RADON TRANSFORM IN THE PROBLEM OF PHASE OPTICAL CONTROL

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The Radon transform, which is the basic transform of computational tomography, a used to reconstruct the phase distribution of the wave field and the coefficients of expansion in a system of Zernike polynomials from the measurements of the tilts of the wavefront with the help of the Hartmann sensor is discussed. The algorithms which can be used to reconstruct the phase front and to determine the amplitudes of its modes for arbitrary dimensions and positions of the subapertures are constructed. The accuracy of reconstruction is evaluated in the numerical experiment.

When creating adaptive optical systems and devices intended for quality control of the optical parts a considerable attention is devoted to the development of the wavefront sensors. In particular, the Hartmann sensor can be used to obtain the estimate of average derivative of the phase at the points of the subaperture centers.[1,2] These data should be converted into the values of phase. The existing algorithms of such conversion are based on the representation of derivatives in terms of finite differences with subsequent solution of a system of linear algebraic equations. The other numerical methods can be also used. This approach to a certain degree makes it difficult to estimate the quality of the wavefront reconstruction in the presence of noise and makes it impossible to choose effectively dimensions and shapes of subapertures and positions of sensors.

To solve this problem, we propose in the present paper the integral representation relating the phase to the values of its partial derivatives. On this basis the analytical relations are derived for evaluation of the tilt, defocusing, astigmatism and other modes of the expansion of phase in a system of Zernike polynomials[1] avoiding the stage of reconstruction of the phase itself. The effectiveness of this approach is studied in the numerical experiment.

To derive the corresponding relation for reconstruction of the phase $S(x, y)$ from its partial derivatives $\mu(x, y) = \partial S(x, y)/\partial x$, $\nu(x, y) = \partial S(x, y)/\partial y$, we will write the inverse Radon transform[3]

$$S(x,y) = -\frac{1}{2\pi^2}\int_0^{2\pi} d\theta \int_0^{\infty} \frac{\partial \hat{S}(\theta,p)}{\partial p}\frac{dp}{p - x\cos\theta - y\sin\theta}, \qquad (1)$$

where

$$\hat{S}(\theta,p) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'\,dy'\,S(x',y')\,\delta(p - x'\cos\theta - y'\sin\theta) \qquad (2)$$

is the direct Radon transform. Integral (1) is defined in the sense of its principal value. The meaning of the variables entering into formulas (1) and (2) is clear from Fig. 1. Let us write the expression for $\partial\hat{S}(\theta, \rho)/\partial\rho$ in terms of the derivative with respect to the direction

$$\frac{\partial \hat{S}(p,\theta)}{\partial p} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'\,dy' \left[\frac{\partial S(x',y')}{\partial x'}\cos\theta + \frac{\partial S(x',y')}{\partial y'}\sin\theta\right] \delta(p - x'\cos\theta - y'\sin\theta). \qquad (3)$$

Substituting Eq. (3) into Eq. (1), using the property of δ–function, and integrating over the angular variable θ, we obtain

$$S(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy \frac{1}{(x-x')^2 + (y-y')^2} \times \{(x-x')\,\mu(x',y') + (y-y')\,\nu(x',y')\}. \qquad (4)$$

The functions $\nu(\rho)$ and $\mu(\rho)$ are finite, i.e.,

$$\mathrm{supp}\,\nu(\rho) = \mathrm{supp}\,\mu(\rho) \subset |\rho| \le R, \qquad \rho = \{x,y\}, \qquad (5)$$

where R is the aperture radius, supp f is the carrier of the function f.

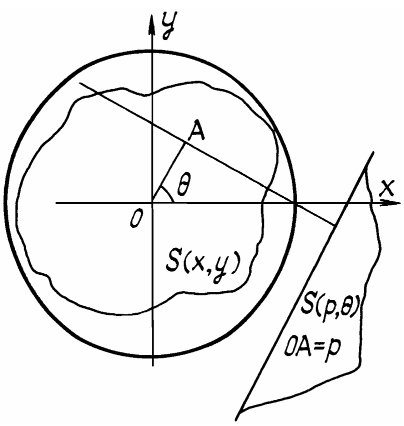

FIG. 1.

