DESCRIPTION OF THE AEROSOL PARTICLES DISTRIBUTION OVER A UNIT VOLUME

A.I. Borodulin, B.M. Desyatkov, and S.S. Kotlyarova

Scientific Research Institute of Aerobiology, Novosibirsk
Received January 15, 1997

The discrete one-point distribution functions for concentration of aerosol particles in the atmosphere are proposed on the basis of natural physical assumptions. In particular, the binomial law and Poisson distribution are grounded. They are compared with the continuous analog of the distribution function from our previous results. Examples of practical use of the obtained results are considered. Applicability limits of the approach proposed are discussed.

As a rule traditional methods of studying aerosol transfer in the atmosphere enable one to obtain only mathematical expectations of the admixture concentrations. Since the aerosol dispersal occurs in a turbulent medium, these data are not sufficient for solving some practical problems. In the general case, it is necessary to know the concentration distribution functions, namely, the statistical description of the propagation process. The following relation for the concentration distribution function $F(C)$ of aerosol contaminations, at a given point with the coordinates x, y, z and at the moment t in time, is proposed in Ref. 1:

$$F(C) = 1 + \frac{1}{2}\left[\operatorname{erf}\left(\frac{C - \bar{C}}{\alpha}\right) - \operatorname{erf}\left(\frac{C + \bar{C}}{\alpha}\right)\right], \qquad (1)$$

where C is the number concentration of particles; $\bar{C}$ is the mathematical expectation of the concentration; α is the second parameter of the distribution law; erf denotes the probability integral.[2]

The expression (1) is the exact solution to the Kolmogorov equation.[3] It was obtained under the assumption that the variation of concentration is a stochastic Markovian process. The function (1) agrees with the results of laboratory experiments performed in a wind tunnel as well as with independent data of several field experiments.[1]

Note that Eq. (1) describes the continuous spectrum of concentration values. However, it may happen, during the aerosol dispersal, so that at some moment in time at a given point the number of particles k in a unit volume around this point may occur to be very low so that Eq. (1) is inapplicable. In this case, one should turn to methods of statistical description with a discrete spectrum of values of particles' number in a unit volume.

The aim of this paper is to justify discrete analogs of the continuous distribution function (1) on the basis of natural physical assumptions, and to analyze them.

Because a linear relation between k and C, the function $F(C)$ can be transformed to the distribution function $F_v(k)$ for k particles in a certain volume V. To do that, one should perform substitutions $k = CV$ and $\bar{k} = \bar{C}V$ in Eq. (1) (the upper bar means averaging). Then the second parameter of the distribution law for the number of particles is $\beta = V\alpha$.

The distribution functions $F(C)$ and $F_v(k)$ depend on two parameters. So, it is necessary and sufficient to give two concentration moments in order to apply them in practice. For instance, mathematical expectation of concentration can be obtained by solving a semi-empirical equation of turbulent diffusion,[4] and the second parameter α is obviously connected with the concentration variance σ_C^2 which can be obtained by solving a similar equation, see Ref. 5.

Thus, with the allowance for the above-stated remarks, the distribution $F_v(k)$ (in fact, continuous) can be used to approximate a discrete distribution law for particles' number, if one uses integer k.

Let us consider the limiting case when only one particle is travelling. Let W_0 be the probability of the particle being inside a given volume V at a moment t. Then the probability of being outside the volume is $1 - W_0$ (there are no particles in the volume V). If a source emits n similar particles travelling independently, the probability of observing $k \le n$ particles in the volume is, obviously[2]

$$p(k) = C_k^n\, W_0^k\, (1 - W_0)^{n-k}, \quad k = 0, 1, 2, \ldots, n, \qquad (2)$$

where C_k^n are the binomial coefficients.[2] The discrete distribution function $P(k)$ corresponding to the probabilities (2) has the form

