# THE THEORY OF THE VECTOR OPTICAL TRANSFER OPERATOR

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*Methods of the influence functions (IFs) and the spatial-frequency characteristics (SFCs) are used to solve the vector boundary-value problem of the theory of polarized radiation transfer in a three-dimensional plane layer. Generalized solution is first obtained for the calculation of the parameter of the Stokes vector with the help of the vector IFs (VIFs) or the vector SFCs (VSFCs). The vector optical transfer operators (VOTOs) are constructed whose kernels are tensors of IFs (TIFs) or SFCs (TSFCs). Basic models for calculation of VIFs and VSFCs are described.*

## INTRODUCTION

On the basis of the scalar kinetic equations of the radiative transfer theory, basic models of the influence functions (IFs), spatial-frequency characteristics (SFCs), and optical transfer operator (OTO) of the system atmosphere – underlying surface have been formulated in our works[1–3] by rigorous methods of the perturbation theory and the theory of fundamental solutions and theoretical calculational investigations into the propagation of the optical and millimeter (MM) waves[4–7] with the consideration of the polarization and depolarization mechanisms in the scattering, absorbing, and radiating Earth's atmosphere, including hydrometeors (cloudiness, rain, fog, and so on) with various sources of insulation and reflecting boundary characteristics, the vector optical transfer operator[8] (VOTO) is constructed in the present paper. Particular cases of the VOTO were considered in our previous papers.[1,9,10]

In the last few years, investigations of the MM wave range (MMWR) for which the quasioptical approximation[8] is true, have been busted. The term “millimeter waves” is the contemporary of radio. MMWR came into use only some years after Hertz carried out a series of experiments (1886–1888) and discovered radio waves. Hertz established the wave nature of propagation of the electromagnetic energy, thereby confirming the Maxwell theoretical conclusions. He proved that radio wave reflection and refraction laws are true and studied their polarization nature.

Because of increasing demands for communication channels, the working frequencies of ground-based and satellite systems constantly increase and communication technique is complicated due to the use of digital devices and orthogonally polarized channels. This progress calls for the development of our knowledge about interaction between hydrometeors and radiation; in so doing, the knowledge should be more detailed for extinction calculation. On the basis of information about scattering and absorbing properties and emissivity of hydrometeors, it is necessary to investigate various effects of multiple and incoherent scattering on the polarized and cross-polarized components, effects of scattering on noise and transfer characteristics of channels, and so on. Polarization properties of signals in combination with multifrequency measurements are used in the method of radar meteorology to measure the intensity of precipitation and to investigate the phase state, shape, size distribution, and motion of precipitation particles in clouds. The main purpose of the review done in Ref. 6 is theoretical analysis of the problem. Background radio wave radiation and, according to the Kirchhoff law, absorption of the atmosphere interfer with the reception of signals transmitted through the atmosphere at frequencies above 10 GHz. To consider the effect of the troposphere on the MMW propagation and to develop methods of compensation for background radiation it is necessary to investigate spectral polarization and spatial-angular characteristics of the atmospheric radio wave radiation.

Study of properties of the underlying surface[4] is urgent for the design of radio engineering communication systems, detection and ranging, and remote sensing. In the process of propagation of MMW above the snow, ice, or plant cover, sandy or water surface, asphalt or concrete cover, and so on the field fluctuations and interferences are introduced.

Electromagnetic fields (EMFs) and vital activity of organisms are interrelated.[7] The development of human civilization, due to advances in electronics and radio engineering, creates significant additional electromagnetic background radiation of anthropogenic origin in which all organisms of the Earth should live. Considerable recent attention has been focused on biophysical effects of low-intensity MMWs between 30 and 300 GHz corresponding to the wavelengths in air between 10 and 1 mm.

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Natural MMW sources with significant intensities are absent. Artificial sources appeared only ~40 years ago. "Millimeter" problem is only a part of the global problem on the impact of weak and ultraweak EMFs on living organisms. Strong interactions, by essence, are the energetic impacts and any biological effects are produced exclusively by heating of an object, that is, they are thermal. Hypotheses are set up that weak impacts may affect not only the regulating functions of an organism, but also its biological protection systems.

## PROBLEM FORMULATION

A problem of transfer of optical or millimeter radiation with wavelength $\lambda$ through a plane layer infinite in horizontal direction ($-\infty < x,\ y < \infty$) and having finite height ($0 \le z \le H$) is considered in three-dimensional Euclidean space; radius-vector $r = (x, y, z)$ has the projection $r_\perp = (x, y)$ onto the horizontal plane. The system atmosphere − underlying surface at the level $z = H$ is considered nonmultiplicative (without multiplication). A set of all directions $s = (\mu, \varphi)$, where $\mu = \cos\Theta \in [-1, 1]$, $\Theta \in [0, \pi]$ is the zenith angle measured from the internal normal to the upper boundary of the layer $z = 0$, which coincides with the $z$ axis, and $\varphi \in [0, 2\pi]$ is the azimuth angle measured from the positive direction of the $x$ axis, forms the unit sphere $\Omega = \Omega^+ \cup \Omega^-$. Here, $\Omega^+$ and $\Omega^-$ are the semispheres for the directions of propagation of downwelling transmitted radiation ($\mu \ge 0$) and upwelling reflected radiation ($\mu \le 0$), respectively. For convenient presentation of the boundary conditions, we introduce the set

$$t = \{z, r_\perp, s\colon z = 0,\ s \in \Omega^+\},\quad b = \{z, r_\perp, s\colon z = H,\ s \in \Omega^-\}.$$

Under assumption of stationary state of the medium and constant insolation source power, the field of quasimonochromatic polarized radiation is most adequately described by the four-component vector $\mathbf{\Phi}(r, s)$, whose components are the Stokes parameters. By the common name of the Stokes parameters, a wide variety of interrelated quantities describing the radiation polarization state[1,11–14] is meant. For macroscopic isotropic and plane stratified medium, the Stokes vector is determined as a solution to the general vector boundary-value problem (GVBVP) of the transfer theory

$$K\mathbf{\Phi} = \mathbf{F}(z, s),\quad \mathbf{\Phi}|_t = \mathbf{F}^0(r_\perp, s),$$
$$\mathbf{\Phi}|_b = \varepsilon R\mathbf{\Phi} + \mathbf{F}^H(r_\perp, s) \qquad (1)$$

with the following linear operators:
the transfer operator

$$D \equiv (s, \mathrm{grad}) + \sigma(z) = D_z + \left(s_\perp, \frac{\partial}{\partial r_\perp}\right),$$

$$D_z \equiv \mu \frac{\partial}{\partial z} + \sigma(z);$$

the integral of collisions

$$S\mathbf{\Phi} \equiv \sigma_s(z) \int_\Omega P(z, s, s')\, \mathbf{\Phi}(z, r_\perp, s')\, \mathrm{d}s';$$

the reflection operator

$$[R\mathbf{\Phi}](H, r_\perp, s) \equiv \int_{\Omega^+} q(r_\perp, s, s^+)\, \mathbf{\Phi}(H, r_\perp, s^+)\, \mathrm{d}s^+ \qquad (2)$$

which describes a single event of radiation interaction with the underlying surface; the parameter $0 \le \varepsilon \le 1$ indicates the event of radiation interaction with the boundary $z = H$; $q(r_\perp, s, s^+)$ is the reflection phase matrix; the integrodifferential operator $K \equiv D - S$; the one-dimensional operator $K_z = D_z - S$; $P(z, s, s')$ is the scattering phase matrix[1]; $\sigma(z)$ and $\sigma_s(z)$ are the vertical profiles of the extinction and scattering coefficients; $\mathbf{F}$, $\mathbf{F}^0$, and $\mathbf{F}^H$ are the insolation sources.