Relation (4) determines the phase in terms of its partial derivatives. Taking into account Eq. (5) then Eq. (4) assumes the following form in the polar coordinate system:

$$S(x,y) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi' \int_0^{\infty} d\rho' \,[\cos\varphi' \mu(x-\rho'\cos\varphi', y-\rho'\sin\varphi') +$$

$$+ \sin\varphi' \nu(x-\rho'\cos\varphi', y-\rho'\sin\varphi')]\theta(R^2 - |\rho-\rho'|^2), \quad (6)$$

where $\theta(z) = \begin{cases} 1, z \ge 0 \\ 0, z < 0 \end{cases}$ is the characteristic Heaviside unit function.

The expansion of the phase of the wavefront in a system of the Zernike polynomials[1] is of interest

$$S(x,y) = \sum_{k=1} a_k \Psi k(x,y), \quad (7)$$

where Ψ_k are the Zernike modes describing the classic aberrations of optical systems. The Zernike modes are orthogonal within the circle of radius R. The aberration coefficients a_k characterize the range of variation of the function $S(x,y)$.

Using integral representation (4) relating the phase to its gradient, we can derive the integral relation which can be used to obtain the coefficients a_k in terms of the components of the gradient $\nabla S(x,y)$.

Let us define the components of the vector $\mathbf{a} = (a_1, a_2, a_3..., a_n)$:

$$a_k = \langle S, \Psi_k \rangle \, \| \Psi_k \|^{-1}, \quad (8)$$

where

$$\| \Psi_k \| = \sqrt{\langle \Psi_k, \Psi_k \rangle} = \sqrt{\int\int \Psi_k^2(\rho,\varphi)\,\rho\, d\rho\, d\varphi}$$

is the norm and $\langle f, g \rangle$ is the scalar product.

Let us substitute Eq. (4) into Eq. (8) and after double integration over angular and radial variables we obtain the relations for the components of the vector **a** in terms of the gradient of the function $S(\rho, \varphi)$: for the components of the vector **a** describing the contribution of the modes invariant under rotation

$$a_n^0 = \sqrt{\frac{n+1}{\pi^3}}\left\{\langle \mu, \mathrm{Re}\, P_0 \rangle + \langle \nu, \mathrm{Im}\, P_0 \rangle\right\}, \quad (9)$$

for the components of the vector **a** describing the contribution of the even modes

$$a_n^{ev} = \sqrt{\frac{n+1}{\pi^3}}\left\{\langle \mu, \mathrm{Re}\, P_1 \rangle + \langle \nu, \mathrm{Re}\, P_2 \rangle\right\}, \quad (10)$$

and for the components of the vector **a** describing the contribution of the odd modes

$$a_n^{odd} = \sqrt{\frac{n+1}{\pi^3}}\left\{\langle \mu, \mathrm{Im}\, P_1 \rangle + \langle \nu, \mathrm{Im}\, P_2 \rangle\right\}. \quad (11)$$

Here

$$P_1 = e^{j(m+1)\varphi} \sum_{s=0}^{\frac{n-m}{2}} \frac{A(m,n,s)\,\rho^{n-2s-1}}{n-2s+m+2} +$$

$$+ e^{j(m-1)\varphi} \sum_{s=0}^{\frac{n-m}{2}} \frac{A(m,n,s)\,(\rho^{n-2s+1} - \rho^{m-1})}{n-2s-m+2}, \quad (12)$$

$$P_2 = e^{-j(m+1)\varphi} \sum_{s=0}^{\frac{n-m}{2}} \frac{A(m,n,s)\,\rho^{n-2s+1}}{n-2s-m+2} +$$

$$+ e^{-j(m-1)\varphi} \sum_{s=0}^{\frac{r-m}{2}} \frac{A(m,n,s)\,(\rho^{n-2s+1} - \rho^{m-1})}{n-2s-m+2}, \quad (13)$$

$$P_0 = e^{j\varphi} \sum_{s=0}^{n/2} \frac{A(0,n,s)\,\rho^{n-2s+1}}{n+2s+2}, \quad (14)$$

