Laser guide stars and models of atmospheric turbulence

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The quality of image of an extraterrestrial object formed by an astronomical optical system through the turbulent atmosphere is analyzed. Relative increase of the Strehl parameter at adaptive correction is calculated using the technique of laser guide stars. The efficiency of adaptive correction of distortions for different types of guide sources is compared. A specialized wave front sensor is applied, which operates with the use of a wide laser beam as a reference wave and allows the continuous phase of the reference wave to be reconstructed. The calculations are performed for different models of the vertical dependence of the structure parameter of refractive index of the turbulent atmosphere. The estimates obtained show that the formed field is quite close to a plane wave in its parameters. That is why we obtain high correction and large increase of the Strehl parameter, which indicates indirectly a good correction of higher modal components, which are poorly corrected with the use of traditional schemes of formation of a laser guide star with the aid of a focused laser beam. The comparative calculations for different models of the vertical profile of the structure parameter of refractive index have shown serious differences in the behavior of correlation radii for plane and spherical waves.

Introduction

Atmospheric turbulence is a serious limitation for astronomical observations. Application of adaptive optical systems can improve significantly the quality of images observed through the atmosphere. In adaptive systems, information on the distribution of a medium turbulent inhomogeneities along the propagation path is obtained from measurements with the use of reference sources. A reference source as some object with the known amplitude-phase distribution located at a known distance can be formed directly on the surface of the object, whose image is analyzed in the optical system, or it can lie at an infinite distance (natural guide star), or in the atmosphere (the so-called laser guide star of LGS). In this paper, we consider a new method of formation of a laser guide star.

Initial statements

Let an object (natural star) observed by an astronomical telescope lie at infinity. The wave formed by the object on the telescope entrance aperture is a plane wave. Assume that the guide source is in the plane x. The entrance aperture of the telescope forming an image of the extraterrestrial source is approximated by the exponential function $W(\rho) = \exp(-\rho^2/2R^2)$, the action of the telescope can be replaced by an equivalent lens introducing the phase term $\exp(-ik\rho^2/2f)$, where f is the equivalent focal length of the telescope optical system; $2R$ is the diameter of the telescope entrance aperture. Consider the case of adaptive correction of distortions based on the phase conjugation algorithm with the use of wave front measurements from the laser guide star. For traditional LGS, it is usually assumed that its visible size does not allow it to be resolved by the telescope optical system and therefore can be considered as a point source.

The phase of the wave with the wave number $k = 2\pi/\lambda$ for a point guide source in the plane of the telescope entrance aperture $x = 0$ can be written as

$$S_{\text{ref}}(0,\boldsymbol{\rho}) = kx + \frac{k\rho^2}{2x} + S_{\text{sph}}(0,\boldsymbol{\rho};x,0), \tag{1}$$

where $S_{\text{sph}}(0,\boldsymbol{\rho};x,0)$ is the random phase of the spherical wave caused by turbulence during the wave propagation from the plane x to the point $\boldsymbol{\rho}$, lying in the plane of the entrance aperture $x = 0$. It is assumed here that a point source lies on the optical axis of the telescope. This means that the conditions of operation of the adaptive optical system allow the guide source to be formed at the same optical axis, at which the studied object is observed. If the image of the natural star is formed in the focal plane of the telescope ($x = -f$), we obtain the field in the following form:

$$U(-f,\boldsymbol{\rho}) = \iint d^2\rho_1 \exp(-\rho_1^2/2R^2)\exp(-\rho_1^2/2f)\times$$
$$\times G_0(0,\boldsymbol{\rho}_1;-f,\boldsymbol{\rho})\exp[iS_{\text{pl}}(\boldsymbol{\rho}_1)]. \tag{2}$$

Here $G_0(0,\boldsymbol{\rho}_1;-f,\boldsymbol{\rho})$ is the Green function for the free space; $S_{\text{pl}}(\boldsymbol{\rho}_1)$ are phase fluctuations caused by atmospheric turbulence for the plane wave at the telescope entrance aperture. Equation (2) is written on the assumption that the field distribution can be

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represented by the Kirchhoff diffraction integral[1] under the condition that the plane wave with the phase distortions $S_{\text{pl}}(\rho_1)$ is incident on the entrance aperture. For most astronomical applications, it is possible to take into account only phase fluctuations in the incident wave, while amplitude fluctuations contribute smaller as compared to the phase ones. It can be shown[5] that as a result of the phase adaptive correction with the use of the fluctuation part of the spherical reference wave (1) from LGS, the corrected field in the focal plane takes the form

$$U(-f,\rho) = \iint d^2\rho_1 \exp(-\rho_1^2/2R^2) G_0(0,\rho_1;-f,\rho) \times$$
$$\times \exp[iS_{\text{pl}}(\rho_1) - iS_{\text{sph}}(x,0;0,\rho_1) - ik\rho_1^2/2f]. \quad (3)$$

It should be noted that in Eq. (3) the integral is calculated within the entrance aperture of the telescope, that is, in a circle with an area πR^2.