Boundary problem (1) is linear and we seek for its solution in the form of the superposition

$$\mathbf{\Phi}(z, r_\perp, s) = \mathbf{\Phi}_0 + \mathbf{\Phi}_R;\quad \mathbf{\Phi}_0 = \mathbf{\Phi}_u^b + \mathbf{\Phi}_u^0 + \mathbf{\Phi}_u^H + \mathbf{\Phi}_a,$$

where $\mathbf{\Phi}_0$ is the background radiation in the layer and $\mathbf{\Phi}_R$ is the illumination produced by the reflecting boundary.

Directly transmitted radiation from the source on the boundary $z = 0$ is the solution of the Couchy vector problem (CVP)

$$D\mathbf{\Phi}_u^0 = 0,\quad \mathbf{\Phi}_u^0|_t = \mathbf{F}^0(r_\perp, s),\quad \mathbf{\Phi}_u^0|_b = 0;\quad s \in \Omega^+. \qquad (3)$$

Directly transmitted radiation from the source on the boundary $z = H$ is the solution of the CVP

$$D\mathbf{\Phi}_u^H = 0,\quad \mathbf{\Phi}_u^H|_t = 0,\quad \mathbf{\Phi}_u^H|_b = \mathbf{F}^H(r_\perp, s);\quad s \in \Omega^-. \qquad (4)$$

Directly transmitted radiation from internal sources is the solution of the CVP

$$D_z\mathbf{\Phi}_u^b = \mathbf{F},\quad \mathbf{\Phi}_u^b|_t = 0,\quad \mathbf{\Phi}_u^b|_b = 0;\quad s \in \Omega. \qquad (5)$$

Multiply scattered background radiation is the solution of the problem with the vacuum boundary conditions

$$K\mathbf{\Phi}_a = S\mathbf{\Phi}_u^b + S\mathbf{\Phi}_u^0 + S\mathbf{\Phi}_u^H,\ \mathbf{\Phi}_a|_t = 0,\ \mathbf{\Phi}_a|_b = 0.$$

It includes three components: $\mathbf{\Phi}_a = \mathbf{\Phi}_a^b + \mathbf{\Phi}_a^0 + \mathbf{\Phi}_a^H$ that are the solutions of the following first vector boundary-value problems (FVBVPs):

$$K_z\mathbf{\Phi}_a^b = S\mathbf{\Phi}_u^b,\ \mathbf{\Phi}_a^b|_t = 0,\ \mathbf{\Phi}_a^b|_b = 0, \qquad (6)$$

$$K\mathbf{\Phi}_a^0 = S\mathbf{\Phi}_u^0,\ \mathbf{\Phi}_a^0|_t = 0,\ \mathbf{\Phi}_a^0|_b = 0, \qquad (7)$$

$$K\mathbf{\Phi}_a^H = S\mathbf{\Phi}_u^H,\ \mathbf{\Phi}_a^H|_t = 0,\ \mathbf{\Phi}_a^H|_b = 0. \qquad (8)$$

The background radiation in the layer with transparent or absolutely black (nonreflecting) boundaries is the solution of the FVBVP

$$K\mathbf{\Phi}_0 = \mathbf{F}, \quad \mathbf{\Phi}_0|_t = \mathbf{F}^0, \quad \mathbf{\Phi}_0|_b = \mathbf{F}^{\mathrm{H}}. \tag{9}$$

It includes three background components: $\mathbf{\Phi}_0 = \mathbf{\Phi}_0^{\mathrm{b}} + \mathbf{\Phi}_0^0 + \mathbf{\Phi}_0^{\mathrm{H}}$, each can be calculated individually. If problem (9) is one-dimensional (horizontally homogeneous), it can be solved numerically[1] for the total background $\mathbf{\Phi}_0$ or for each component $\mathbf{\Phi}_0^{\mathrm{b}}$, $\mathbf{\Phi}_0^0$, or $\mathbf{\Phi}_0^{\mathrm{H}}$ of practical or methodical interest. In case of using the IF method for horizontally homogeneous problem (9) or the IF or SFC method for three-dimensional horizontally homogeneous plane layer with horizontal inhomogeneity of either $\mathbf{F}^0$ or $\mathbf{F}^{\mathrm{H}}$, or $\mathbf{F}^0$ and $\mathbf{F}^{\mathrm{H}}$ simultaneously, problem (9) must be solved individually for the components $\mathbf{\Phi}_0^{\mathrm{b}}$, $\mathbf{\Phi}_0^0$, and $\mathbf{\Phi}_0^{\mathrm{H}}$, respectively.

Let us consider the background radiation of the layer $\mathbf{\Phi}_0^0 = \mathbf{\Phi}_{\mathrm{u}}^0 + \mathbf{\Phi}_{\mathrm{a}}^0$ – the solution of the FVBVP

$$K\mathbf{\Phi}_0^0 = 0, \quad \mathbf{\Phi}_0^0|_t = \mathbf{F}^0(r_\perp, s), \quad \mathbf{\Phi}_0^0|_b = 0, \tag{10}$$

which comprises directly transmitted radiation $\mathbf{\Phi}_{\mathrm{u}}^0$ from the source – the solution of CVP (3) – and multiply scattered radiation $\mathbf{\Phi}_{\mathrm{a}}^0$ in the layer with transparent boundaries – the solution of FVBVP (7).

The background radiation of the layer $\mathbf{\Phi}_0^{\mathrm{H}} = \mathbf{\Phi}_{\mathrm{u}}^{\mathrm{H}} + \mathbf{\Phi}_{\mathrm{a}}^{\mathrm{H}}$ is the solution of the FVBVP

$$K\mathbf{\Phi}_0^{\mathrm{H}} = 0, \quad \mathbf{\Phi}_0^{\mathrm{H}}|_t = 0 \quad \mathbf{\Phi}_0^{\mathrm{H}}|_b = \mathbf{F}^{\mathrm{H}}(r_\perp, s). \tag{11}$$

It comprises directly transmitted radiation $\mathbf{\Phi}_{\mathrm{u}}^{\mathrm{H}}$ from the source – the solution of CVP (4) – and multiply scattered radiation $\mathbf{\Phi}_{\mathrm{a}}^{\mathrm{H}}$ in the layer with nonreflecting boundaries – the solution of FVBVP (8).

The background radiation $\mathbf{\Phi}_0^{\mathrm{b}} = \mathbf{\Phi}_{\mathrm{u}}^{\mathrm{b}} + \mathbf{\Phi}_{\mathrm{a}}^{\mathrm{b}}$ is the solution of one-dimensional FVBVP

$$K_z\mathbf{\Phi}_0^{\mathrm{b}} = \mathbf{F}, \quad \mathbf{\Phi}_0^{\mathrm{b}}|_t = 0, \quad \mathbf{\Phi}_0^{\mathrm{b}}|_b = 0. \tag{12}$$

It comprises directly transmitted radiation $\mathbf{\Phi}_{\mathrm{u}}^{\mathrm{b}}$ from the internal sources – the solution of CVP (5) – and multiply scattered radiation $\mathbf{\Phi}_{\mathrm{a}}^{\mathrm{b}}$ in the layer with vacuum boundaries – the solution of FVBVP (6).

