POLYNOMIAL EXPANSION OF ATMOSPHERIC ABERRATIONS

K.V. Shishakov and V.I. Shmal'gauzen

Department of Physics, M.V. Lomonosov Moscow State University
Received May 21, 1990

The problem of choosing the optimal orthonormalized basis functions for the reconstruction of atmospheric aberrations from measurements is studied. A system of polynomials that makes it possible to decompose the aberrations for annular receiving apertures is proposed. It is shown that for a circular aperture the error in reproducing the wavefront with these polynomials is approximately 10% *smaller than that obtained with Zernike polynomials.*

The methods of adaptive optics are used to make laser systems operating under conditions of atmospheric distortions more efficient.[1] The Measurement of the distorted wavefront and subsequent correction of the wavefront make it possible to reduce substantially the jitter, scintillation, and spreading of optical beams and images.[2]

To describe the wave distortions of light fields on atmospheric paths the light fields are usually expanded in modes in a system of basis functions. Theoretically the most accurate representation is obtained using the Karunen-Love functions, whose expansion coefficients are uncorrelated.[3] The choice of basis functions for the problem of wavefront reconstruction from measurements is discussed in Ref. 4. In practical applications, because the Karunen-Love functions are given in a tabular form and are difficult to calculate, these functions are replaced by orthonormalized polynomials.[5] Thus, for example, for circular apertures the Zernike polynomials are usually employed. At the same time, the use of Zernike polynomials for receiving apertures of a different form is not always desirable because the conditions of orthonormality are not satisfied and the expansion coefficients are more strongly correlated. We shall study the problem of constructing basis functions in a class of polynomials for such cases.

We expand the phase profile $\varphi(r)$ of the arriving light field in a system of basis functions $G_i(r)$ with the expansion coefficients β_i ($i = \overline{1, n}$). The quality of the optical system in the presence of atmospheric aberrations is described by the Strehl number St:[2]

$$St = \left| \int A(r) e^{i\Delta\varphi(r)} d^2 r \right|^2 \Big/ \left| \int A(r) d^2 r \right|^2, \tag{1}$$

where $A(r)$ is the modulus of the complex amplitude of the light wave and $\Delta\varphi(r)$ is the residual spatial error of the approximation $\delta\varphi = \varphi - \sum_{i=1}^{n} \beta_i G_i$.

Let $|\Delta\varphi| \ll 1$. In this case the expression (1) has a simple approximate representation:

$$St \simeq 1 - \Delta, \quad \Delta = \frac{1}{S} \int A(r)\, [\Delta\varphi(r)]^2 d^2 r = \frac{1}{S}(\Delta\varphi,\ \Delta\varphi), \tag{2}$$

where $S = \int A(r) d^2 r$ and the parentheses denote a scalar product of the enclosed functions with weight $A(r)$. From the expression obtained it follows that in problems of optimal wavefront reconstruction it is best to judge the strength of the residual aberrations according to the criterion Δ, i.e., according to the weighted scalar square of the functions $\Delta\varphi(r)$.

We shall study the question of the choice of the optimal orthonormalized basis functions $G_i(r)$ $\left(\frac{1}{S}(G_i G_j) = \delta_{ij} \right)$ for reconstruction of atmospheric aberrations from measurements. We represent the expansion coefficients β_i ($i = \overline{1, n}$) as a linear superposition of the signals ξ_j ($j = \overline{1, m}$, $m \geq n$) from the sensors of the measuring apparatus: $\beta_i = \sum_{j=1}^{n} B_{ij} \xi_j$, where $.B_{ij}$ are constant coefficients, calculated from the condition that the average error $\langle \Delta \rangle$ be minimized:

$$\sum_{j=1}^{m} B_{ij} \langle \xi_j \xi_k \rangle = \frac{1}{S} \langle \xi_k (\varphi,\ G_i) \rangle, \quad i = \overline{1, n}\ , \quad k = \overline{1, m}\ . \tag{3}$$

In this case the error $\langle \Delta \rangle$ is determined by the expression

$$\langle \Delta \rangle = \frac{1}{S} \langle (\varphi,\ \varphi) \rangle - $$
$$- \frac{1}{S^2} \sum_{i=1}^{n} \iint A(r) A(\rho) L(r,\ \rho) G_i(r) G_i(\rho) d^2 r\, d^2 \rho\ , \tag{4}$$

where

$$L(r,\ \rho) = \sum_{j=1}^{m} \sum_{k=1}^{m} c_{jk} \langle \xi_j \varphi(r) \rangle \langle \xi_k \varphi(\rho) \rangle, \tag{5}$$

0235-6880/90/12 1129-4 $02.00

and C_{jk} are elements of the matrix that is the inverse of the matrix with the elements $\langle \xi_j \xi_k \rangle$.

