# ADVANTAGES OF GROUP ANALYSIS OF DIFFERENTIAL EQUATIONS IN SOLVING THE PROBLEMS ON OPTIMAL CONTROL OF LASER SYSTEMS

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*The use of group analysis of differential equations in solving the problem of pulsed laser intensity control in addition to the optimal control theory has made it possible to answer questions whether is the intensity control with varying Q-factor or laser pumping preferable and what kind of a Q-switch, active or passive, is advantageous.*

Creation of multifunctional laser systems with radiation parameters varied within a wide range is important both in laboratory setups and in the industrial systems of information transfer and processing. When designing such systems one inevitably faces some difficulties, connected with the determination of the set of control parameters of those accessible for control as well as with the estimate of the functions controlling these parameters.

Within the framework of the optimal control theory it is not always possible to overcome the above difficulties since the traditional statement of the problems of optimal control assumes the availability of a set of versions of control, at the same time the selection of these versions is arbitrary. To determine the best control, for each version the optimization problem should be solved. Disadvantages of such an approach are evident, namely, incompleteness of the original set of control versions and the lack of effectiveness of the selection procedure.

The complement of the optimal control theory by the method of group analysis of differential equations enables one to solve the optimal control problem in a new way and to avoid the above disadvantages. Using the complement of the optimal control theory one can obtain the classes of control with the preset characteristics: specific types of particular solutions of differential equations, describing the system dynamics, correspond to a particular control class. This method also gives *a priori* information about the versatility of one or another control class. A criterion of versatility is the number of types of particular solutions appropriate to a given version of the control.

Methods of group analysis of differential equations, namely, the problem of group classification of differential equations solved using the above methods, can be applied to any mathematical models of laser systems if they represent a set of differential equations. The sole exception is provided by the sets of the first order ordinary differential equations since in this case the integration of the problem of a group classification of a certain set of differential equations resulted from the solution, defined as a set of determining equations, is not easier than the integration of the initial set of differential equations.

Most of the mathematical models describing laser dynamics are nonlinear. Therefore the numerical methods and computers are used for solving the above models. The results obtained have disadvantages, namely, insignificant realibility and low degree of generality, incompleteness, and lack of clarity. Analytical methods are usually used to compensate for the above-mentioned disadvantages.

At the existing stage of the applied mathematics development, among analytical methods only the group analysis can meet the highest requirements. Being an object of study for mathematicians and widely used, the group analysis is not used in physics. And, recent investigations demonstrate[1] that the productivity of this method is rather high, and the field of its application is being constantly extended.[2]

The group analysis of differential equations makes it possible to obtain a description of the general structure of the set of all solutions, to separate out the definite classes of solutions whose determination is simple as compared with the general solution, to derive the solutions from the known ones, to select for definite purposes the analytical form of parameters or functions playing the role of an arbitrary element so that the differential equation allows for the group of transformations with definite characteristics or a more extended group.

The latter is the most important for the laser system control problems since their mathematical models contain the controlling parameters being arbitrary elements in the problem of group classification.

An algorithm of solution of the group classification problem is as follows. First we construct a basic group for the considered set of differential equations consisting of an ensemble of local single parameter Lie groups for the local space transforms of dependent and independent variables involved in a given set of differential equations admitted by this set. Then we

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assume the arbitrariness of a selected controlling parameter. As a result, we obtain a kernel of the basic groups for the set of differential equations, being equal to the intersection of all basic groups when the controlling parameter runs the set of all possible values.

At the stage of constructing the basic group, the equations can be obtained containing only the controlling parameter (or an arbitrary element). They are denoted as classifying ones. Using an additional group called the equivalence group, solutions of the classifying equations are subdivided into the classes equivalent with respect to the operation of the equivalence group. The result of solution of the problem of group classification is the table involving the kernel of the basic groups and its expansions produced by specifping the arbitrary element.

If this table is available, we can not only analyze the possibilities of one or another control but also calculate the above-mentioned results of the group analysis.