The problem for the adjacent illumination is the general vector boundary-value problem

$$K\mathbf{\Phi}_{\mathrm{R}} = 0, \quad \mathbf{\Phi}_{\mathrm{R}}|_t = 0, \quad \mathbf{\Phi}_{\mathrm{R}}|_b = \varepsilon R\mathbf{\Phi}_{\mathrm{R}} + \varepsilon\mathbf{E}(r_\perp, s), \tag{13}$$

where $\mathbf{E}(r_\perp, s) = R\mathbf{\Phi}_0$ is the brightness (illuminance, irradiance) of the boundary produced by the background radiation. It comprises four components of the source:

$$\mathbf{E}(r_\perp, s) = \mathbf{E}_{\mathrm{u}}^{\mathrm{b}} + \mathbf{E}_{\mathrm{u}}^0 + \mathbf{E}_{\mathrm{u}}^{\mathrm{H}} + \mathbf{E}_{\mathrm{a}}, \quad \mathbf{E}_{\mathrm{u}}^{\mathrm{b}} = R\mathbf{\Phi}_{\mathrm{u}}^{\mathrm{b}}, \quad \mathbf{E}_{\mathrm{u}}^0 = R\mathbf{\Phi}_{\mathrm{u}}^0,$$

$$\mathbf{E}_{\mathrm{u}}^{\mathrm{H}} = R\mathbf{\Phi}_{\mathrm{u}}^{\mathrm{H}}, \ \mathbf{E}_{\mathrm{a}} = R\mathbf{\Phi}_{\mathrm{a}},$$

that produce four components of illumination

$$\mathbf{\Phi}_{\mathrm{R}} = \mathbf{\Phi}_{\mathrm{Ru}}^{\mathrm{b}} + \mathbf{\Phi}_{\mathrm{Ru}}^0 + \mathbf{\Phi}_{\mathrm{Ru}}^{\mathrm{H}} + \mathbf{\Phi}_{\mathrm{Ra}}$$

that are the solutions of the following general vector boundary-value problems:

$$K\mathbf{\Phi}_{\mathrm{Ru}}^{\mathrm{b}} = 0, \ \mathbf{\Phi}_{\mathrm{Ru}}^{\mathrm{b}}|_t = 0, \ \mathbf{\Phi}_{\mathrm{Ru}}^{\mathrm{b}}|_b = \varepsilon R\mathbf{\Phi}_{\mathrm{Ru}}^{\mathrm{b}} + \varepsilon\mathbf{E}_{\mathrm{u}}^{\mathrm{b}}, \tag{14}$$

$$K\mathbf{\Phi}_{\mathrm{Ru}}^0 = 0, \ \mathbf{\Phi}_{\mathrm{Ru}}^0|_t = 0, \ \mathbf{\Phi}_{\mathrm{Ru}}^0|_b = \varepsilon R\mathbf{\Phi}_{\mathrm{Ru}}^0 + \varepsilon\mathbf{E}_{\mathrm{u}}^0, \tag{15}$$

$$K\mathbf{\Phi}_{\mathrm{Ru}}^{\mathrm{H}} = 0, \ \mathbf{\Phi}_{\mathrm{Ru}}^{\mathrm{H}}|_t = 0, \ \mathbf{\Phi}_{\mathrm{Ru}}^{\mathrm{H}}|_b = \varepsilon R\mathbf{\Phi}_{\mathrm{Ru}}^{\mathrm{H}} + \varepsilon\mathbf{E}_{\mathrm{u}}^{\mathrm{H}}, \tag{16}$$

$$K\mathbf{\Phi}_{\mathrm{Ra}} = 0, \ \mathbf{\Phi}_{\mathrm{Ra}}|_t = 0, \ \mathbf{\Phi}_{\mathrm{Ra}}|_b = \varepsilon R\mathbf{\Phi}_{\mathrm{Ra}} + \varepsilon\mathbf{E}_{\mathrm{a}}. \tag{17}$$

In the solutions of problems (14)–(17), we can consider pairwise components

$$\mathbf{\Phi}_{\mathrm{Ru}}^{\mathrm{b}} = \mathbf{\Phi}_{\mathrm{Ru0}}^{\mathrm{b}} + \mathbf{\Phi}_{\mathrm{Rus}}^{\mathrm{b}}, \ \mathbf{\Phi}_{\mathrm{Ru}}^0 = \mathbf{\Phi}_{\mathrm{Ru0}}^0 + \mathbf{\Phi}_{\mathrm{Rus}}^0,$$

$$\mathbf{\Phi}_{\mathrm{Ru}}^{\mathrm{H}} = \mathbf{\Phi}_{\mathrm{Ru0}}^{\mathrm{H}} + \mathbf{\Phi}_{\mathrm{Rus}}^{\mathrm{H}}, \ \mathbf{\Phi}_{\mathrm{Ra}} = \mathbf{\Phi}_{\mathrm{Ra0}} + \mathbf{\Phi}_{\mathrm{Ras}},$$

where $\mathbf{\Phi}_{\mathrm{Ru0}}^{\mathrm{b}}$, $\mathbf{\Phi}_{\mathrm{Ru0}}^0$, $\mathbf{\Phi}_{\mathrm{Ru0}}^{\mathrm{H}}$, and $\mathbf{\Phi}_{\mathrm{Ra0}}$ are the directly transmitted radiation components from the boundary $z = H$. They are the solutions of CVPS

$$D\mathbf{\Phi}_{\mathrm{Ru0}}^{\mathrm{b}} = 0, \ \mathbf{\Phi}_{\mathrm{Ru0}}^{\mathrm{b}}|_t = 0, \ \mathbf{\Phi}_{\mathrm{Ru0}}^{\mathrm{b}}|_b = \mathbf{E}_{\mathrm{u}}^{\mathrm{b}};$$

$$D\mathbf{\Phi}_{\mathrm{Ru0}}^0 = 0, \ \mathbf{\Phi}_{\mathrm{Ru0}}^0|_t = 0, \ \mathbf{\Phi}_{\mathrm{Ru0}}^0|_b = \mathbf{E}_{\mathrm{u}}^0;$$

$$D\mathbf{\Phi}_{\mathrm{Ru0}}^{\mathrm{H}} = 0, \ \mathbf{\Phi}_{\mathrm{Ru0}}^{\mathrm{H}}|_t = 0, \ \mathbf{\Phi}_{\mathrm{Ru0}}^{\mathrm{H}}|_b = \mathbf{E}_{\mathrm{u}}^{\mathrm{H}};$$

$$D\mathbf{\Phi}_{\mathrm{Ra0}} = 0, \quad \mathbf{\Phi}_{\mathrm{Ra0}}|_t = 0, \quad \mathbf{\Phi}_{\mathrm{Ra0}}|_b = \mathbf{E}_{\mathrm{a}} . \tag{18}$$