Nearly all laser guide stars are formed now by focusing laser radiation from the ground. However, the use of focused laser beams for formation of a laser guide star has a serious limitation connected with the fact that the point guide source and the plane wave (formed from some actual star) are imaged in different planes and phase fluctuations for plane and spherical waves have different values. As a result, the phase conjugate correction[5] incompletely compensates distortions. In the literature, this phenomenon is called focal anisoplanatism.[2]

Oriented wavefront sensor

Some approaches were proposed to prevent manifestations of focal anisoplanatism. For example, a wide collimated beam is proposed to use LGS formation,[2,3] which irradiates a rather large area and forms a secondary source with the size somewhat exceeding the diameter of the telescope aperture. In this case, it is proposed[4] to use the wavefront sensor of such a design that each its subaperture sees only a limited part of the whole irradiated LGS surface. We can consider such a guide star as a diffusely luminous surface,[3,4] and since the field of view of each subaperture is much smaller than the whole luminous area, the LGS jitter almost does not contribute to the measured displacement of an individual fragment (for an individual subaperture). Thus, an individual aperture does not see edges of the irradiated surface of the secondary source, and therefore the jitter of the secondary source itself does not contribute to the measured jitter of its image in the focal plane of the telescope. As a result, the measured image jitter of an individual subaperture is caused only by radiation propagation through turbulent fluctuations of the atmosphere on the path from LGS to the telescope.

In our papers,[3,4,6] it was proved that for each subaperture we have an individual secondary source formed by the subaperture field of view. The spatial coherence radius of the field of this secondary partial source (diffusely reflected radiation) can be estimated as $\rho_{\text{coh}} \approx \lambda/\theta$, where $\theta = d/x$ is the angle, at which the irradiated part of the secondary source is seen (d is the LGS area visible within the field of view of an individual subaperture). If we select the angle $\theta = d/x < \lambda/r_0^{\text{sph}}$, that is, the field of view angle of each subaperture is smaller than the coherence angle for the spherical wave, then we obtain[1,5,6] that within the coherence radius r_0^{sph} the field of this partial secondary source can be considered as coherent, the phase reconstructed from such measurements can be joined smoothly with neighboring subapertures, and the continuous phase can be obtained as a result. If a square subaperture array is used and the distance between neighboring subapertures is taken equal to the coherence radius for the spherical wave ($d = r_0^{\text{sph}}$), then the phase reconstructed in neighboring subapertures can be joined smoothly.

Calculation of mean intensity distribution of the field in the telescope focal plane

To estimate the efficiency of the proposed new scheme of LGS formation, we will calculate comparatively the mean intensity distribution for the field in the focal plane of the telescope without correction, with correction by the traditional scheme, that is, based on Eq. (3), and by the new approach with the use of a collimated beam forming the guide star. In the last case, a large irradiated surface serves as a guide source. With the aid of a specialized wavefront sensor,[3,4] we divide this surface into a system of spherical sources (for simplicity, we use a square grid, that is, obtain $N \times N = N^2$ spherical waves). Then the field (2) after correction by a system of spherical waves[4] under the condition that the phase of the guide source is reconstructed as a continuous one can be written as follows:

$$U(-f,\rho) = \sum_{j=1}^{N^2} \iint d^2\rho_1 G_0(0,\rho_1;-f,\rho) \exp(-(\rho_1-\rho_j)^2/2d^2) \times$$
$$\times \exp[iS_{\text{pl}}(\rho_1) - iS_{\text{sph}}(\rho_1;\rho_j) - ik\rho_1^2/2f], \quad (4)$$

where $S_{\text{sph}}(\rho_1;\rho_j)$ are phase fluctuations of an individual spherical wave at the point ρ_1 on the entrance aperture of the telescope; the source of this wave lies in the LGS plane at the point ρ_j. It should be noted that the integral in Eq. (4) is calculated over the area of an individual subaperture $\Sigma = \pi d^2/4$, whose center lies at the point ρ_j. Totally, we have $N \times N = N^2$ of such partial reference waves. Since the optical field formed by the secondary system of sources on the telescope entrance aperture remains coherent (for an individual subaperture) within the zone $d < r_0^{\text{sph}}$, after summation and averaging over the whole telescope aperture and over turbulent atmospheric fluctuations, we obtain from Eq. (4) the following equation for the mean intensity distribution of the corrected field in the focal plane:

$$<I(-f,\rho)> = \sum_{j=1}^{N^2}\sum_{l=1}^{N^2}\iint d^2\rho_1 d^2\rho_2 \exp(-(\rho_1-\rho_j)^2/2d^2)\times$$
$$\times\exp(-(\rho_2-\rho_l)^2/2d^2)G_0(0,\rho_1;-f,\rho)\times$$
$$\times G_0^*(0,\rho_2;-f,\rho)\exp(-ik\rho_1^2/2f+ik\rho_2^2/2f)\times$$
$$\times<\exp\{i[S_{\rm pl}(\rho_1)-S_{\rm pl}(\rho_2)]-i[S_{\rm sph}(\rho_1;\rho_j)-S_{\rm sph}(\rho_2;\rho_l)]\}>. \quad (5)$$

