Conditions for the appearance and density of real-plane zeros in a fluctuating wave

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It is established that a scalar quasi-monochromatic wave vanishes in the observation plane as a linear function of two variables, if the probability density of the logarithmic derivative of the wave amplitude decreases at infinity as x^{-3}. The coefficient with the negative third index of the Laurent series for this probability density determines the number of real-plane zeros. The general results obtained can be reduced to well-known particular cases.

Introduction

In the process of propagation, as the distance increases and fluctuations in the turbulent medium become stronger, a wave acquires some special properties. At interference in optics, these properties manifest themselves as bifurcations of the interference fringes at those parts of the observation plane, where the intensity of the object wave decays.[1,2] At diffraction of light coming from such areas, for example, on the subaperture of a Hartman diaphragm, focal spots form doublets[3] and more complicated structures. These phenomena can be explained within the framework of the mathematical model of a wave, when its complex representation is approximated locally by a polynomial and, in the general case, by an integer exponential-type function (IETF) of several variables. The well-known factorization theorems relate the IETF and polynomial properties to the distribution of zero points of these functions over real and complex planes.[4,5] From this point of view, special properties arise as the wave (its amplitude or intensity) vanishes at isolated points of the observation plane. These zero points are the centers of optical vortices,[6] whose characteristic feature is that the wave phase is uncertain at the center, while in some vicinity of the center the phase varies monotonically around the center and is minimum.[7] In this case, the phase is no longer a continuous function of two variables in the observation plane, and the wave front dislocations are formed.

Since the appearance of zero is a discrete event, the properties associated with it arise, as the wave process goes over some threshold, and zero is an indicator of this event. Once the zero has appeared, we have a more complex state from both the experimental and theoretical points of view. In this case, the probability density of wave amplitude fluctuations changes qualitatively, some light propagation models become inefficient, and description of the wave process in terms of amplitude and phase, as well as their recording, leads to ambiguous results.[8,9]

Zeros and associated optical vortices and wave front dislocations are very interesting physical objects. Their study is important for understanding the nature of wave processes and for various applications. The goal of this paper is to reveal general conditions enabling the appearance of zeros and determining their density.

1. Preliminary considerations

Zeros lie in the observation plane at isolated points corresponding to intersection of zero lines of the real and imaginary parts of the complex wave model. In the simplest case, this function vanishes as linear terms of the Taylor series, but, according to the Weierstrass preparation theorem, there exist zeros with a more complex structure. For example, the function can vanish on a closed curve in the observation plane or on a line originating from infinity and ending in the infinity.[5] However, for them to arise, zero lines of the real and imaginary parts of the wave model should coincide completely in the observation plane, but this is statistically improbable.

In the experiment with a light-scattering plate, the interference pattern observed seemed as it was caused by isolated zero of a second-order polynomial of two variables.[2] The numerical statistical experiment on light propagation through a turbulent medium also gave the real and imaginary parts of the wave and its phase that could correspond to such zero.[3] It should be noted that higher-order polynomials of two variables vanish, as points of self-intersection of zero lines of the real and imaginary parts coincide. This event is also statistically improbable, because any finite area holds an infinite number of points.

In the observation plane cross sections, higher-order polynomials of two variables degenerate into different-order polynomials of one variable. They can

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be factorized in different cross sections passing through the zero point by different number of binomials. In this sense, zero of a higher-order polynomial is anisotropic.

At insufficient resolution in both numerical and field experiments, a cluster of first-order zeros can look as one higher-order zero, but obvious cases of complex zeros are usually connected with special initial conditions,[10] therefore higher-order zeros should be considered as deterministic objects. In this connection, from here on we use the linear approximation to describe zero and to study it as a statistical object.

Consider a wave propagating through a turbulent medium and experiencing fluctuations. Let us present the mathematical model of the wave in the complex form as an analytical signal

$$U(x, y, z, t) + iV(x, y, z, t), \qquad (1)$$

determined in the direction of propagation z and in time.[11] This model includes quasi-monochromatic, parabolic, and scalar approximation for the light wave.

