THE TECHNIQUE FOR CALCULATION OF LIGHT SCATTERING CHARACTERISTICS OF NONABSORBING RADIALLY INHOMOGENEOUS PARTICLES

N.N. Belov and S.O. Suslov

L.Ya. Karpov Physicochemical Scientific–Research Institute, Moscow

Received June 24, 1991

The application of the technique of phase functions to optics of nonabsorbing spherical particles with radially nonuniform refractive index is considered. Radial dependence of the phase functions is obtained. The relations for calculating the light scattering characteristics of particles with a smooth radial behavior of the refractive index are derived. The method is shown to be sufficiently accurate.

Calculations of light–scattering characteristics of small spherical particles with radially inhomogeneous refractive index recently[1–3] attracted an increased attention of researchers. However, this problem can be analytically solved only for some specific kinds of radial dependences of the refractive index,[1,2] and therefore it is necessary to solve it numerically. Direct integration of the equations, which result from theory, is hard due to strong oscillations of the solutions.[2] Techniques used today are not quite exact and require a lot of computer time.[4]

The phase functions technique,[5] which is widely used in quantum mechanics, is applied to investigation of light scattering by nonabsorbing radially inhomogeneous particles. The possibility of applying it to calculating optical characteristics of spherical particles was considered in Ref. 4. This paper deals with further development of this technique.

Let us consider the light scattering by a spherical particle with the radius a and the real refractive index $n(r)$. Let us also assume that $n(r)$ is a continuous differentiable function defined on the interval $0 \le r \le a$. Let the wave number k be equal to $2\pi/\lambda$ and the diffraction parameter of the particle x be ka. As in the case of a homogeneously spherical particle, the light scattering in our case is completely determined by coefficients a_l and b_l of the scattering series (see Refs. 2 and 6) which for the sphere with radial inhomogenity of the refractive index have the form[2]

$$a_l = \frac{\psi_l(x)\, W_l'(x) - n^2(x)\, \psi_l'(x)\, W_l(x)}{\xi_l(x)\, W_l'(x) - n^2(x)\, \xi_l'(x)\, W_l(x)}\,;$$

$$b_l = \frac{\psi_l(x) G_l'(x) - \psi_l'(x) G_l(x)}{\xi_l(x) G_l'(x) - \xi_l'(x) G_l(x)}\,, \qquad (1)$$

where $\rho = kr$, ψ_l is the Riccati–Bessel function, and ξ_l is the Riccati–Hankel function of the first kind

$$\psi_l(\rho) = (\pi\rho/2)^{1/2} J_{l+1/2}(\rho); \qquad \xi_l(\rho) = \psi_l(\rho) + i\chi_l(\rho)\,, \qquad (2)$$

and the functions $G_l(\rho)$ and $W_l(\rho)$ are determined by equations[1,2]

$$G_l''(\rho) + \left[n^2(\rho) - l(l+1)/\rho^2\right] G_l(\rho) = 0\,;$$

$$W_l''(\rho) - \left[\ln(n^2(\rho))\right]' W_l'(\rho) + \left[n^2(\rho) - l(l+1)/\rho^2\right] W_l(\rho) = 0. \qquad (3)$$

Solution of Eqs. 3 is the main difficulty of the theory of light scattering by a radially inhomogeneous spherical particle. Let us apply the technique of phase functions well known in quantum mechanics[3] to solving these equations. Let us write functions G_l and W_l in the form

$$G_l(\rho) = A_l^c(\rho)\left[\cos\delta_l^c(\rho)\, \psi_l(\rho) - \sin\delta_l^c(\rho)\chi_l(\rho)\right];$$

$$W_l(\rho) = A_l^w(\rho)\left[\cos\delta_l^w(\rho)\, \psi_l(\rho) - \sin\delta_l^w(\rho)\chi_l(\rho)\right];$$

$$\chi_l(\rho) = (\pi\rho/2)^{1/2} N_{l+1/2}(\rho)\,, \qquad (4)$$

were A_l^w and A_l^c are the amplitude functions, δ_l^w and δ_l^c are the phase functions χ_l are the Riccati–Bessel functions.