The mean intensity distribution for the field in the focal plane of the telescope without adaptive correction can be written analogously:

$$<I(-f,\rho)> = \iint d^2\rho_1 d^2\rho_2 \exp(-(\rho_1^2+\rho_2^2)/2R^2)\times$$
$$\times G_0(0,\rho_1;-f,\rho)G_0^*(0,\rho_2;-f,\rho)\times$$
$$\times\exp(-ik\rho_1^2/2f+ik\rho_2^2/2f)<\exp\{i[S_{\rm pl}(\rho_1)-S_{\rm pl}(\rho_2)]\}>. \quad (6)$$

The angular brackets here denote the operation of averaging over fluctuations of atmospheric turbulence. The relative efficiency of correction based on the guide star formed by a wide collimated beam is estimated in comparison with the traditional scheme. For this purpose, we compare the results of correction with the use of one spherical wave on the optical axis of the telescope and a system of N^2 spherical waves.

Telescope without adaptive correction

First, calculate the mean intensity distribution in the focal plane of the telescope without correction. In the integrand of Eq. (6), precalculate the factor in angular brackets, which is connected with the action of atmospheric turbulence. In the averaging over atmospheric turbulence fluctuations, the idea that fluctuations of the phase S are Gaussian and have a zero mean is used. Then we obtain

$$<\exp(-iS)> = \exp(-\frac{<S^2>}{2}).$$

We use the isotropic model[1] of the spectral density of the refractive index fluctuations, which takes into account the inner scale of turbulence l_0

$$\Phi_n(\kappa,\xi) = 0.033C_n^2(\xi)\kappa^{-11/3}\exp(-\kappa^2/\kappa_m^2),$$
$$\kappa_m = 5.92/l_0.$$

As a result, the term in angular brackets in Eq. (6) under the condition $\kappa_m|\rho_1-\rho_2| \gg 1$, can be written as

$$<\{\ldots\}> = \exp\{-\frac{1}{2}D_s^{\rm pl}(\rho_1-\rho_2)\} =$$
$$= \exp[-3.44|\rho_1-\rho_2|^{5/3}/(r_0^{\rm pl})^{5/3}]. \quad (7)$$

Here $r_0^{\rm pl}$ is the coherence radius for the plane wave propagating from infinity to the plane of the receiving aperture of the telescope. Then we calculate the product of the Green functions of the free space, that is,

$$G_0(0,\rho_1;-f,\rho)G_0^*(0,\rho_2;-f,\rho) =$$
$$= f^{-2}\exp[-ik\rho_1^2/2f + ik\rho_2^2/2f + ik\rho(\rho_1-\rho_2)/f].$$

As a result, the intensity distribution at the telescope focus under the vacuum conditions is described by the equation

$$I_{\rm vac}(-f,\rho) = 4\pi^2R^4f^{-2}\exp(-k^2\rho^2R^2/f^2). \quad (8)$$

In analytical calculations without invoking numerical methods we use the quadratic approximation for Eq. (7). Thus, for the telescope without correction we have

$$<I(-f,\rho)> = 4\pi^2R^4f^{-2}\times$$
$$\times\frac{\exp[-k^2\rho^2R^2/f^2(1+13.76R^2/(r_0^{\rm pl})^2)}{(1+13.76R^2/(r_0^{\rm pl})^2)}. \quad (9)$$

Using Eqs. (8) and (9), we calculate the Strehl parameter, which is the ratio of the mean intensity (9) on the axis of the system in a randomly inhomogeneous medium to the intensity in the vacuum, that is,

$$\mathrm{St} = <I(-f,0)>/I_{\rm vac}(-f,0) = (1+13.76R^2/(r_0^{\rm pl})^2)^{-1}. \quad (10)$$

It should be noted that the Strehl parameter is one of the key parameters for determination of the efficiency of application of an opto-electronic system in a randomly inhomogeneous medium. This parameter determines the penetration of an opto-electronic system. For example, in astronomy it determines the minimal brightness of a star, which can be detected by a telescope. It is clear from Eq. (10) that the value of the system Strehl parameter, when observing through a turbulent medium, depends on the coherence radius of the plane wave

$$r_0^{\rm pl} = 1.707\{k^2\int_0^\infty d\xi C_n^2(\xi)\}^{-3/5}.$$

The coherence radius is calculated over the whole thickness of this randomly inhomogeneous medium.