Let us fix the variable z at some time t and place the origin of the Cartesian coordinate system xOy at the center of a circle in the observation plane lying across the wave propagation direction. In this plane, the model (1) is considered as IETF, and this allows us to represent locally the real and imaginary wave parts by linear terms of the uniformly convergent Taylor series:

$$U(x, y) = u + u'_x x + u'_y y;$$
$$V(x, y) = v + v'_x x + v'_y y. \qquad (2)$$

The constants in these equations are determined at the center of the circle, primes and subscripts denote derivatives with respect to x or y. The coordinates of the point (x_0, y_0), where the wave vanishes as a first-order polynomial, are roots of the system of equations

$$\begin{cases} u + u'_x x_0 + u'_y y_0 = 0, \\ v + v'_x x_0 + v'_y y_0 = 0. \end{cases} \qquad (3)$$

To realize the statistical approach, we should assume, as usually,[12–14] that wave fluctuations are homogeneous and isotropic in the observation plane. The first condition, in particular, suggests that the probability of zero appearance $p(r)$ in some circle with a rather small radius r does not depend on the position of this circle in the observation plane. Then the mean density of zeros can be presented using the well-known method as a limit of the ratio

$$Д = \lim_{r\to 0} \frac{p(r)}{\pi r^2}. \qquad (4)$$

If the assumption of isotropic wave fluctuations is valid, the probability $p(r)$ should be independent of the direction, from which zero came to the circle, at least if the circle is small enough. Therefore, in the system (3) we can assume $y_0 = 0$ and omit subscripts. Then, by presenting the real and imaginary parts of the wave through the amplitude and phase as $u + iv = A \exp i\varphi$, from Eq. (3) we obtain

$$A + (A' + iA\varphi')x_0 = 0,$$

from which we have the following system of equations

$$\begin{cases} A\varphi' x_0 = 0, \\ A + A' x_0 = 0. \end{cases} \qquad (5)$$

One of the solutions to the first equation in the system is $A \neq 0$, $\varphi' = 0$. It corresponds to the case that the wave front of the optical vortex is a right helicoid. Another one solution relates the coordinates of zero to the logarithmic derivative of the amplitude χ' in the following way[12]:

$$x_0 = -\frac{A}{A'} = -(\chi')^{-1}, \quad \chi = \ln A. \qquad (6)$$

The assumption of isotropy leads to the parity of the probability density function $w(x_0)$ of a random coordinate of the zero point x_0. Taking into account this fact and the monotonic relation between the variables in Eq. (6), let us present the probability of zeros in a circle as

$$p(r) = \int_{-r}^{r} w(x_0)\mathrm{d}x_0 = 2\int_{1/r}^{\infty} w(1/\chi')\frac{\mathrm{d}}{\mathrm{d}\chi'}(1/\chi')\,\mathrm{d}\chi' =$$
$$= 2\int_{1/r}^{\infty} w_{\chi'}(\chi')\mathrm{d}\chi', \qquad (7)$$

where $w_{\chi'}$ is the probability density of the variable χ'. Substituting Eq. (7) into Eq. (4), we obtain the following equation for the density of zeros[12]:

$$Д = \lim_{r\to 0} \frac{2}{\pi r^2} \int_{1/r}^{\infty} w_{\chi'}(\chi')\,\mathrm{d}\chi'. \qquad (8)$$

This equation leads, in particular, to a special role of the probability density of the logarithmic amplitude derivative, since it is just its behavior at infinity that determines the density of zeros Д. Let us pass to the study of details of this behavior.

2. Main result

The value of Д is finite, if the integral in Eq. (8) decreases at $r \to 0$ no slower than r^2. To provide for such a behavior, we should assume the continuity of the function $w_{\chi'}$ and its decrease at $\chi' \to \infty$. To study the decrease rate, let us present $w_{\chi'}$ in the vicinity of an infinitely remote point as a part of the Laurent series, which is regular there:

$$w_{\chi'}(\chi') = \sum_{n=2}^{\infty} \ell_{-n}\, \chi'^{-n}. \qquad (9)$$

Since the probability density $w_{\chi'}$ should be an integrable function in the domain of definition, the initial value of the subscript n cannot be less than two. Then, in view of the uniform convergence of the series, it can be integrated by terms. Using this fact, substitute Eq. (9) into Eq. (8) and calculate the integral:

$$\int_{1/r}^{\infty} (\chi')^{-n} \mathrm{d}\chi' = (n-1)^{-1} \left\{ r^{n-1} - (\chi')^{1-n} \Big|_{\infty} \right\} = (n-1)^{-1}\, r^{n-1}. \qquad (10)$$