Following Ref. 5, we impose additional restrictions on the amplitude and phase functions. Let us demand that

$$G_l'(\rho) = A_l^c(\rho)\left[\cos\delta_l^c(\rho)\, \psi_l'(\rho) - \sin\delta_l^c(\rho)\chi_l'(\rho)\right];$$

and

$$W_l'(\rho) = A_l^w(\rho)\left[\cos\delta_l^w(\rho)\, \psi_l'(\rho) - \sin\delta_l^w(\rho)\chi_l'(\rho)\right]; \qquad (5)$$

The necessaity of conditions (5) is grounded because we introduced four functions in Eqs. 4 instead of two unknown functions G_l and W_l. Conditions (5) are equivalent to conditions

$$\left[A_l^c(\rho)\right]'\left[\cos\delta_l^c(\rho)\, \psi_l(\rho) - \sin\delta_l^c(\rho)\chi_l(\rho)\right] -$$

$$- A_l^c(\rho)\left[\delta_l^c\right]'\left[\sin\delta_l^c(\rho)\, \psi_l(\rho) + \cos\delta_l^c(\rho)\, \chi_l(\rho)\right] = 0;$$

$$\left[A_l^w(\rho)\right]'\left[\cos\delta_l^w(\rho)\, \psi_l(\rho) - \sin\delta_l^w(\rho)\chi_l(\rho)\right] -$$

$$- A_l^w(\rho)\left[\delta_l^w\right]'\left[\sin\delta_l^w(\rho)\, \psi_l(\rho) + \cos\delta_l^w(\rho)\, \chi_l(\rho)\right] = 0. \qquad (6)$$

Substitution of Eq. 4 into Eq. 3 gives

$$\left[A_l^c(\rho)\right]'\left[\cos\delta_l^c(\rho)\, \psi_l'(\rho) - \sin\delta_l^c\chi_l'(\rho)\right] -$$

$$- A_l^c(\rho)\left[\delta_l^c\right]'\left[\sin\delta_l^c(\rho)\, \psi_l(\rho) + \cos\delta_l^c(\rho)\, \chi_l(\rho)\right] +$$

$$+ (n^2(\rho) + 1) A_l^c(\rho)\left[\cos\delta_l^c(\rho)\, \psi_l'(\rho) - \sin\delta_l^c\chi_l'(\rho)\right] = 0\,; \qquad (7)$$

$$\left[A_l^w(\rho)\right]'\left[\cos\delta_l^w(\rho)\, \psi_l'(\rho) - \sin\delta_l^w\chi_l'(\rho)\right] -$$

$$-A^w_l(\rho)[\delta^w_l]'[\sin\delta^w_l(\rho)\,\psi_l(\rho)+\cos\delta^w_l(\rho)\,\chi_l(\rho)]+$$

$$+(n^2(\rho)+1)A^w_l\rho)[\cos\delta^w_l(\rho)\,\psi'_l(\rho)-\sin\delta^w_l\chi'_l(\rho)]-$$

$$-[\ln(n^2(\rho))]'A^w_l\rho)[\cos\delta^w_l(\rho)\,\psi'_l(\rho)-\sin\delta^w_l\chi'_l(\rho)]=0\ ;$$

Equations (6) and (7) compose two systems of differential equations of the first order, which are quite sufficient for determing functions A^c_l, δ^c_l and A^w_l, δ^w_l respectively. Using the property of the Riccati–Bessel functions Wronskian $\psi_l\chi'_l-\psi'_l\chi_l=1$ one can remove the derivatives of the functions A^c_l and A^w_l.