Here $\mathbf{\Phi}_{\mathrm{Rus}}^{\mathrm{b}}$, $\mathbf{\Phi}_{\mathrm{Rus}}^0$, $\mathbf{\Phi}_{\mathrm{Rus}}^{\mathrm{H}}$, and $\mathbf{\Phi}_{\mathrm{Ras}}$ are the components of radiation multiply scattered in the layer and re-reflected from the boundary $z = H$. They are the solutions of GVBVPs

$$K\mathbf{\Phi}_{\mathrm{Rus}}^{\mathrm{b}} = 0, \ \mathbf{\Phi}_{\mathrm{Rus}}^{\mathrm{b}}|_t = 0, \ \mathbf{\Phi}_{\mathrm{Rus}}^{\mathrm{b}}|_b = \varepsilon R\mathbf{\Phi}_{\mathrm{Rus}}^{\mathrm{b}} + \varepsilon\mathbf{E}_{\mathrm{Ru0}}^{\mathrm{b}}, \tag{19}$$

$$K\mathbf{\Phi}_{\mathrm{Rus}}^0 = 0, \ \mathbf{\Phi}_{\mathrm{Rus}}^0|_t = 0, \ \mathbf{\Phi}_{\mathrm{Rus}}^0|_b = \varepsilon R\mathbf{\Phi}_{\mathrm{Rus}}^0 + \varepsilon\mathbf{E}_{\mathrm{Ru0}}^0, \tag{20}$$

$$K\mathbf{\Phi}_{\mathrm{Rus}}^{\mathrm{H}} = 0, \ \mathbf{\Phi}_{\mathrm{Rus}}^{\mathrm{H}}|_t = 0, \ \mathbf{\Phi}_{\mathrm{Rus}}^{\mathrm{H}}|_b = \varepsilon R\mathbf{\Phi}_{\mathrm{Rus}}^{\mathrm{H}} + \varepsilon\mathbf{E}_{\mathrm{Ru0}}^{\mathrm{H}}, \tag{21}$$

$$K\mathbf{\Phi}_{\mathrm{Ras}} = 0, \ \mathbf{\Phi}_{\mathrm{Ras}}|_t = 0, \ \mathbf{\Phi}_{\mathrm{Ras}}|_b = \varepsilon R\mathbf{\Phi}_{\mathrm{Ras}} + \varepsilon\mathbf{E}_{\mathrm{Ra0}} \tag{22}$$

with the sources $\mathbf{E}_{\mathrm{Ru0}}^{\mathrm{b}} = R\mathbf{\Phi}_{\mathrm{Ru0}}^{\mathrm{b}}$, $\mathbf{E}_{\mathrm{Ru0}}^0 = R\mathbf{\Phi}_{\mathrm{Ru0}}^0$, $\mathbf{E}_{\mathrm{Ru0}}^{\mathrm{H}} = R\mathbf{\Phi}_{\mathrm{Ru0}}^{\mathrm{H}}$, and $\mathbf{E}_{\mathrm{Ra0}} = R\mathbf{\Phi}_{\mathrm{Ra0}}$.

In case of separation of contributions from multiple scattering to the background radiation, we arrive at separation of the brightness components from a substrate

$$\mathbf{E}_{\mathrm{a}} = \mathbf{E}_{\mathrm{a}}^{\mathrm{b}} + \mathbf{E}_{\mathrm{a}}^0 + \mathbf{E}_{\mathrm{a}}^{\mathrm{H}}, \ \mathbf{E}_{\mathrm{a}}^{\mathrm{b}} = R\mathbf{\Phi}_{\mathrm{a}}^{\mathrm{b}}, \ \mathbf{E}_{\mathrm{a}}^0 = R\mathbf{\Phi}_{\mathrm{a}}^0, \ \mathbf{E}_{\mathrm{a}}^{\mathrm{H}} = R\mathbf{\Phi}_{\mathrm{a}}^{\mathrm{H}},$$

and instead of Eq. (17), we obtain three GVBVPs for the components $\mathbf{\Phi}_{\mathrm{Ra}} = \mathbf{\Phi}_{\mathrm{Ra}}^{\mathrm{b}} + \mathbf{\Phi}_{\mathrm{Ra}}^0 + \mathbf{\Phi}_{\mathrm{Ra}}^{\mathrm{H}}$:

$$K\mathbf{\Phi}^{\rm b}_{\rm Ra} = 0,\ \ \mathbf{\Phi}^{\rm b}_{\rm Ra}|_t = 0,\ \ \mathbf{\Phi}^{\rm b}_{\rm Ra}|_b = \varepsilon R\mathbf{\Phi}^{\rm b}_{\rm Ra} + \varepsilon\mathbf{E}^{\rm b}_{\rm a}, \tag{23}$$

$$K\mathbf{\Phi}^{0}_{\rm Ra} = 0,\ \ \mathbf{\Phi}^{0}_{\rm Ra}|_t = 0,\ \ \mathbf{\Phi}^{0}_{\rm Ra}|_b = \varepsilon R\mathbf{\Phi}^{0}_{\rm Ra} + \varepsilon\mathbf{E}^{0}_{\rm a}, \tag{24}$$

$$K\mathbf{\Phi}^{\rm H}_{\rm Ra} = 0,\ \ \mathbf{\Phi}^{\rm H}_{\rm Ra}|_t = 0,\ \ \mathbf{\Phi}^{\rm H}_{\rm Ra}|_b = \varepsilon R\mathbf{\Phi}^{\rm H}_{\rm Ra} + \varepsilon\mathbf{E}^{\rm H}_{\rm a}. \tag{25}$$

If we consider the individual solution components of the initial general vector boundary-value problem[1] $\mathbf{\Phi} = \mathbf{\Phi}^{\rm b} + \mathbf{\Phi}^0 + \mathbf{\Phi}^{\rm H}$, corresponding to individual problems with the sources $\mathbf{F}$, $\mathbf{F}^0$, and $\mathbf{F}^{\rm H}$, respectively,

$$K\mathbf{\Phi}^{\rm b} = \mathbf{F},\ \ \mathbf{\Phi}^{\rm b}|_t = 0,\ \ \mathbf{\Phi}^{\rm b}|_b = \varepsilon R\mathbf{\Phi}^{\rm b}, \tag{26}$$

$$K\mathbf{\Phi}^{0} = 0,\ \ \mathbf{\Phi}^{0}|_t = \mathbf{F}^0,\ \ \mathbf{\Phi}^{0}|_b = \varepsilon R\mathbf{\Phi}^{0}, \tag{27}$$

$$K\mathbf{\Phi}^{\rm H} = \mathbf{F},\ \ \mathbf{\Phi}^{\rm H}|_t = 0,\ \ \mathbf{\Phi}^{\rm H}|_b = \varepsilon R\mathbf{\Phi}^{\rm H} + \mathbf{F}^{\rm H}, \tag{28}$$

the superpositions take place

$$\mathbf{\Phi}^{\rm b} = \mathbf{\Phi}^{\rm b}_0 + \mathbf{\Phi}^{\rm b}_{\rm R},\ \ \mathbf{\Phi}^{0} = \mathbf{\Phi}^{0}_0 + \mathbf{\Phi}^{0}_{\rm R},\ \ \mathbf{\Phi}^{\rm H} = \mathbf{\Phi}^{\rm H}_0 + \mathbf{\Phi}^{\rm H}_{\rm R},$$

where the background components $\mathbf{\Phi}^{\rm b}_0$, $\mathbf{\Phi}^{0}_0$, and $\mathbf{\Phi}^{\rm H}_0$ are determined by solving problems (12), (10), and (11) and corresponding illuminations $\mathbf{\Phi}^{\rm b}_{\rm R}$, $\mathbf{\Phi}^{0}_{\rm R}$, and $\mathbf{\Phi}^{\rm H}_{\rm R}$ satisfy the general vector boundary-value problems