$$A(m,n,s) = \frac{(-1)^s (n-s)!}{s!\left[\frac{n-m}{2}-s\right]!\left[\frac{n+m}{2}-s\right]!\,(n-2s-m+2)}, \quad (15)$$

where $m < n$ and $n = |m|$ is even. So, the three first Zernike coefficients can be reconstructed from the gradient of the phase distribution with the use of the integral relations:

$$a_1 = -\frac{1}{2}\int_{-\infty}^{\infty}\int [(x^2 + \rho^2/2 + R^2)\mu + xy\nu]\,dx\,dy, \quad (16)$$

$$a_2 = -\frac{1}{2}\int_{-\infty}^{\infty}\int [xy\mu + (y^2 + \rho^2/2 + R^2)\nu]\,dx\,dy,$$

and

$$a_3 = -\frac{\sqrt{3}}{2}\int_{-\infty}^{\infty}\int (\rho^2 - 1)(x\mu + y\nu)\,dx\,dy,$$

where a_1 and a_2 are the tilts of the phase front with respect to the x and y axes and a_3 is defocusing of the phase front. The limits of integration over $x \in [R,R]$ and over $y \in [-\sqrt{R^2 - x^2}, \sqrt{R^2 - x^2}]$. The coefficients a_k can be used to obtain the signals of optimal control of an adaptive optical system with the finite number of degrees of freedom,[2,3] since the problem of calculation of a_k is identical (in the sense of minimizing the standard deviation) to the problem of optimization of approximation of an arbitrary function by Zernike polynomials.

Based on integral representation (4), the algorithm has been developed which can be used to reconstruct the phase front and to determine the coefficients of expansion (7) in terms of the phase gradient ∇S.

In reconstructing the phase in terms of its gradient, the integrals were evaluated by Simpson's quadrature in the polar coordinate system.

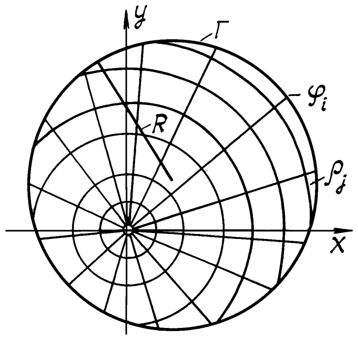

FIG. 2. Scheme of conversion of coordinates and limits of integration in the problem of phase reconstruction according to formula (6).

To ensure a high accuracy, the integral was represented as a sum of two integrals

$$\int_0^{2\pi} \mathrm{d}\rho \int_0^{2R} \mathrm{d}\varphi f(\rho, \varphi)\, \theta(\rho, \varphi) =$$

$$= \int_0^{\rho_{min}} \mathrm{d}\rho \int_0^{2\pi} f(\rho, \varphi)\, \mathrm{d}\varphi + \int_{\rho_{min}}^{\rho_{max}} \mathrm{d}\rho \int_{\varphi_{min}}^{\varphi_{max}} f(\rho, \varphi)\, \mathrm{d}\varphi ,$$

where

$$\varphi_{max} = \varphi + \arccos \frac{\rho^2 + \rho'^2 - R^2}{2\rho\rho'},$$

$$\varphi_{min} = \varphi - \arccos \frac{\rho^2 + \rho'^2 - R^2}{2\rho\rho'},$$

$\rho_{max} = R + \rho$ is the maximum distance between the observation point and the edge of the aperture, $\rho_{min} = R - \rho$ is the minimum one, and (φ, ρ) are the coordinates of the aperture centre in the coordinate system with the origin at the observation point. The limits of integration were determined by intersection of the coordinate grid (θ_j, ρ_j) and the boundary Γ of the carrier of the function $\theta(R^2 - |\rho - \rho'|^2)$ (see Fig. 2). It is obvious that the integration limits are simplest, i.e., $\rho' \in [0,R]$ and $\varphi \in [0,2\pi]$, when the point at which the phase is determined lies in the aperture centre. The integration limits are taken to be $\varphi \in [0,2\pi]$ for $\rho' \in [0,\rho_{min}]$ and $\varphi \in [\varphi_{min},\varphi_{max}]$ for $\rho' \in [\rho_{min},\rho_{max}]$ (see Fig. 2). The number of readings in the interval of angles N_φ was 60 and in the interval of radii N_ρ was 30.