Now consider the limit (8) in a new form:

$$\mathsf{D} = 2\pi^{-1} \lim_{r \to 0} \sum_{n=2}^{\infty} \ell_{-n} \frac{r^{n-3}}{n-1}. \qquad (11)$$

For the density of zeros D to have some finite value, the subscript n should be more than two. However, $\mathsf{D} > 0$ when the initial value of it is three. That is, zeros are possible only if the function $w_{\chi'}$ is asymptotically equivalent to $\ell_{-3}\, \chi'^{-3}$ at infinity

$$\lim_{\chi' \to 0} \chi'^3\, w_{\chi'}(\chi') = \ell_{-3}. \qquad (12)$$

Thus, regardless of the particular form of the probability density function of the logarithmic amplitude derivative, the density of zeros is determined by one coefficient of the Laurent series according to the equation

$$\mathsf{D} = \pi^{-1}\, \ell_{-3}. \qquad (13)$$

3. Comparison with known results

Let us use the general equation (13) and derive the known particular results for the density of zeros of a light wave propagating through a turbulent medium at different fluctuation intensity.

Note that the probability density $w_{\chi'}(\chi')$ represented by a uniformly convergent power series at an infinitely remote point is regular there, therefore the change of variables $\chi' = t^{-1}$ in Eq. (9) is possible. Then the coefficient of the Laurent series ℓ_{-3} can be calculated as the third coefficient of the Taylor series by the equation

$$\ell_{-3} = \frac{1}{3!} \frac{\mathrm{d}^3}{\mathrm{d}t^3}\, w_{\chi'}(t^{-1}) \Big|_{t=0}. \qquad (14)$$

– Consider the case of very strong turbulence. Light wave fluctuations under these conditions become normal, and wave amplitudes become the Rayleigh ones.[15] The function $w_{\chi'}$ and the density of zeros for these conditions were calculated in Ref. 12:

$$w_{\chi'} = \frac{\sigma_{A'}^2}{\langle I \rangle} \left(\chi'^2 + \frac{\sigma_{A'}^2}{\langle I \rangle} \right)^{-3/2}; \quad \mathsf{D} = \frac{\sigma_{A'}^2}{\pi \langle I \rangle}, \qquad (15)$$

where $\sigma_{A'}^2$ is the variance of the amplitude derivative; $\langle I \rangle$ is the mean intensity. It can be easily seen that the probability density from Eq. (15) satisfies the asymptotic condition (12), and so zeros are possible. Find, following Eq. (14), that the general equation (13) transforms into the well-known equation for the density of zeros of a normal random process from Eq. (15)

$$\ell_{-3} = \frac{\sigma_{A'}^2}{3! \langle I \rangle} \frac{\mathrm{d}^3}{\mathrm{d}t^3} \left(t^{-2} + \frac{\sigma_{A'}^2}{\langle I \rangle} \right)^{-3/2} \Bigg|_{t=0} = \frac{\sigma_{A'}^2}{\langle I \rangle} \Rightarrow \mathsf{D} = \frac{\sigma_{A'}^2}{\pi \langle I \rangle}. \qquad (16)$$

– Another one technique for determination of the density of zeros is also efficient for the considered case of very strong turbulence.[13,14] It uses the relation of the number of zeros to the probability that the zero lines intersect the real and imaginary parts of the wave in the observation plane. In this connection, we need multidimensional joint probability densities of wave components and their gradients. The result was obtained for a normal random process. It was found that the density of zeros is determined by the width of the spatial spectrum of the wave θ in the form

$$\mathsf{D} = k^2 \langle \theta^2 \rangle / 2\pi, \qquad (17)$$

where k is the wave number.

Reduce Eq. (17) to the equation for the density of zeros of a normal process from Eq. (15). In the observation plane cross sections, the wave, with the allowance for the narrow band of its angular spectrum, can be represented by an analytical signal, which is a normal random process with the variance σ^2. The probability density of the intensity of this process and the mean intensity can be determined by the equations:

$$w_I = \frac{1}{2\sigma^2} \exp\left(-\frac{I}{2\sigma^2} \right), \ \sigma^2 = \langle I \rangle. \qquad (18)$$

