phase element relative to the polarizer for $\delta_r = 130^\circ$. My calculation by Eq. (7) for $\delta_r = \lambda/4$, $\theta = 90^\circ$, $\varphi_1 = 45^\circ$, $\varphi_2 = 75^\circ$, $\varphi_3 = 105^\circ$, and $\varphi_4 = 135^\circ$ yielded cond $M = 1.53$.

For comparison let us present the conditionality numbers for the other techniques of measuring the Stokes parameters. The technique[1] requiring four measurements of the intensity, when one measurement is carried out without the phase element (this makes the technical realization more difficult), has the conditionality number 1.3. The technique[5] requiring six measurements, when one measurement is also carried out without the phase element, gives the conditionality number 1.06.

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