By analogy to the procedure used to obtain the Karunen-Love functions,[3] we determine the functions $G_i(r)$ as the eigenfunctions of the integral equation

$$\lambda_i^2 G_i(r) = \frac{1}{S} \int A(\rho) L(r, \rho) G_i(\rho) d^2\rho , \tag{6}$$

where λ_i^2 are the eigenvalues.

It can be shown that in this case the following expressions are valid:

$$\langle \Delta \rangle = \frac{1}{S} \langle (\varphi, \varphi) \rangle - \sum_{i=1}^{n} \lambda_i^2;$$

$$\langle \beta_i \beta_j \rangle = \frac{1}{S^2} \sum_{k=1}^{m} \sum_{l=1}^{m} c_{kl} \langle \xi_k(\varphi, G_i) \rangle \langle \xi_l(\varphi, G_j) \rangle = \lambda_i^2 \delta_{ij} \tag{7}$$

From Eq. (6) it is not difficult to see that the functions $G_i(r)$ will be polynomials if the kernel $L(r, \rho)$ is expanded in a series of spatial polynomials. If the measurements ξ_j depend linearly on the phase of the profile, this is equivalent to polynomial approximation of the correlation function $\langle \varphi(r)\varphi(\rho) \rangle$.

Thus the problems of optimal approximation of atmospheric aberrations and the problems of wavefront reconstruction from measurements lead to essentially the same equations for the basis functions. Only the weighting functions are different.

We shall study the construction of the basis functions in problems of the polynomial expansion of atmospheric aberrations on circular and annular receiving apertures with a uniform distribution of the light intensity. We shall also assume that the phase distortions are measured perfectly over the entire aperture. In this case the expansion coefficients β_i assume the simple form $\beta_i = \frac{1}{S}(\varphi, G_i)$, and Eq. (6) reduces to the following equation:[3]

$$\lambda_i^2 G_i(r) = \frac{1}{S} \int_{\Omega} \langle \varphi(r)\varphi(\rho) \rangle G_i(\rho) d^2\rho , \tag{8}$$

where Ω is the region of the aperture with outer diameter D and inner diameter aD; $A(r) = 1$, $r \in \Omega$; $A(r) = 0$, $r \notin \Omega$.

In the expression (8) the spatial correlation function $\langle \varphi(r)\varphi(\rho) \rangle$ has a linear representation in terms of the structure function $D_\varphi(r - \rho)$:[3]

$$\langle \varphi(r)\varphi(\rho) \rangle = -\frac{1}{2}\Big[D_\varphi(r - \rho) + \frac{1}{S^2} \int_{\Omega}\int_{\Omega} D_\varphi(r' - \rho') d^2r' d^2\rho' - \frac{1}{S}\int_{\Omega} D_\varphi(r' - r) d^2r' - \frac{1}{S}\int_{\Omega} D_\varphi(\rho' - \rho) d^2\rho' \Big]. \tag{9}$$

For the Kolmogorov model of atmospheric turbulence $D_\varphi(r') = 6.88(r'/r_0)^{5/3}$ where $r' = |r - \rho|$ and r_0 is Fried's correlation radius.[5]

We shall obtain the first ten polynomials $G_i(r)$. For this we expand $D_\varphi(r)$ in the following form:[6]

$$D_\varphi(r) = \left(\frac{D}{r_0}\right)^{5/3} \left[10.788 \left(\frac{r}{D}\right)^2 - 10.795 \left(\frac{r}{D}\right)^4 + 11.620 \left(\frac{r}{D}\right)^4 - 4.768 \left(\frac{r}{D}\right)^8 \right]. \tag{10}$$

In this case the solutions of the integral equation (8) will be circular polynomials:[7]

$$G_1 = 1, \quad G_j = \sum_{k=0}^{3} c_{kj} \left(\frac{r}{D}\right)^{q+2k} \begin{cases} \cos q\theta \\ \sin q\theta \end{cases}, \quad j \ge 2, \tag{11}$$

where $q = 0, 1, 2, \ldots$ and C_{kj} are constant coefficients.

For them the integral equation (8) reduces to a matrix equation of dimension 4, and in addition all integrals can be calculated in terms of analytic functions. The coefficients C_{kj} obtained are presented in Table I. The dependence of the quantities λ_j^2 on the parameter a is studied in Ref. 7.