As a result, we obtain differential equations for the phase functions:

$$\frac{d}{dr}\,\delta^c_l=(n^2(\rho)-1)[\cos\delta^c_l(\rho)\,\psi_l(\rho)-\sin\delta^c_l(\rho)\chi_l(\rho)]^2; \tag{8}$$

$$\frac{d}{dr}\,\delta^w_l=(n^2(\rho)-1)[\cos\delta^w_l(\rho)\,\psi_l(\rho)-\sin\delta^w_l(\rho)\chi_l(\rho)]^2-$$

$$-[\ln(n^2(\rho))]'[\cos\delta^w_l(\rho)\,\psi_l(\rho)-\sin\delta^w_l(\rho)\chi_l(\rho)]\times$$

$$\times[\cos\delta^w_l(\rho)\,\psi'_l(\rho)-\sin\delta^w_l(\rho)\chi'_l(\rho)]$$

with boundary conditions

$$\delta^c_l(0)=\delta^w_l(0)=0 \tag{9}$$

which result from the limitation of functions G_l and W_l of the origin of the coordinate system.

Equations (8) can be solved only numerically excluding some specific kinds of functions $n(r)$. Substituting them into Eq. 1, we obtain the expression for the coefficients of the scattering series

$$a_l=\frac{\psi_l(x)\,w'_l(x)-n^2(x)\psi'_l(x)\,w_l(x)}{\xi_l(x)\,w'_l(x)-n^2(x)\xi'_l(x)\,w_l(x)}\ ;$$

$$b_l=\frac{\psi_l(x)\,g'_l(x)-\psi'_l(x)\,g_l(x)}{\xi_l(x)\,g'_l(x)-\xi'_l(x)\,g_l(x)}\ , \tag{10}$$

where

$$g_l(x)=\cos\delta^c_l(x)\,\psi_l(x)-\sin\delta^c_l(x)\chi_l(x),$$

$$w_l(x)=\cos\delta^w_l(x)\,\psi_l(x)-\sin\delta^w_l(x)\chi_l(x),$$

$$g'_l(x)=\cos\delta^c_l(x)\,\psi'_l(x)-\sin\delta^c_l(x)\chi'_l(x),$$

$$w'_l(x)=\cos\delta^w_l(x)\,\psi'_l(x)-\sin\delta^w_l(x)\chi'_l(x)\ .$$

As can be seen from Eqs. 10, the coefficients of the scattering series depend only on the phase functions but not on the amplitude functoins. This fact significantly simplifies the solution of the problem.

We have calculated light–scattering characteristics of particles with radial function of the refractive index, for which the analytical solution of Eqs. 3 is known. The angular scattering characteristics of particles with the diffraction parameter $x=5.0$ and the refractive index profile $n(\rho)=(2-(\rho/x)^2)^{1/2}$ (Luneberg lens[2]) are shown in Fig. 1. Figures 1a and 1b show the angular dependences of the values

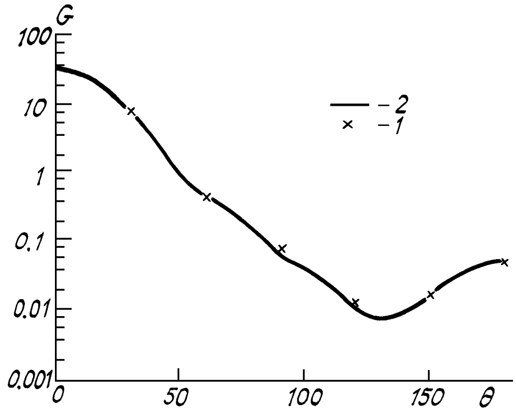
$$G_1(\theta)=\frac{4}{x^2}\,|S_1(\theta)|^2 \ \text{ and } \ G_2(\theta)=\frac{4}{x^2}\,|S_2(\theta)|^2$$

respectively. The calculations were made analytically[2] and by the technique of phase functions, where[6]