$$K\mathbf{\Phi}^{\rm b}_{\rm R} = 0,\ \ \mathbf{\Phi}^{\rm b}_{\rm R}|_t = 0,\ \ \mathbf{\Phi}^{\rm b}_{\rm R}|_b = \varepsilon R\mathbf{\Phi}^{\rm b}_{\rm R} + \varepsilon\mathbf{E}^{\rm b}_0, \tag{29}$$

$$K\mathbf{\Phi}^{0}_{\rm R} = 0,\ \ \mathbf{\Phi}^{0}_{\rm R}|_t = 0,\ \ \mathbf{\Phi}^{0}_{\rm R}|_b = \varepsilon R\mathbf{\Phi}^{0}_{\rm R} + \varepsilon\mathbf{E}^{0}_0, \tag{30}$$

$$K\mathbf{\Phi}^{\rm H}_{\rm R} = 0,\ \ \mathbf{\Phi}^{\rm H}_{\rm R}|_t = 0,\ \ \mathbf{\Phi}^{\rm H}_{\rm R}|_b = \varepsilon R\mathbf{\Phi}^{\rm H}_{\rm R} + \varepsilon\mathbf{E}^{\rm H}_0 \tag{31}$$

with the sources $\mathbf{E}^{\rm b}_0 = R\mathbf{\Phi}^{\rm b}_0$, $\mathbf{E}^{0}_0 = R\mathbf{\Phi}^{0}_0$, and $\mathbf{E}^{\rm H}_0 = R\mathbf{\Phi}^{\rm H}_0$. It is evident that the sources in problems (13), (17), and (18) are in essence the superpositions

$$\mathbf{E} = \mathbf{E}^{\rm b}_0 + \mathbf{E}^{0}_0 + \mathbf{E}^{\rm H}_0,\ \ \mathbf{E}^{\rm b}_0 = \mathbf{E}^{\rm b}_{\rm u} + \mathbf{E}^{\rm b}_{\rm a},\ \ \mathbf{E}^{0}_0 = \mathbf{E}^{0}_{\rm u} + \mathbf{E}^{0}_{\rm a},$$

$$\mathbf{E}^{\rm H}_0 = \mathbf{E}^{\rm H}_{\rm u} + \mathbf{E}^{\rm H}_{\rm a},\ \ \mathbf{E}_{\rm a} = \mathbf{E}^{\rm b}_{\rm a} + \mathbf{E}^{0}_{\rm a} + \mathbf{E}^{\rm H}_{\rm a},\ \ \mathbf{E}^{\rm b}_{\rm a} = R\mathbf{\Phi}^{\rm b}_{\rm a},$$

$$\mathbf{E}^{0}_{\rm a} = R\mathbf{\Phi}^{0}_{\rm a},\ \ \mathbf{E}^{\rm H}_{\rm a} = R\mathbf{\Phi}^{\rm H}_{\rm a}.$$

If in the solutions of problems (29)–(31) we consider the components for the directly transmitted, unscattered, and multiply scattered radiation

$$\mathbf{\Phi}^{\rm b}_{\rm R} = \mathbf{\Phi}^{\rm b}_{\rm R0} + \mathbf{\Phi}^{\rm b}_{\rm Rs},\ \ \mathbf{\Phi}^{0}_{\rm R} = \mathbf{\Phi}^{0}_{\rm R0} + \mathbf{\Phi}^{0}_{\rm Rs},\ \ \mathbf{\Phi}^{\rm H}_{\rm R} = \mathbf{\Phi}^{\rm H}_{\rm R0} + \mathbf{\Phi}^{\rm H}_{\rm Rs},$$

$$\mathbf{\Phi}^{\rm b}_{\rm R0} = \mathbf{\Phi}^{\rm b}_{\rm Ru0} + \mathbf{\Phi}^{\rm b}_{\rm Ra},\ \ \mathbf{\Phi}^{0}_{\rm R0} = \mathbf{\Phi}^{0}_{\rm Ru0} + \mathbf{\Phi}^{0}_{\rm Ra},$$

$$\mathbf{\Phi}^{\rm H}_{\rm R0} = \mathbf{\Phi}^{\rm H}_{\rm Ru0} + \mathbf{\Phi}^{\rm H}_{\rm Ra},$$

we will obtain the following set of problems for their determination:

$$D\mathbf{\Phi}^{\rm b}_{\rm R0} = 0,\ \ \mathbf{\Phi}^{\rm b}_{\rm R0}|_t = 0,\ \ \mathbf{\Phi}^{\rm b}_{\rm R0}|_b = \mathbf{E}^{\rm b}_0; \tag{32}$$

$$K\mathbf{\Phi}^{\rm b}_{\rm Rs} = 0,\ \ \mathbf{\Phi}^{\rm b}_{\rm Rs}|_t = 0,\ \ \mathbf{\Phi}^{\rm b}_{\rm Rs}|_b = \varepsilon R\mathbf{\Phi}^{\rm b}_{\rm Rs} + \varepsilon\mathbf{E}^{\rm b}_{\rm R0}; \tag{33}$$

$$D\mathbf{\Phi}^{0}_{\rm R0} = 0,\ \ \mathbf{\Phi}^{0}_{\rm R0}|_t = 0,\ \ \mathbf{\Phi}^{0}_{\rm R0}|_b = \mathbf{E}^{0}_0; \tag{34}$$

$$K\mathbf{\Phi}^{0}_{\rm Rs} = 0,\ \ \mathbf{\Phi}^{0}_{\rm Rs}|_t = 0,\ \ \mathbf{\Phi}^{0}_{\rm Rs}|_b = \varepsilon R\mathbf{\Phi}^{0}_{\rm Rs} + \varepsilon\mathbf{E}^{0}_{\rm R0}; \tag{35}$$

$$D\mathbf{\Phi}^{\rm H}_{\rm R0} = 0,\ \ \mathbf{\Phi}^{\rm H}_{\rm R0}|_t = 0,\ \ \mathbf{\Phi}^{\rm H}_{\rm R0}|_b = \mathbf{E}^{\rm H}_0; \tag{36}$$

$$K\mathbf{\Phi}^{\rm H}_{\rm Rs} = 0,\ \ \mathbf{\Phi}^{\rm H}_{\rm Rs}|_t = 0,\ \ \mathbf{\Phi}^{\rm H}_{\rm Rs}|_b = \varepsilon R\mathbf{\Phi}^{\rm H}_{\rm Rs} + \varepsilon\mathbf{E}^{\rm H}_{\rm R0}, \tag{37}$$

where $\mathbf{E}^{\rm b}_{\rm R0} = R\mathbf{\Phi}^{\rm b}_{\rm R0}$, $\mathbf{E}^{0}_{\rm R0} = R\mathbf{\Phi}^{0}_{\rm R0}$, and $\mathbf{E}^{\rm H}_{\rm R0} = R\mathbf{\Phi}^{\rm H}_{\rm R0}$.

Boundary-value problems (14)–(17), (19)–(25), (26)–(31), (33), (35), and (37) with the sources on the boundary $z = H$, describing different approximations for the illumination calculation in the layer due to the influence of the reflecting boundary, are the particular cases of general vector boundary-value problem (13).

The parameter ε (sometimes we assume that ε = 1) is introduced to specify the form of the dependence of the solution of problem (13) on the characteristics of the reflection operator $R$. Because the source $E(r_\perp, s)$ is usually determined as a contribution from a single reflection of the background radiation, the power of the parameter ε corresponds to the multiplicity of the radiation interaction with the boundary.