Traditional correction with the use of a focused beam

It is not difficult to show analogously that for the field formed by a natural star the following equation can be resulted from the adaptive correction with the use of one guide star lying on the telescope axis[5–8]:

$$<I(-f,\rho)> = \iint d^2\rho_1 d^2\rho_2 \exp(-\rho_1^2/2R^2)\times$$
$$\times\exp(-\rho_2^2/2R^2)G_0(0,\rho_1;-f,\rho)G_0^*(0,\rho_2;-f,\rho)\times$$
$$\times\exp(-ik\rho_1^2/2f+ik\rho_2^2/2f)<\exp\{i[S_{\rm pl}(\rho_1)-S_{\rm pl}(\rho_2)]-$$
$$-i[S_{\rm sph}(x,0;0,\rho_1)-S_{\rm sph}(x,0;0,\rho_2)]\}>, \qquad (11)$$

and the term in the angular brackets is

$$<\{\ldots\}> = \exp\{-\frac{1}{2}D_s^{\rm pl}(\rho_1-\rho_2)-\frac{1}{2}D_s^{\rm sph}(\rho_1-\rho_2)+$$
$$+<S_{\rm pl}(\rho_1)S_{\rm sph}(x,0;0,\rho_1)> - <S_{\rm pl}(\rho_1)S_{\rm sph}(x,0;0,\rho_2)> +$$
$$+<S_{\rm pl}(\rho_2)S_{\rm sph}(x,0;0,\rho_2)> - <S_{\rm pl}(\rho_2)S_{\rm sph}(x,0;0,\rho_1)>\}. \qquad (12)$$

Here $D_s^{\rm pl}(\rho_1-\rho_2)$, $D_s^{\rm sph}(\rho_1-\rho_2)$ are structure functions of the phase for the plane and spherical waves. To calculate components of Eq. (12), let us write the equation for the phase in the plane and spherical waves (with a center at the origin of coordinates) in the approximation of geometric optics. Since the wave propagates top-down, for the plane wave we have[1]:

$$S_{\rm pl}(0,\rho) = k\int_0^x d\xi \iint d^2 n(\kappa, x-\xi)\exp(i\kappa\rho + ik\alpha\rho), \qquad (13)$$

where α is the wave front tilt angle for the star relative to the telescope axis. If $\alpha = 0$, we deal with the normally incident wave. We can also write the equation for the spherical wave propagating from ρ_0 in the plane x:

$$S_{\rm sph}(0,\rho) = k\int_0^x d\xi\, n_1[\xi,\ \rho\xi/x + \rho_0(1-\xi/x)]. \qquad (14)$$

Hereinafter, we use the spectral representation for fluctuations of the refractive index

$$n_1(\xi, \mathbf{R}) = \iint d^2 n(\kappa, \xi)\exp(i\kappa\mathbf{R}),$$

then for fluctuations in the spherical wave

$$S_{\rm sph}(0,\rho) = k\int_0^x d\xi \iint d^2 n(\kappa, x-\xi)\times$$
$$\times\exp[i\kappa\rho\xi/x + i\kappa\rho_0(1-\xi/x)].$$

We continue calculation of component terms of Eq. (12) and introduce for brevity

$$\Delta_j(\rho_1) = S_{\rm pl}(\rho_1) - S_{\rm sph}(\rho_1, \rho_j). \qquad (15)$$

Here ρ_j, $j = 1,\ldots,N$ are coordinates of the sources of spherical waves. Then we obtain

$$<[\Delta_j(\rho_1)-\Delta_j(\rho_2)]^2> = <[S_{\rm pl}(\rho_1)-S_{\rm pl}(\rho_2)]^2> +$$
$$+<[S_{\rm sph}(\rho_1,\rho_j)-S_{\rm sph}(\rho_2,\rho_j)]^2> -$$
$$-2<[[S_{\rm pl}(\rho_1)-S_{\rm pl}(\rho_2)][S_{\rm sph}(\rho_1,\rho_j)-S_{\rm sph}(\rho_2,\rho_j)]>. \qquad (16)$$

We use the isotropic model [1] of the spectral density of fluctuations of the refractive index, and then for the condition $\kappa_m|\rho_1-\rho_2| \gg 1$ the first two terms of Eq. (12) have the form:

$$D_s^{\rm pl}(\rho_1-\rho_2) + D_s^{\rm sph}(\rho_1-\rho_2) =$$
$$= 2.94k^2\{\int_0^\infty d\xi C_n^2(\xi) + \int_0^x d\xi(\xi/x)^{5/3}C_n^2(x-\xi)\}|\rho_1-\rho_2|^{5/3}, \qquad (17)$$

and mutual terms from Eq. (12) yield

$$-2<[S_{\rm pl}(\rho_1)-S_{\rm pl}(\rho_2)][S_{\rm sph}(\rho_1,\rho_j)-S_{\rm sph}(\rho_2,\rho_j)]>=$$
$$=-8\pi^2 0.033k^2\int_0^x d\xi C_n^2(x-\xi)\int_0^\infty d\kappa\kappa\kappa^{-11/3}\exp(-\kappa^2/\kappa_m^2)[\ldots], \qquad (18)$$

where for convenience, summing up all these terms, write them in the form $[\ldots] = I_1 + I_2 + I_3 + I_4$, where

$$I_1 = -2\pi J_0[\kappa(1-\xi/x)|\rho_1-\rho_j|],$$
$$I_2 = -2\pi J_0[\kappa(1-\xi/x)|\rho_2-\rho_j|],$$
$$I_3 = 2\pi J_0[\kappa|\rho_j(1-\xi/x)-\rho_2+\rho_1\xi/x|],$$
$$I_4 = 2\pi J_0[\kappa|\rho_j(1-\xi/x)-\rho_1+\rho_2\xi/x|].$$