The probability density of the amplitude derivative of a normal process also has the Gaussian form:

$$w_{A'} = \frac{1}{\sigma b \sqrt{2\pi}} \exp\left(-\frac{A'^2}{2\sigma^2 b^2} \right), \quad \sigma_{A'}^2 = \sigma^2 b^2 = b^2 \langle I \rangle, \qquad (19)$$

where $\sigma_{A'}^2$ is the variance of the amplitude derivative; b is the halfwidth of the spatial spectral density.[15] Take into account that b is measured in the unit of reciprocal length and $2b^2 = k^2 \langle \theta^2 \rangle$. As a result, we obtain from Eq. (17)

$$\mathrm{D} = \frac{k^2}{2\pi}\left\langle\theta^2\right\rangle = \frac{b^2}{\pi} = \frac{\sigma^2_{A'}}{\pi\langle I\rangle} . \tag{20}$$

Thus, in the case of a normal random process, the general equation (13) for the density of zeros reduces to the well-known equation (15), which transforms into the earlier result (17).

– Consider the case of weak turbulence. As a consequence of Rytov approximation for the parabolic equation, fluctuations of the amplitude logarithm have the normal distribution.[9] Then the probability density of the logarithmic amplitude derivative is also normal:

$$w_{\chi'} = \frac{1}{\sqrt{2\pi}\sigma_{\chi'}} \exp\left(\frac{-\chi'^2}{2\sigma^2_{\chi'}}\right). \tag{21}$$

This probability density decreases at infinity faster than χ'^{-3}. Therefore, according to the condition (12), zeros should not arise. Proceeding as before [see Eq. (14)], we obtain the result known from Ref. 12:

$$\ell_{-3} = \frac{1}{3!\sqrt{2\pi}\sigma_{\chi'}}\frac{\mathrm{d}^3}{\mathrm{d}t^3}\exp\left(\frac{-1}{2\sigma^2_{\chi'}t^2}\right)\Bigg|_{t=0} = 0 \Rightarrow \mathrm{D} = 0. \tag{22}$$

– Analyze the most complicated intermediate case, when the light wave propagates through strong turbulence. Note first that the probability density $w_{\chi'}(\chi')$ can be expressed through the function $w_2(A, A')$ – the joint probability density of the amplitude A and its derivative A' – by the known method[12]:

$$w_{\chi'}(\chi') = \int\limits_0^\infty w_2(A, A\,\chi')\,A\,\mathrm{d}A . \tag{23}$$

After substitution of Eq. (23) into Eq. (14), we can change the order of integration over A and differentiation with respect to t taking into account the independence of these variables and the absence of singularities, namely:

$$\ell_{-3} = \frac{1}{3!}\left(\frac{\mathrm{d}^3}{\mathrm{d}t^3}\int\limits_0^\infty w_2(A, At^{-1})\,A\,\mathrm{d}A\right)\Bigg|_{t=0} =$$

$$= \frac{1}{3!}\left(\int\limits_0^\infty \frac{\mathrm{d}^3}{\mathrm{d}t^3} w_2(A, At^{-1})\,A\,\mathrm{d}A\right)\Bigg|_{t=0} , \tag{24}$$

where $t^{-1} = \chi'$ as before. Then, from Eq. (24), we obtain the same result as in Ref. 12, namely

$$\ell_{-3} = \frac{\alpha\sigma^2_{A'}}{(\alpha-1)\langle I\rangle} \Rightarrow \mathrm{D} = \frac{\alpha\sigma^2_{A'}}{\pi(\alpha-1)\langle I\rangle} . \tag{25}$$

Calculations are given in Appendix.

Conclusion

With the most general assumptions on the character of fluctuations and on the mathematical model of the wave process, we have found the earlier unknown conditions, under which the wave amplitude becomes zero. Only a certain behavior of the probability density of the logarithmic amplitude derivative asymptotically equivalent to x^{-3} at infinity offers the possibility for zeros to appear. It turned out that the probability of zeros is determined by one parameter regardless of the functional form of the probability density. This parameter is the coefficient with the negative third subscript of the Laurent series for the probability density of the logarithmic amplitude derivative. Calculations of this coefficient for special functions demonstrate that the obtained general result provides for the known particular cases.