Let us introduce a parametric series (perturbation series) in the multiplicity of the radiation reflection from the underlying surface

$$\mathbf{\Phi}_{\rm R}(z, r_\perp, s) = \sum_{k=1}^{\infty} \varepsilon^k\, \mathbf{\Phi}_k(z, r_\perp, s), \tag{38}$$

with the terms that satisfy the system of radiative transfer equations connected with the recursion relations

$$k = 1:\ \ K\mathbf{\Phi}_1 = 0,\ \ \mathbf{\Phi}_1|_t = 0,\ \ \mathbf{\Phi}_1|_b = \mathbf{E}(r_\perp, s), \tag{39}$$

$$k \ge 2:\ \ K\mathbf{\Phi}_k = 0,\ \ \mathbf{\Phi}_k|_t = 0,\ \ \mathbf{\Phi}_k|_b = R\mathbf{\Phi}_{k-1}(H, r_\perp, s). \tag{40}$$

For application of the finite-difference methods, special measures for limitation of the layer extension in the horizontal plane are necessary. Most effective and natural is the approach based on the determination of the influence functions by the Fourier transform method.[1, 15] In so doing, three-dimensional problem is reduced to a problem for one-dimensional layer of finite height. Simultaneously, factorization of the solution takes place: the coordinate $r_\perp$ is substituted by the parameter $p$ and the problem for the SFC is solved for the fixed values of this parameter.

## SOLUTIONS OF THE FIRST VECTOR BOUNDARY-VALUE PROBLEMS BY THE VIF AND VSFC METHODS

Each problem of the system of equations (39)–(40) is the FVBVP of the form

$$K\mathbf{\Phi} = 0,\ \ \mathbf{\Phi}|_t = 0,\ \ \mathbf{\Phi}|_b = \mathbf{f}(r_\perp, s) \tag{41}$$

with absolutely transparent (nonreflecting) boundaries having the zero albedos and the source on the level $z = H$:

$$\mathbf{f}(r_\perp, s) = \begin{cases} \mathbf{E}(r_\perp, s) \text{ for } k = 1, \\ [R\mathbf{\Phi}_{k-1}](H, r_\perp, s) \text{ for } k \ge 2, \end{cases}$$

which is analogous to problems (11), (26), and (27).

The first boundary-value problem (9) for the total background $\mathbf{\Phi}_0$ is solved by the IF or SFC methods individually for each component (10), (11), and (12). The total value of the background is then obtained as a superposition of values of its individual components by virtue of linearity of boundary-value problem (9).

In the present paper, we restrict ourselves to the consideration of the first vector boundary-value problem (41) for the three-dimensional radiative transfer equation, which with the help of the Fourier transform in the coordinate $r_\perp$

$$\mathbf{g}(p) \equiv F[\mathbf{f}(r_\perp)](p) = \int_{-\infty}^{\infty} \mathbf{f}(r_\perp) \exp[i(p, r_\perp)] \, dr_\perp,$$

where the spatial frequency $p = \{p_x, p_y\}$ takes only real values ($-\infty < p_x, \; p_y < \infty$), reduces to the boundary-value problem for the one-dimensional parametric complex radiative transfer equation ($\mathbf{B} \equiv F[\mathbf{\Phi}]$)

$$L(p)\,\mathbf{B} = 0, \quad \mathbf{B}|_t = 0, \quad \mathbf{B}|_b = \mathbf{g}(p, s) \qquad (42)$$

with the operator

$$L(p) \equiv D_z - i(p, s_\perp) - S,$$

$$(p, s_\perp) = p_x \sin\vartheta \cos\varphi + p_y \sin\vartheta \sin\varphi.$$

Various possible polarization states of a plane transverse-electric wave in general are described by the vector $\mathbf{\Phi}$ comprising four real components $\Phi_1$, $\Phi_2$, $\Phi_3$, and $\Phi_4$, named the Stokes parameters. These components have the dimensionality of the intensity and are the coefficients in the expansion of the vector $\mathbf{\Phi} = \{\Phi_m\}$, $m = 1, \ldots, M$, $M = 4$, in unit vectors $\mathbf{i}_m$ of a certain coordinate system

$$\mathbf{\Phi} = \mathbf{i}_1\,\Phi_1 + \mathbf{i}_2\,\Phi_2 + \mathbf{i}_3\,\Phi_3 + \mathbf{i}_4\,\Phi_4, \qquad (43)$$

which depends on the method of description of the polarized radiation.[1, 8–14]

The solution of vector problem (41) we search for the fixed coordinate system, that is, we believe that all parameters of the Stokes vector have expansions analogous to Eq. (43). The polarization states of the source $\mathbf{f} = \{f_n\}$, $n = 1, \ldots, N$, $N \le 4$, and radiation $\mathbf{\Phi}$ may differ. Depending on the optical properties of a scattering, absorbing, and polarizing medium, as a result of transfer, the radiation in the layer may be polarized for unpolarized source; its state and/or degree of polarization may change for the polarized source; beginning with a certain multiplicity of scattering, the number of nonzero components of parameters of the Stokes vector may change: $N \le M$ or $N \ge M$ is possible.[1, 8–10]

In general, when the parameters of the Stokes vector of the source $\mathbf{f}$ comprise noncoincident parameters $f_n$, the solution of linear BVKVP (41) may be expressed in the form of the superposition

$$\mathbf{\Phi}(r, s) = \sum_{n=1}^{N} \mathbf{\Phi}_n(r, s),$$

whose components are the solution of the set of FVBVPs

$$K\mathbf{\Phi}_n = 0, \quad \mathbf{\Phi}_n|_t = 0, \quad \mathbf{\Phi}_n|_b = \mathbf{t}_n\, f_n(r_\perp, s) \qquad (44)$$

with the vectors $\mathbf{t}_n = \{\delta_{mn}\}$, $m = 1, \ldots, M$, $n = 1, \ldots, N$, where $\delta_{mn}$ is the Kronecker delta symbol. In analogy with the scalar problem of the transfer theory,[1–3] for horizontally inhomogeneous anisotropic function of the source $f_n$ on the boundary $z = H$, when the spatial and angular variables cannot be separated and the functional

$$f_n(r_\perp, s) = \frac{1}{2\pi} \int_{\Omega^-} \delta(s - s^-)\, ds^- \int_{-\infty}^{\infty} f_n(r'_\perp, s^-)\, \delta(r_\perp - r'_\perp)\, dr'_\perp,$$

can be introduced, the solution of vector problem (44) for fixed $n = 1, \ldots, N$ has the form of the vector linear functional

$$\mathbf{\Phi}_n = (\mathbf{\Theta}_n, f_n) = \frac{1}{2\pi} \int_{\Omega^-} ds^- \int_{-\infty}^{\infty} f_n(r'_\perp, s^-)\, \mathbf{\Theta}_n(s^-; z, r_\perp - r'_\perp, s)\, dr'_\perp.$$

The vector influence functions (VIFs) $\mathbf{\Theta}_n(s^-; z, r_\perp, s) = \{\Theta_{mn}\}$, $n = 1, \ldots, N$, whose components are the Stokes parameters $\Theta_{mn}(s^-; z, r_\perp, s)$, $m = 1, \ldots, M$, are the solution of the set of FVBVPs

$$K\mathbf{\Theta}_n = 0, \quad \mathbf{\Theta}_n|_t = 0, \quad \mathbf{\Theta}_n|_b = \mathbf{t}_n\, f_\delta(s^-; r_\perp, s) \qquad (45)$$

with the parameter $s^-$ and the source function $f_\delta(s^-; r_\perp, s) = \delta(r_\perp)\delta(s - s^-)$.