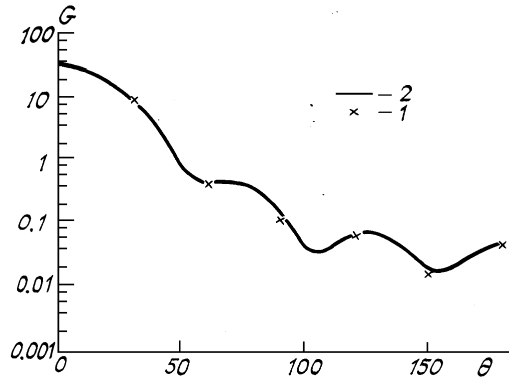
$$S_1(\theta)=\sum_n\frac{2n+1}{n(n+1)}\,(a_n\pi_n(\theta)+b_n\tau_n(\theta))\ ,$$

$$S_2(\theta)=\sum_n\frac{2n+1}{n(n+1)}\,(a_n\tau_n(\theta)+b_n\pi_n(\theta))\ . \tag{11}$$

Quite good agreement between the analytical and numerical results is observed.



a



b

FIG. 1. Angular dependence of the values $G_1(\theta)$ *(a), and* $G_2(\theta)$ *(b): 1) analytical calculations, and 2) calculations by formulas (10).*

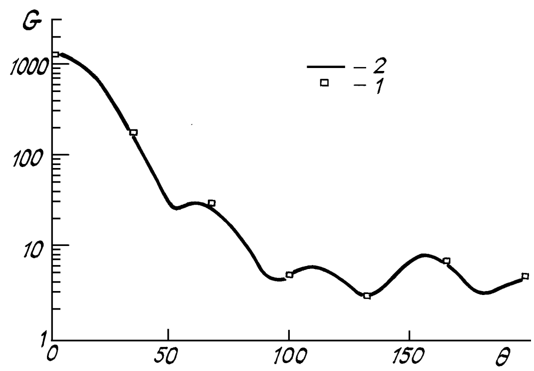


FIG. 2. Angular dependence of the value $G(\theta)$*: 1) analytical calculations and 2) calculations by the technique of phase functions.*

The angular dependence of the value $G(\theta) = |S_1(\theta)|^2 + |S_2(\theta)|^2$ obtained analytically and calculated by the technique of phase functions for the radial dependence of the refractive index $n(\rho) = 1.5/(1 + 0.0051\rho^2)$ and diffraction parameter $x = 5.0$ are shown in Fig. 2. Good agreement between the results of analytical and numerical calculations shows that the technique of phase functions makes it possible to calculate optical characteristics of particles quite accurately.

Thus, we propose a new technique for calculation light–scattering characteristics of nonabsorbing spherical particles with radially dependent refractive index based on the application of the technique of phase functions to the solution of the basic Eq. (3) arising in optics of radially inhomogeneous particles. Equations (9), allowing one to calculate light scattering characteristics of particles with smooth radial dependence of the refractive index, are derived. Comparison of the results obtained using the proposed technique with the results of analytical calculations for the refractive index with different radial behavior shows the technique under consideration possesses sufficient accuracy.

REFERENCES

1. A.P. Prishivalko, V.A. Babenko, and V.M. Kuz'min, *Light Scattering and Absorption by Inhomogeneous and Anysotropic Spherical Particles* (Nauka i technika, Minsk, 1984), 262 pp.
2. H. Kerker, *The Scattering of Light and Other Electromagnetic Radiation* (Acad. Press., N.Y. and London, 1969), 666 pp.
3. D.W. Mackowski, R.A. Altenkirch, and M.P. Mengus, Appl. Opt. **29**, No. 10, 1551–1559 (1990).
4. L. Shafai, Can. J. Phys., **50**, No. 8, 743–749 (1972).
5. V.V. Babikov, *The Technique of Phase Functions in Quantum Mechanics* (Nauka, Moscow, 1988), 256 pp.
6. C.F. Bohren and D.R. Huffman, *Absorption and Scattering of Light by Small Particles* (Interscience–Wirley, N.Y., 1986), 660 pp.