Appendix

Calculation of the density of zeros for the case of strong turbulence

In the case of a strong turbulence, the joint probability density of the wave amplitude and its derivative can be presented by the product of the K-distribution and the normal distribution[12]:

$$w_2(A, A') = \frac{4A^\alpha}{\sigma_{A'}\sqrt{2\pi}\,\Gamma(\alpha)}\left(\frac{\alpha}{\langle I\rangle}\right)^{(\alpha+1)/2} \times$$

$$\times K_{\alpha-1}\left(2\sqrt{\frac{\alpha}{\langle I\rangle}}\,A\right)\exp\left(-\frac{A'^2}{2\,\sigma^2_{A'}}\right), \tag{A1}$$

where $\Gamma(\alpha)$ is the gamma function; $K_{\alpha-1}$ is the MacDonald function. The parameter α can be expressed through the scintillation index β as

$$\alpha = 2/(\beta^2 - 1),\ \beta^2 > 1.$$

Substitution of At^{-1} in place of A' allows the coefficient ℓ_{-3} to be calculated. Following Eq. (14), let us find the third derivative of the exponential factor in Eq. (A1) with respect to t:

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}\exp\left(-\frac{A^2}{2\sigma^2_{A'}t^2}\right) =$$

$$= \left(\frac{A^6}{\sigma^6_{A'}t^9} - \frac{9A^4}{\sigma^4_{A'}t^7} + \frac{12A^2}{\sigma^2_{A'}t^5}\right)\exp\left(-\frac{A^2}{2\sigma^2_{A'}t^2}\right). \tag{A2}$$

Having substituted this result into Eq. (23), we obtain the sum of three integrals over A. They have the same form and are tabulated in Ref. 16:

$$\ell_{-3} = \frac{2}{3\sigma^2_{A'}\sqrt{2\pi}\,\Gamma(\alpha)}\left(\frac{\alpha}{\langle I\rangle}\right)^{(\alpha+1)/2} \times$$

$$\times\left(\frac{I_{\alpha+8}}{\sigma^6_{A'}t^9} - \frac{9I_{\alpha+6}}{\sigma^4_{A'}t^7} + \frac{12I_{\alpha+4}}{\sigma^2_{A'}t^5}\right)\Bigg|_{t=0} ; \tag{A3}$$

$$I_{\tilde{\alpha}} = \int_0^\infty A^{\tilde{\alpha}-1} \exp(-qA^2) K_\nu(cA) \mathrm{d}A = \frac{1}{2c} q^{(1-\tilde{\alpha})/2} \times$$
$$\times \Gamma\left(\frac{\tilde{\alpha}+\nu}{2}\right) \Gamma\left(\frac{\tilde{\alpha}-\nu}{2}\right) \exp\left(\frac{c^2}{8q}\right) W_{\frac{1-\tilde{\alpha}}{2},\frac{\nu}{2}}\left(\frac{c^2}{8q}\right), \quad \text{(A4)}$$

where $q = \sigma_{A'}^{-2}\, t^{-2}$; $c = 2\sqrt{\alpha/\langle I\rangle}$; $\nu = \alpha - 1$.

At $|t| \to 0$, the variable $z = (c^2/8q) \to 0$, then the Whuttaker function $W_{\kappa,\mu}$ has the following asymptotic representation[17]:

$$W_{\kappa,\mu}(z) = z^{\frac{1}{2}+\mu} \exp(-z/2)\, U(1/2 + \mu - \kappa, 1 + 2\mu, z) =$$
$$= z^{\frac{1}{2}-\mu} \exp(-z/2) \frac{\Gamma(2\mu)}{\Gamma(1/2+\mu-\kappa)}, \quad \text{(A5)}$$

where U is the second solution of the Kummer equation; $\kappa = (1-\tilde{\alpha})/2$; $\mu = (1-\alpha)/2$. For small $|t|$ Eq. (A3) simplifies significantly and takes the form

$$I_{\tilde{\alpha}} = \frac{(\sigma_{A'} t)^{\tilde{\alpha}-\alpha+1}}{4} \Gamma(\alpha-1) \times$$
$$\times \Gamma\left(\frac{\tilde{\alpha}-\alpha+1}{2}\right) \left(\frac{\langle I\rangle}{\alpha}\right)^{(\alpha-1)/2}. \quad \text{(A6)}$$

As before, $\tilde{\alpha}$ takes the values: $\alpha + 8$, $\alpha + 6$, $\alpha + 4$.

Now it is easy to obtain the final equation (25) by substituting the integral (A6) into Eq. (A3) and assuming $t = 0$.

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