The components of the Stokes vector $\mathbf{\Phi}_n = \{\Phi_{mn}\}$ are calculated with the help of the linear scalar functional

$$\Phi_{mn} = (\Theta_{mn}, f_n) =$$

$$= \frac{1}{2\pi} \int_{\Omega^-} ds^- \int_{-\infty}^{\infty} \Theta_{mn}(s^-; z, r_\perp - r'_\perp, s)\, f_n(r'_\perp, s^-)\, dr'_\perp. \qquad (46)$$

Now we introduce the tensor $\Pi$ specified by $N$ Stokes vectors $\mathbf{\Theta}_n$ $n = 1, \ldots, N$, and agree to write conventionally the IF tensors (IFTs) in the form of a table (matrix)

$$\Pi = \left\{ \begin{array}{c} \Theta_{11} \dots \Theta_{1n} \dots \Theta_{1N} \\ \dots\dots\dots\dots \\ \Theta_{m1} \dots \Theta_{mn} \dots \Theta_{mN} \\ \dots\dots\dots\dots \\ \Theta_{M1} \dots \Theta_{Mn} \dots \Theta_{MN} \end{array} \right\} . \qquad (47)$$

The first subscript $m = 1, \dots, M$, $M \le 4$ of the components $\Theta_{mn}$ of the tensor $\Pi$ denotes the serial number of the Stokes parameter of VIF $\mathbf{\Theta}_n$ and the second subscript $n = 1, \dots, N$, $N \le 4$, denotes the serial number of the source vector $\mathbf{t}_n$ in the set of problems (44) describing the model for calculation of VIF $\mathbf{\Theta}_n$ and therefore components of tensor $\Pi$ (47).

In analogy with the definition of a scalar product of the tensor of the vector **a** from the right, which is also named the linear vector function of the vector[16] **a**, we introduce the linear vector functional of the vector **a** as a scalar product of the influence function tensor $\Pi$ on the vector $\mathbf{a} = \{a_n\}$, $n = 1, \dots, N$, $N \le 4$, from the right, the result of which is a new vector

$$\mathbf{B} = (\Pi, \mathbf{a}) = \{b_m\}, \quad m = 1, \dots, M, \quad M \le 4, \qquad (48)$$

with the components

$$b_m = \sum_{n=1}^{N} (\Theta_{mn}, a_n) =$$
$$= (\Theta_{m1}, a_1) + \dots + (\Theta_{mn}, a_n) + \dots + (\Theta_{mN}, a_N).$$

With the help of definition (48), the solution of BVKVPs (41) is expressed as a linear vector functional of the vector **f** in the form

$$\mathbf{\Phi} = (\Pi, \mathbf{f}) = \{\Phi_m\}, \qquad m = 1, \dots, M, \qquad M \le 4, \quad (49)$$

where the components of the Stokes vector

$$\Phi_m = \sum_{n=1}^{N} (\Theta_{mn}, f_n) = \sum_{n=1}^{N} \Phi_{mn} =$$
$$= (\Theta_{m1}, f_1) + \dots + (\Theta_{mn}, f_n) + \dots + (\Theta_{mN}, f_N)$$

are the linear combinations of the linear scalar functionals $\Phi_{mn} = (\Theta_{mn}, f_n)$ determined by formula (46).

The solution of complex vector problem (42) is represented by the superposition

$$\mathbf{B}(z, p, s) = \sum_{n=1}^{N} \mathbf{B}_n(z, p, s),$$

whose components are the solution of the set of FVBVPs for the complex transfer equation

$$L(p)\, \mathbf{B}_n = 0, \quad \mathbf{B}_n|_t = 0, \quad \mathbf{B}_n|_b = \mathbf{t}_n\, g_n(p, s). \qquad (50)$$

The solution of problem (50) for fixed $n = 1, \dots, N$ is obtained in the form of the vector linear functional

$$\mathbf{B}_n(z, p, s) = (\mathbf{\Psi}_n, g_n) = \frac{1}{2\pi} \int_{\Omega^-} \mathbf{\Psi}_n(s^-; z, p, s)\, g_n(p, s^-)\, \mathrm{d}s^-,$$

whose kernel in the vector spatial-frequency characteristic (VSFC) $\mathbf{\Psi}_n(s^-; z, p, s)$ with the parameters $s^-$ and $p$. It is the solution of FVBVPs for the complex transfer equation

$$L(p)\, \mathbf{\Psi}_n = 0, \quad \mathbf{\Psi}_n|_t = 0, \quad \mathbf{\Psi}_n|_b = \mathbf{t}_n\, g_\delta(s^-; p, s) \qquad (51)$$

with the source function $g_\delta(s^-; p, s) = F[f_\delta(s^-; r_\perp, s)] = \delta(s - s^-)$.

The boundary-value problem (51) is the Fourier transform of boundary problem (45) and VIF and VSFC are related with the expressions

$$\mathbf{\Theta}_n = F^{-1}[\mathbf{\Psi}_n], \quad \mathbf{\Psi}_n = F[\mathbf{\Theta}_n].$$

The components of the Stokes vector $\mathbf{B}_n = \{B_{mn}\}$ are calculated with the help of the linear scalar functional

$$B_{mn}(z, p, s) = (\Psi_{mn}, g_n) =$$
$$= \frac{1}{2\pi} \int_{\Omega^-} \Psi_{mn}(s^-; z, p, s)\, g_n(p, s^-)\, \mathrm{d}s^-. \qquad (52)$$

Now we introduce the tensor $\Gamma$ determined by $N$ Stokes vector $\mathbf{\Psi}_n$, $n = 1, \dots, N$, and agree to write the tensor SFC (TSFC) in the form of table (matrix)

$$\Gamma = \left\{ \begin{array}{c} \Psi_{11} \dots \Psi_{1n} \dots \Psi_{1N} \\ \dots\dots\dots\dots \\ \Psi_{m1} \dots \Psi_{mn} \dots \Psi_{mN} \\ \dots\dots\dots\dots \\ \Psi_{M1} \dots \Psi_{Mn} \dots \Psi_{MN} \end{array} \right\} . \qquad (53)$$

With the help of definition (48), the solutions of the FVBVP for parametric complex transfer equation (42) are determined as a linear vector functional of the vector **g** in the form

$$\mathbf{B} = (\Gamma, \mathbf{g}) = \{B_m\}, \quad m = 1, \dots, M, \quad M \le 4, \qquad (54)$$

where the components of the Fourier transform of the Stokes vector

$$B_m = \sum_{n=1}^{N} (\Psi_{mn}, g_n) = \sum_{n=1}^{N} B_{mn} =$$
$$= (\Psi_{m1}, g_1) + \dots + (\Psi_{mn}, g_n) + \dots + (\Psi_{mN}, g_N)$$

are the linear combinations of the linear scalar functionals $B_{mn} = (\Psi_{mn}, g_n)$ determined by formula (52).

To solve diffraction problems for spherical particles and problems of acoustics and scattering of electromagnetic radiation, the $T$-matrix method[17, 18] is widely used. The linear transform ($T$-matrix) relates the coefficients of expansion of incident and scattered fields. The $T$-matrix depends on the choice of the coordinate system, but in a fixed coordinate system the $T$-matrix is the invariant under the parameters of incident radiation.