To check the formulas and to simulate the reconstruction process, we took the function $S_1(x,y)$ comprising the sum of the first ten Zernike polynomials. These polynomials are given in Table I.

TABLE I

κ	a_κ		Zernike polynomials, Ψ_k
	exact	calculate.	
1	5	5.006	$2x$
2	5	5.006	$2y$
3	1.2	1.201	$\sqrt{3}[2(x2 + y2) - 1]$
4	6	6.008	$2\sqrt{6}\, xy$
5	6	6.008	$2\sqrt{6}\, (x^2 - y^2)$
6	0.8	0.80002	$2\sqrt{2}\, y\, [3(x^2 + y^2) - 2]$
7	0.96	0.96055	$2\sqrt{2}\, x\, [3(x^2 + y^2) - 2]$
8	0.95	0.95003	$2\sqrt{2}\, y\, [3(x^2 - y^2)$
9	−1.5	−1.502	$2\sqrt{2}\, x\, (x^2 - 3y^2)$
10	−1.71	−1.711	$6\sqrt{5}\, [(x^2 + y^2)^2 - (x^2 + y^2) + 1/6]$

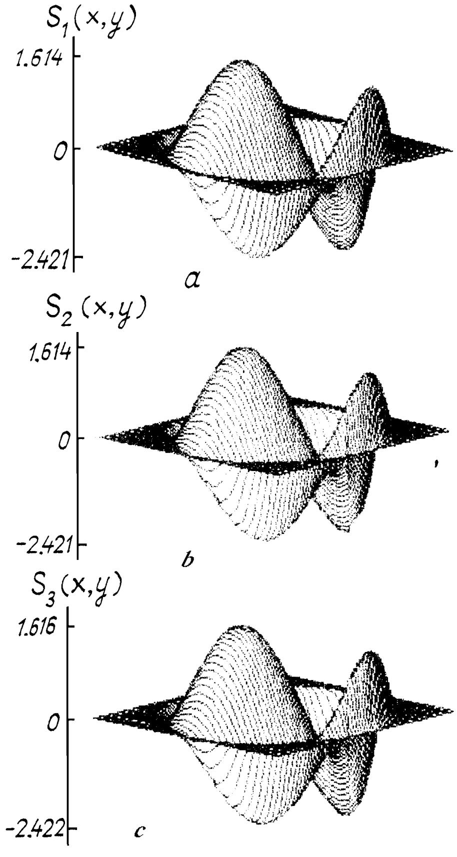

FIG. 3. Phase distribution over the aperture: a) initial distribution $S_1(x,y\}$ b) distribution $S_2(x,y)$ reconstructed from formula (6), and c) distribution $S_3(x,y)$ reconstructed from the calculated Zernike coefficients.

Figure 3 shows the shape of the function. After we had determined the gradient of the function $\nabla S_1(x,y)$ according to formula (5), we reconstructed the phase $S_2(x,y)$ and the Zernike coefficients were then evaluated from formulas (9) – (11). The values of the reconstructed coefficients a are also given in Table I. The function $S_3(x,y)$ was reconstructed from these coefficients

$$S_3(x, y) = \sum_{k=1}^{N} a_k \Psi_k .$$

Figure. 3 shows the results of reconstruction of the initial function $S_1(x,y)$ and of the functions $S_2(x,y)$ and $S_3(x,y)$. The error of calculation of a_k according to the quadrature Gauss formula does not exceed 10 %.

Thus, the numerical experiment has shown the effictiveness of the proposed approach to the determination of the phase and corresponding coefficients of its expansion in a system of Zernike polynomials. In our opinion, it is interesting to study the accuracy and resolution of the proposed approach under conditions of fixed tilts of the phase front in real measuring systems. In addition, we are going to take into account an averaging of the tilts within the subapertures as well as configurations and positions of the subapertures. We are going to consider this problem on the basis of the calculation of the Strehl factor and to construct the point spread function (PSF) of the system "meter–algorithm".

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