It is seen that to calculate all four terms in Eq. (18), it is necessary to calculate the following integral:

$$-2<[S_{\rm pl}(\rho_1)-S_{\rm pl}(\rho_2)]\cdot[S_{\rm sph}(\rho_1,\rho_j)-S_{\rm sph}(\rho_2,\rho_j)]>=$$
$$=-8\pi^2 0.033k^2\int_0^x d\xi C_n^2(x-\xi)\int_0^\infty d\kappa\kappa\kappa^{-11/3}\exp(-\kappa^2/\kappa_m^2)[\ldots]. \qquad (19)$$

Let the inequality

$$\frac{(1-\xi/x)^2|\rho_1-\rho_j|^2\kappa_m^2}{4} \gg 1,$$

keeps true at the most part of the path. Then we can use the following asymptotic in Eq. (19):

$${}_1F_1\left(-5/6, 1; -\frac{(1-\xi/x)^2|\rho_1-\rho_j|^2\kappa_m^2}{4}\right) \approx$$
$$\approx \frac{(1-\xi/x)^{5/3}}{\Gamma(11/6)}\frac{|\rho_1-\rho_j|^{5/3}}{2^{5/3}}\kappa_m^{5/3}.$$

As a result, we get

$$\int_0^\infty (\ldots)\mathrm{d}\kappa = \frac{\Gamma(-5/6)}{2^{8/3}\Gamma(11/6)}(1-\xi/x)^{5/3}\left|\rho_1-\rho_j\right|^{5/3}.$$

After summation of all six terms of Eq. (16), we have

$$<[\Delta_j(\rho_1)-\Delta_j(\rho_2)]^2>=$$

$$=-8\pi^2 0.033k^2\int_0^x \mathrm{d}\xi C_n^2(x-\xi)\frac{\Gamma(-5/6)}{2^{8/3}\Gamma(11/6)}\times$$

$$\times[\left|\rho_1-\rho_2\right|^{5/3}+(\xi/x)^{5/3}\left|\rho_1-\rho_2\right|^{5/3}+$$

$$+(1-\xi/x)^{5/3}\left|\rho_1-\rho_j\right|^{5/3}+(1-\xi/x)^{5/3}\left|\rho_2-\rho_j\right|^{5/3}-$$

$$-\left|\rho_j(1-\xi/x)-\rho_2+\rho_1(\xi/x)\right|^{5/3}-$$

$$-\left|\rho_j(1-\xi/x)-\rho_1+\rho_2(\xi/x)\right|^{5/3}]. \quad (20)$$

Then perform the first check. Assuming $\rho_1=\rho_2$ in Eq. (20), we obtain for the integrand

$$<[\Delta_j(\rho_1)-\Delta_j(\rho_2)]^2>=$$

$$=-8\pi^2 0.033k^2\int_0^x \mathrm{d}\xi C_n^2(x-\xi)\frac{\Gamma(-5/6)}{2^{8/3}\Gamma(11/6)}\times$$

$$\times[\left|\rho_1-\rho_1\right|^{5/3}+(\xi/x)^{5/3}\left|\rho_1-\rho_1\right|^{5/3}+$$

$$+(1-\xi/x)^{5/3}\left|\rho_1-\rho_j\right|^{5/3}+(1-\xi/x)^{5/3}\left|\rho_1-\rho_j\right|^{5/3}-$$

$$-\left|\rho_j(1-\xi/x)-\rho_1+\rho_2(\xi/x)\right|^{5/3}-$$

$$-\left|\rho_j(1-\xi/x)-\rho_2+\rho_1(\xi/x)\right|^{5/3}]\equiv 0.$$

Consequently, Eq. (20) is true. Then we consider the term in square brackets in Eq. (20):

$$[\ldots]=[\left|\rho_1-\rho_2\right|^{5/3}+(\xi/x)^{5/3}\left|\rho_1-\rho_2\right|^{5/3}+$$

$$+(1-\xi/x)^{5/3}\left|\rho_1-\rho_j\right|^{5/3}+(1-\xi/x)^{5/3}\left|\rho_2-\rho_j\right|^{5/3}-$$

$$-\left|\rho_j(1-\xi/x)-\rho_2+\rho_1(\xi/x)\right|^{5/3}-$$

$$-\left|\rho_j(1-\xi/x)-\rho_1+\rho_2(\xi/x)\right|^{5/3}].$$

The coefficient is $-8\pi^2 0.033\frac{\Gamma(-5/6)}{2^{8/3}\Gamma(11/6)}=2.82$. It can be easily shown that for the case of a traditional guide star, when the reference spherical wave is at the axis of the system, that is, $\rho_j=0$, we have in the quadratic approximation

$$<[\Delta_j(\rho_1)-\Delta_j(\rho_2)]^2>\approx 2.82\int_0^x \mathrm{d}\xi C_n^2(x-\xi)[\left|\rho_1-\rho_2\right|^2+$$

$$+(\xi/x)^2\left|\rho_1-\rho_2\right|^2+(1-\xi/x)^2\left|\rho_1\right|^2+(1-\xi/x)^2\left|\rho_2\right|^2-$$

$$-\left|\rho_2-\rho_1(\xi/x)\right|^2-\left|\rho_1-\rho_2(\xi/x)\right|^2]=$$

$$=2.82k^2\int_0^x \mathrm{d}\xi C_n^2(\xi)(\xi/x)^2\left|\rho_1-\rho_2\right|^2. \quad (21)$$