The Stokes parameters of the incident and scattered radiation are related by a linear transform. The tensors of IF and SFC are associated with the $T$ matrices: as $T$ matrices, TIF and TSFC depend on the representation of the parameters of the Stokes vector and are invariant under the Stokes parameters of the radiation source.

In addition to models of VIF (45) and VSFC (51), the set of basic models comprises: the vector influence function[1, 10]

$$\mathbf{\Theta}_{rn}(z, r_\perp, s) = \frac{1}{2\pi} \int_{\Omega^-} \mathbf{\Theta}_n(s^-; z, r_\perp, s)\, \mathrm{d}s^-$$

– the solution of the first boundary-value problem

$$K\mathbf{\Theta}_{rn} = 0,\ \ \mathbf{\Theta}_{rn}|_t = 0,\ \mathbf{\Theta}_{rn}|_b = \mathbf{t}_n\, \delta(r_\perp);$$

the vector spatial-frequency characteristic

$$\mathbf{\Psi}_{rn}(z, p, s) = F[\mathbf{\Theta}_{rn}] = \frac{1}{2\pi} \int_{\Omega^-} \mathbf{\Psi}_n(s^-; z, p, s)\, \mathrm{d}s^-$$

– the solution of the first vector complex boundary-value problem

$$L(p)\, \mathbf{\Psi}_{rn} = 0,\ \ \mathbf{\Psi}_{rn}|_t = 0,\ \ \mathbf{\Psi}_{rn}|_b = \mathbf{t}_n;$$

the vector influence function

$$\mathbf{\Theta}_{zn}(s^-; z, s) = \int_{-\infty}^{\infty} \mathbf{\Theta}_n(s^-; z, r_\perp, s)\, \mathrm{d}r_\perp$$

– the solution of the first one-dimensional vector boundary-value problem

$$K_z\mathbf{\Theta}_{zn} = 0,\ \ \mathbf{\Theta}_{zn}|_t = 0,\ \ \mathbf{\Theta}_{zn}|_b = \mathbf{t}_n\, \delta(s - s^-);$$

the vector transmission function[1, 10]

$$\mathbf{W}_n(z, s) = \frac{1}{2\pi} \int_{\Omega^-} \mathrm{d}s^- \int_{-\infty}^{\infty} \mathbf{\Theta}_n(s^-; z, r_\perp, s)\, \mathrm{d}r_\perp = \int_{-\infty}^{\infty} \mathbf{\Theta}_{rn}(z, r_\perp, s)\, \mathrm{d}r_\perp = \frac{1}{2\pi} \int_{\Omega^-} \mathbf{\Theta}_{zn}(s^-; z, s)\, \mathrm{d}s^-$$

– the solution of the first one-dimensional vector boundary-value problem

$$K_z\, \mathbf{W}_n = 0,\ \ \mathbf{W}_n|_t = 0,\ \ \mathbf{W}_n|_b = \mathbf{t}_n\, .$$

## SOLUTION OF GENERAL VECTOR BOUNDARY-VALUE PROBLEMS BY THE VIF AND VSFC METHODS

Let us take advantage of the above-formulated models for VIF and VSFC and represent solutions of the first vector boundary-value problems in the form of the vector linear functionals, whose kernels are the TIF and TSFC, to construct the solution of general vector boundary-value problem (13). The Fourier transform of reflection operator (2) we determine from the formula ($v \equiv F[q]$)

$$[T\mathbf{B}](H, p, s) \equiv F[R\mathbf{\Phi}] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \mathrm{d}p' \int_{\Omega^+} v(p - p', s, s^+)\, \mathbf{B}(H, p', s^+)\, \mathrm{d}s^+.$$

Now we introduce operations of radiation interaction with the boundary expressed in terms of TIF (47)

$$[G\mathbf{f}](s^-; H, r_\perp, s) \equiv R(\Pi, \mathbf{f}) = \int_{\Omega^+} g(r_\perp, s, s^+)\, (\Pi, \mathbf{f})\, \mathrm{d}s^+$$

and in terms of TSFC (53)

$$[Q\mathbf{g}](s^-; H, p, s) \equiv F[G\mathbf{f}] = T(\Gamma, \mathbf{g}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \mathrm{d}p' \int_{\Omega^+} v(p - p', s, s^+)\, (\Gamma, \mathbf{g})\, \mathrm{d}s^+.$$

The components of series (38) we express in terms of the TIF

$$\mathbf{\Phi}_1 = (\Pi, \mathbf{E});\ \ \mathbf{\Phi}_k = (\Pi, R\mathbf{\Phi}_{k-1}) = (\Pi, G^{k-1}\, \mathbf{E}).$$

As a result, we obtain the asymptotically accurate solution of GVBVP (13)

$$\mathbf{\Phi}_\mathrm{R} = (\Pi,\ Y\mathbf{E});\quad Y\mathbf{E} \equiv \sum_{k=0}^{\infty} (\Pi, G^k\, \mathbf{E}) \tag{55}$$

– the sum of the Neumann series in the multiplicity of reflection from the boundary.

In terms of the Fourier transforms, we derive

$$\mathbf{B}_\mathrm{R}(z, p, s) \equiv F[\mathbf{\Phi}_\mathrm{R}] = \sum_{k=0}^{\infty} \varepsilon^k\, \mathbf{B}_k(z, p, s), \tag{56}$$

where the terms are the solution to the system of recursion problems ($\mathbf{W} = F[\mathbf{E}]$)

$$k = 1:\ \ L(p)\, \mathbf{B}_1 = 0,\ \ \mathbf{B}_1|_t = 0,\ \ \mathbf{B}_1|_b = \mathbf{W}(p, s);$$
$$k \geq 2:\ \ L(p)\, \mathbf{B}_k = 0,\ \ \mathbf{B}_k|_t = 0,\ \ \mathbf{B}_k|_b = T\mathbf{B}_{k-1}(H, p, s),$$

They are determined as functionals

$$\mathbf{B}_1 = (\Gamma, \mathbf{W}); \;\; \mathbf{B}_k = (\Gamma, T\mathbf{B}_{k-1}) = (\Gamma, Q^{k-1}\,\mathbf{W}).$$

The sum of series (56) is the Fourier transform of the asymptotically accurate solution to GVBVP (13) in a class of slowly increasing functions[15]

$$\mathbf{B}_\mathrm{R} = (\Gamma, Z\mathbf{W}); \quad Z\mathbf{W} \equiv \sum_{k=0}^{\infty} (\Gamma, Q^k\,\mathbf{W}) \qquad (57)$$

that is, the sum of the Neumann series in the multiplicity of radiation interaction with the boundary (in terms of the Fourier transforms).

Representation of the solution of the general vector boundary-value problem (13) in the form of functionals (55) and (57) that define the explicit relations of the solution with the sources and reflection characteristics of the underlying surface we call the vector optical transfer operator (VOTO). The Neumann series $Y\mathbf{E}$ and $Z\mathbf{W}$ determine the scenario and the Fourier transform of the scenario of optical image of the spatially inhomogeneous anisotropic reflecting boundary of the layer formed as a result of multiple scattering of radiation within the layer and radiation re-reflection from the layer bottom considering the polarization or depolarization mechanisms not only within the layer, but also on its boundary.

The suggested constructive approach is efficient for mathematical modeling of optical and millimeter radiation transfer through natural media and for solving multidimensional problems of radiative correction in case of remote sensing of objects on the Earth's surface, in the theory of vision and image transfer through turbid media.

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