Below we shall demonstrate the applicability of this approach to the problem of control of laser radiation intensity for a laser described by the following set of equations:

$$\partial_z S^+ + \partial_t S^- = S^+ N - a(t, S^+, S^-)\, S^+;$$
$$\partial_z S^+ + \partial_t S^- = S^- N - a(t, S^+, S^-)\, S^-; \qquad (1)$$
$$\partial_t N = b(t, S^+, S^-) - b(t, S^+, S^-)\, N - (S^+ + S^-)\, N,$$

where $S^+$ and $S^-$ are the normalized densities of light fluxes, propagating along the optical axis $z$ in the opposite directions; $N$ is the normalized inverse population; the controlling parameters are: $a(t, S^+, S^-)$ is the loss parameter; $b(t, S^+, S^-)$ is the pumping parameter; $t$ is time.

The results of solution of the problem of group classification are presented by the following classification schemes:

$$L_0^1 \subset L^2\ (f(S^+, S^-)),$$
(2)
$$L_0^1 \subset L^2\,(f(S^+, S^-)) \subset L^5\ (S^+ + S^- + S^+ \ln S^+ - S^- \ln S^-),$$

where $L_0$ is the space basis of vector field tangents of the kernel of basic groups admissible by the system. The dimension of this space is shown by the superscript; $L$ is the space expansion of $L_0$ due to specialization of the arbitrary element, whose form is given in parentheses.

Below we write the coordinates of the vectors generating the space of vector field tangents. If we know these coordinates, then, solving the set of fie equations, one can determine uniquely the single-parameter groups of transformations admissible by the set:

$\zeta_1(1, 0, 0, 0, 0)$,
$\zeta_2(0, 1, 0, 0, 0)$,
$\zeta_3(0, 0, S^+, 0, 0)$, (3)
$\zeta_4(0, 0, 0, S^-, 0)$,
$\zeta_5(2z, z + t, 3S^+ (\ln S^+ - 1), S^+ + S^- (\ln S^- - 1), N - a)$.

The first of the classification schemes (2) corresponds to the solution of the problem with the arbitrary element $a$, the solution of the problem with the arbitrary element $b$ is sought following the second scheme.

From the above schemes it follows that the influence of the pumping parameter $b$ on the symmetry of the set of differential equations is stronger than that of the loss parameter $a$.

In the general case when the parameter $a$ or $b$ is an arbitrary function $f$ of $t$, $S^+$, $S^-$ that correspond to the mixed $Q$-switch or pumping, the set of equations admits only one transformation corresponding to the shift transformation along the optical axis. The spatially homogeneous solution is invariant relative to this transformation. So in this case the set of equations has low degree of symmetry, and the range of particular solutions is poor.

The second type of the functional dependence of the parameters $a$ and $b$, which gives the solution to the set of classifying equations, is again an arbitrary one relative to $S^+$ and $S^-$ being concrete relative to $t$: Q-switching must be only passive, that is, it must not depend on time explicitly. In this case the equations allow the two-parameter set of transformations that is not only a shift along the axis $z$, but also the time shift. The set of particular solutions of the system in this case is larger: in addition to the spatially inhomogeneous solutions, the stationary ones and traveling-wave solutions are added to the invariant solutions. Thus, narrowing of the set of controls by excluding active Q-switch makes possible the better control over the laser radiation intensity.

The latter type of the functional dependence refers only to the arbitrary element $b$. This definite type of a passive Q-switch of pumping enables one to expand the range of particular solutions. Self-similar solutions are added to the above-mentioned ones, corresponding to the subgroup of expansions with the tangent vectors $\zeta_3$ and $\zeta_4$. Besides, the invariant solutions have not yet been found concerning the subgroup with the tangent vector $\zeta_5$ and concerning the combination of subgroups with the tangent vectors $\zeta_3$, $\zeta_4$, and $\zeta_5$. All the above-said makes it possible to conclude that the present control is most promising from the standpoint of completeness of data on the possible consequences of the control.

## REFERENCES

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