As a result of correction with the use of one spherical reference wave, the integrand of Eq. (11) includes a factor of the form

$$\exp\left[-\frac{\left|\rho_1-\rho_2\right|^2}{(r_0^{\mathrm{pl}})^2}\frac{\int_0^x \mathrm{d}\xi C_n^2(\xi)(\xi/x)^2}{\int_0^\infty \mathrm{d}\xi C_n^2(\xi)}\right]. \quad (22)$$

Thus, for distribution of mean intensity in the telescope focal plane in the traditional correction scheme obtain the equation

$$<I(-f,\rho)>=\frac{4\pi^2R^4}{f^2[1+4R^2/(\tilde{r}_0^{\mathrm{pl}})^2]}\times$$

$$\times\exp\{-k^2\rho^2R^2/[f^2(1+4R^2/(\tilde{r}_0^{\mathrm{pl}})^2)]\}, \quad (23)$$

in which the coherence radius of the field $\tilde{r}_0^{\mathrm{pl}}$ (at correction with the spherical reference wave) is introduced in the form

$$\tilde{r}_0^{\mathrm{pl}}=r_0^{\mathrm{pl}}\left[\frac{\int_0^x \mathrm{d}\xi C_n^2(\xi)(\xi/x)^2}{\int_0^\infty \mathrm{d}\xi C_n^2(\xi)}\right]^{-1/2}. \quad (24)$$

As a result, the Strehl parameter for the telescope corrected with the aid of focused LGS is

$$\mathrm{St}=[1+4R^2/(\tilde{r}_0^{\mathrm{pl}})^2]^{-1}. \quad (25)$$

Thus, in Eq. (25) compared to Eq. (10), we can see a significant increase of the Strehl parameter as a result of application of adaptive correction based on traditional LGS. Consequently, adaptive correction with the use of traditional focused LGS is in fact equivalent to the increase in the size of the coherent part of the telescope aperture, and this increase appears to be equal to

$$\left[\frac{\int_0^x \mathrm{d}\xi C_n^2(\xi)(\xi/x)^2}{\int_0^\infty \mathrm{d}\xi C_n^2(\xi)}\right]^{-1/2}$$

and can be calculated using the models of the vertical profile of the structure parameter of the atmospheric refractive index $C_n^2(\xi)$.

Models of atmospheric turbulence

For numerical calculations, we use the models of the vertical profile of the refractive index structure parameter[7–15] and calculate the coherence radii for the plane and spherical waves under conditions of vertical propagation (in this case, the variable ξ corresponds to propagation along the vertical). Compare phase fluctuations in the plane and spherical waves and calculate the following integrals:

$$D_s^{\rm pl}(\rho_1, \rho_2) = 2.82k^2|\rho_1-\rho_2|^{5/3}\int\limits_0^x d\xi C_n^2(\xi) =$$

$$= 6.88|\rho_1-\rho_2|^{5/3}(r_0^{\rm pl})^{-5/3}, \qquad (26)$$

$$D_s^{\rm sph}(\rho_1, \rho_2) = 2.82k^2|\rho_1-\rho_2|^{5/3}\int\limits_0^x d\xi C_n^2(\xi)(1-\xi/x)^{5/3} =$$

$$= 6.88|\rho_1-\rho_2|^{5/3}(r_0^{\rm sph})^{-5/3}, \qquad (27)$$

as well as introduce the following designations:

$$r_0^{\rm pl} = \{0.41k^2\int\limits_0^x d\xi C_n^2(\xi)\}^{-3/5}$$

for the coherence radius in the plane wave;

$$r_0^{\rm sph} = \{0.41k^2\int\limits_0^x d\xi (1-\xi/x)^{5/3} C_n^2(\xi)\}^{-3/5}$$

for the coherence radius in the spherical wave.

Calculating the ratio of the correlation radii for plane and spherical waves, we obtain that the coherence radius in the spherical wave is higher than in the plane wave:

$$\frac{r_0^{\rm sph}}{r_0^{\rm pl}} = \left\{\frac{\int\limits_0^\infty d\xi C_n^2(\xi)}{\int\limits_0^x d\xi(1-\xi/x)^{5/3}C_n^2(\xi)}\right\}^{3/5}. \qquad (28)$$

The calculated coherence radii (for a height x =100 km) are summarized in Table 1.