TABLE I.

q	i	a	c_{01}	c_{11}	c_{21}	c_{31}
		0	2.255	-0.543	0.276	-0.082
		0.25	2.188	-0.524	0.267	-0.080
1	2.3	0.5	2.010	-0.448	0.234	-0.074
		0.75	1.771	-0.307	0.171	-0.068
		0	-2.237	7.105	-5.082	1.513
		0.25	-2.688	8.069	-5.874	1.836
0	4	0.5	-4.453	11.642	-8.949	3.288
		0.75	-7.750	16.274	-13.649	6.971
		0	5.024	-4.791	1.605	
		0.25	4.867	-4.642	1.556	
2	5.6	0.5	4.457	-4.273	1.451	
		0.75	4.018	-3.948	1.451	
		0	-7.439	16.105	-6.302	-0.183
		0.25	-7.424	16.044	-6.315	-0.180
1	7.8	0.5	-9.310	19.692	-8.169	-0.202
		0.75	-25.521	52.554	-24.824	-0.449
		0	5.342	-3.203		
		0.25	5.173	-3.101		
3	9.10	0.5	4.660	-2.801		
		0.75	4.057	-2.511		
		0	1.192	-4.913	-5.197	11.995
		0.25	1.843	-7.132	-2.788	11.017
0	11	0.5	6.136	-20.247	8.646	7.935
		0.75	46.925	-129.729	96.088	-11.136

We shall now compare the polynomials obtained with other functions employed for describing atmospheric aberrations on circular receiving apertures ($a = 0$). The radial sections of the polynomials G_j are shown in Fig. 1. The true Karunen-Love functions are presented in Ref. 3. They are virtually identical to the functions G_j ($j \le 10$). The calculations showed that using the first ten Karunen-Love functions reduces the error $\langle \Delta \rangle$ by an amount less than 1% of the analogous error for G_j. The Zernike polynomials describe atmospheric aberrations less accurately than the functions G_j. The ratios of the correlation coefficients $\langle \beta_i \beta_j \rangle$ for the polynomials G_k and the Zernike polynomials Z_k ($k \le 10$), preorthogonalized for annular apertures,[7] are presented in Table II. One can see from the table that the polynomials $G_k(r)$ ($k \le 10$) substantially (by an order of magnitude) reduce the correlation coefficients $\langle \beta_2 \beta_8 \rangle$ and $\langle \beta_3 \beta_7 \rangle$, and they reduce the average error of the approximation by approximately 10%.

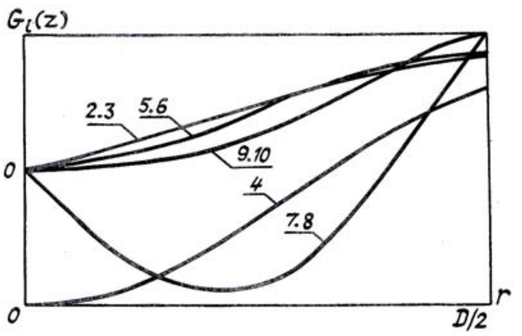

FIG. 1. The radial sections of the polynomials G_j.

TABLE II.

a	$\langle\beta_i\beta_j\rangle_G / \langle\beta_i\beta_j\rangle_Z$						$\langle\Delta\rangle_G / \langle\Delta\rangle_Z$
	2-2, 3-3	4-4	5-5 6-6		9-9 10-10	2-8 3-7	
0	1.0010	1.037	1.030	1.070	0.994	−0.153	0.90
0.5	1.0007	1.016	1.023	1.059	0.955	−0.086	0.92

In order to reduce further the quantities $\langle \beta_2 \beta_8 \rangle$ and $\langle \beta_3 \beta_7 \rangle$ and to increase the accuracy of the expansion additional terms must be included in the expression (10)[6]. There are no fundamental difficulties in doing this.

REFERENCES

1. V.E. Zuev, ed., *Signals and Noise in Laser Ranging* [in Russian] (Radio i Svyaz', Moscow, 1985), 364 pp.
2. V.P. Lukin, *Atmospheric Adaptive Optics* [in Russian] (Nauka, Novosibirsk, 1986), 156 pp.
3. J.Y. Wang and J.K. Markey, J. Opt. Soc. Am. **88**, No. 1, 78 (1978).
4. K.T. Protasov, Opt. Atm. **3**, No. 1, 42–46 (1990).
5. J. Noll, J. Opt. Soc, Am. **66**, No. 3, 207–211 (1976).
6. K.V. Shishakov and V.I. Shmal'gausen, Opt. Atm. **2**, No. 2, 125–127 (1989).
7. K.V. Shishakov and V.I. Shmal'gausen, Vestn. Mosk. Univ. Fiz. Astron. **30**, No. 4, 89–91 (1989).