Table 1

Model	$r_0^{\rm pl}$, cm	$r_0^{\rm sph}$, cm
Profile of Geophysical Laboratory of USAF	20.80	22.90
Model for the Cerro Paranal Observatory	13.16	14.80
Hufnagel–Stanley profile of turbulence	5.01	8.10
Modified Hufnagel–Stanley profile	8.03	18.70
Greenwood turbulence profile	12.92	13.10
Profile for nighttime atmospheric conditions	19.91	21.97

The data of Table 1 allow us to estimate the size of the coherent area or the allowable size of the subaperture, as well as the Strehl parameter in the telescope without correction. These results coincide with the earlier obtained ones.[5,16–18] Table 2 summarizes numerically calculated sizes of the increased coherent part of the telescope aperture due to action of adaptive correction. The calculations were conducted by Eq. (24) for three most widely used models for three heights of LGS formation: 20, 40, and 100 km.

Table 2

x, km	Greenwood model	Modified Hufnagel–Stanley profile	US profile for nighttime conditions
20	6.08	5.19	7.07
40	11.32	10.15	13.74
100	27.74	25.82	27.42

These calculations show that the increase in the telescope aperture coherent part for different models of the vertical profile of the refractive index structural parameter for a height x = 100 km ranges from 25 to 27 times. Thus, if the coherence radius for the plane wave is equal, say, to 20 cm, then the traditional correction increases the size of the telescope aperture coherent part roughly to 5 m. As a result, it can be stated that the traditional correction with the use of a single point guide star significantly increases the telescope efficiency, but for rather large telescopes (larger than 10 m) the traditional scheme fails to ensure the complete correction.

In addition, as can be seen from Table 1, there are serious differences in coherence radii for the plane and spherical waves. The wavefront sensor[3,4] employs just these differences in the coherence radii. In this sensor, the subaperture size is equal to the coherence radius for spherical waves, which is somewhat larger than that for a plane wave.

Correction with a guide star being a wide collimated beam

Let us calculate the Strehl parameter for the telescope operating through the turbulent atmosphere with correction based on a wide collimated beam. The use of a specialized wavefront sensor is assumed. Using Eq. (5), we can write the mean intensity of the corrected field in the following form:

$$\langle I(-f,\rho)\rangle = \sum_{l=1}^{N^2}\sum_{j=1}^{N^2}\iint d^2\rho_1 d^2\rho_2 G_0(0,\rho_1;-f,\rho)\times$$

$$\times G_0^*(0,\rho_2;-f,\rho)\exp(-(\rho_1-\rho_j)^2/2R^2)\times$$

$$\times\exp(-(\rho_2-\rho_l)^2/2R^2)\exp(-ik\rho_1^2/2f+ik\rho_2^2/2f)\times$$

$$\times\langle\exp\{i[S_{\rm pl}(\rho_1)-S_{\rm pl}(\rho_2)]-$$

$$-i[S_{\rm sph}(\rho_1;\rho_j)-S_{\rm sph}(\rho_2;\rho_l)]\}\rangle, \qquad (29)$$

where ρ_j, ρ_l $(j, l = 1, \ldots, N^2)$ are coordinates of sources of spherical waves. Repeating the operation of averaging as earlier, we have

$$<[\Delta_j(\rho_1) - \Delta_l(\rho_2)]^2> =$$
$$= -8\pi^2 0.033k^2 \int_0^x d\xi C_n^2(x-\xi) \frac{\Gamma(-5/6)}{2^{8/3}\Gamma(11/6)} \times$$
$$\times[|\rho_1-\rho_2|^{5/3} + |(1-\xi/x)(\rho_1-\rho_2) + (\xi/x)(\rho_j-\rho_l)|^{5/3} +$$
$$+(\xi/x)^{5/3}|\rho_1-\rho_j|^{5/3} + (\xi/x)^{5/3}|\rho_2-\rho_l|^{5/3} -$$
$$-|\rho_j(\xi/x) - \rho_2 + \rho_1(1-\xi/x)|^{5/3} -$$
$$-|\rho_l(\xi/x) - \rho_1 + \rho_2(1-\xi/x)|^{5/3}]. \quad (30)$$

Then, using the quadratic approximation in place of the 5/3 dependence for the term in the square brackets in Eq. (28), obtain

$$[\ldots] \approx [|\rho_1-\rho_2|^2 + |(1-\xi/x)(\rho_1-\rho_2) + (\xi/x)(\rho_j-\rho_l)|^2 +$$
$$+(\xi/x)^2|\rho_1-\rho_j|^2 + (\xi/x)^2|\rho_2-\rho_l|^2 -$$
$$-|\rho_j(\xi/x) - \rho_2 + \rho_1(1-\xi/x)|^2 -$$
$$-|\rho_l(\xi/x) - \rho_1 + \rho_2(1-\xi/x)|^2] =$$
$$= (\xi/x)^2[(\rho_1-\rho_2) - (\rho_j-\rho_l)]^2.$$

As a result, write for the corrected mean intensity

$$<I(-f,\rho)> = \frac{\exp(-ik\rho(\rho_j-\rho_l)/f)}{f^2} \times$$
$$\times\sum_{l=1}^{N^2}\sum_{j=1}^{N^2} \iint d^2\rho_1 d^2\rho_2 \exp(-\rho_1^2/2d^2) \times$$
$$\times\exp(-(\rho_2^2/2d^2)\exp[-ik\rho(\rho_1-\rho_2)/f -$$
$$-1{,}41k^2 \int_0^x d\xi C_n^2(\xi)(\xi/x)^2|(\rho_1-\rho_2)|^2]. \quad (31)$$

Note that for off-axis points $(\rho \neq 0)$, an oscillating factor of the form $\exp[-ik\rho_j(\rho_j - \rho_l)/f]$ appears in Eq. (31) for terms with $j \neq l$ in the integrand. Therefore, these terms are strongly suppressed (similarly to N^{-2}, where N is the dimension of the subaperture array). However, for the system axis ($\rho = 0$) Eq. (31) transforms into

$$<I(-f,0)> = \frac{N^4}{f^2} \iint d^2\rho_1 d^2\rho_2 \exp(-\rho_1^2/2d^2)\exp(-\rho_2^2/2d^2) \times$$
$$\times\exp[-ik\rho(\rho_1-\rho_2)/f - 1.41k^2 \int_0^x d\xi C_n^2(\xi)(\xi/x)^2|(\rho_1-\rho_2)|^2]. \quad (32)$$

Equation (32) fully coincides with Eq. (23) for the mean intensity distribution when correcting by the traditional scheme, but in the last equation the integration is performed not over the whole entrance aperture of the telescope, but over the subaperture area, whose size is equal to $2R/N = d \approx r_0^{\text{sph}}$, and then almost the whole receiving aperture of the telescope becomes coherent. As a result of calculations for adaptive correction with the use of a wide collimated beam, the Strehl parameter of such a telescope is equal to

$$\text{St} = [1 + R^2 N^{-2} (\tilde{r}_0^{\text{pl}})^{-2}]^{-1}. \quad (33)$$

Thus, increasing the number of subapertures N of the initial telescope aperture, it is possible to make the Strehl parameter arbitrary infinitely close to unity for any telescope, that is, almost any aperture can be made coherent.

Conclusions

Let us summarize the results of our calculations and reduce them to simple equations. Take into account the use of quadratic approximation in calculations, that is, the change of the 5/3 law by the 2 law. Now perform the inverse substitution. Then from Eq. (10) we obtain that the Strehl parameter for the telescope without correction is

$$\text{St} \approx \left[1 + 4\pi^2 \frac{\int_0^\infty d\xi C_n^2(\xi)(2R)^{-1/3}}{(\lambda/2R)^2} \right]^{-1}. \quad (34)$$

In the system with correction, which employs traditional focused LGS, the transformation of Eq. (25) yields

$$\text{St} \approx \left[1 + 4\pi^2 \frac{\int_0^x d\xi C_n^2(\xi)(\xi/x)^2(2R)^{-1/3}}{(\lambda/2R)^2} \right]^{-1}. \quad (35)$$

Finally, at correction with the collimated beam as LGS, using a special wavefront sensor, from Eq. (33) we obtain

$$\text{St} \approx \left[1 + 4\pi^2 \frac{\int_0^x d\xi C_n^2(\xi)(\xi/x)^2(2R)^{-1/3}}{N^{5/3}(\lambda/2R)^2} \right]^{-1}. \quad (36)$$

The results of analytical and numerical calculations have shown a high efficiency of application of a laser guide star in the form of a wide collimated beam. The specialized sensor of the Hartman type[4] allows the reference wave phase to be reconstructed as a continuous function. The estimates show that the resulting field of the guide source is rather close to a plane wave in its parameters. Therefore, we obtain a high correction and a significant increase of the Strehl parameter, which indicates indirectly a good correction of higher modal components, which are poorly corrected with the use of traditional schemes of LGS formation with a focused laser beam. The comparative calculations for different models of the vertical profile of the structure parameter of refractive index have shown serious differences in the behavior of correlation radii for plane and spherical waves.

It should be noted that accounting for the influence of amplitude and phase fluctuations naturally decreases the achievable level of correction. The resulting Strehl parameter will be somewhat lower than that determined by Eqs. (35) and (36).

One more feature of the proposed scheme of LGS formation should be noted. The wide collimated beam and the specialized wavefront sensor,[3,4] the each subaperture of which sees only a limited area of LGS, erase the problem[16–18] of correction of the global wavefront tilt with the use of LGS, since the jitter of the initial beam, caused by the upward propagation from the telescope aperture, does not contribute to the jitter of subaperture images. Therefore, the summation of local wavefront tilts over the whole subaperture array of the wavefront sensor can also give a signal for correction of the global wavefront tilt. This somewhat facilitates, in general, rather complicated scheme of correction with the use of LGS, since it is not necessary to use of not only the laser guide star, but also a natural star, which gives a signal for correction of the global wavefront tilt.

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