0235-6880/97/10 777-04 $02.00

$$P(k) = \sum_{i=0}^{k} C_i^n \, W_0^{\,i} \, (1 - W_0)^{n-i}. \quad (3)$$

This is the binomial distribution.[2] It is a two-parameter function with the mathematical expectation $\bar{k} = nW_0$ and variance $\sigma_k^2 = nW_0\,(1 - W_0)$.[2]

If the initial value of the concentration expectation is given as delta function at the point of the source's action, then, according to Ref. 4, the field $\bar{C}_0$ obtained can be treated as the probability density of a single particle being at a given point at a moment t. Then

$$W_0 = \int\limits_V \bar{C}_0 \, \mathrm{d}V = \int\limits_V \bar{C} \, \mathrm{d}V \left(\int\limits_\Omega \bar{C} \, \mathrm{d}V \right)^{-1},$$

where Ω is the domain where the admixture is dispersed; $\bar{C}$ is the concentration field from an arbitrary source.

Let the number n of particles emitted by the source tend to infinity. Then, if W_0 tends to zero (in fact, beginning from $W_0 < 0.1$) and the product nW_0 is finite, the binomial distribution (3) can be approximated by the Poisson distribution[2]

$$p(k) = \exp\,(-\,\bar{k})\, \frac{(\bar{k})^k}{k!}\,; \quad \left(P(k) = \sum_{i=0}^{k} p(i) \right). \quad (4)$$

For this distribution, mathematical expectation of the number of particles is equal to the variance $\bar{k} = \sigma_k^2$ (see Ref. 2). This distribution, in contrast to the above-mentioned ones, is a single-parameter function.

Now let us compare the approximating function $F_v(k)$ with the distributions (3) and (4) under the assumption that they have similar mathematical expectations and variances. After calculations, we obtain

$$\frac{\sigma_k^2}{(\bar{k})^2} = \mathrm{erf}\left(\frac{\bar{k}}{\beta}\right)\left[1 + \frac{1}{2}\left(\frac{\beta}{\bar{k}}\right)^2\right] + \frac{1}{\pi^{1/2}}\left(\frac{\beta}{\bar{k}}\right)\exp\left[-\left(\frac{\bar{k}}{\beta}\right)^2\right] - 1. \quad (5)$$

On the other hand, according to properties of the binomial and Poisson distributions, we have

$$\frac{\sigma_k^2}{(\bar{k})^2} = \begin{cases} \dfrac{1}{\bar{k}} - \dfrac{1}{n} & \text{for the binomial distribution,} \\ \dfrac{1}{\bar{k}} & \text{for the Poisson distribution.} \end{cases} \quad (6)$$

Thus, specifying the mathematical expectation of the number of particles in a certain volume and, if necessary, the total number of the emitted particles, one can obtain the parameter β by solving Eq. (5) with the allowance for Eq. (6). Note that this procedure is valid only if concentration variation is rather weak within the chosen volume V. Under this condition, the probability W_0 is approximately equal to the product of $\bar{C}_0$ by V. Thus, the estimation of W_0 is based on a single-point characteristic for which the relation (5) is valid.

The dependence (5), with the allowance for the relation between the mathematical expectation and variance for the Poisson law demonstrates that the limiting value of the parameter β equals 0.89 when the average number of particles k in a certain volume tends to zero. In the other case, when the number of particles grows, β asymptotically tends to $(2\bar{k})^{1/2}$. The calculation demonstrate that the dependence of β on k obtained by the relation (5) and the asymptotic behavior coincide already at $k > 3$. The view of the dependence of β on $\bar{k}$ may be gotten from the data presented in Table I.

TABLE I. The dependence of the parameter β on the average number of particles k under the condition that the mean values and variances of the discrete and continuous distributions are equal.

$\bar{k}$	β	$\bar{k}$	β
10000.0	141.2	1.30	1.73
1000.0	22.4	1.00	1.56
100.0	14.1	0.69	1.39
10.0	4.5	0.28	1.11
8.0	4.0	0.10	0.97
6.2	3.5	0.01	0.90
4.5	3.0	0.001	0.89
3.2	2.6	0.0001	0.89
2.1	2.1	0.00001	0.89

The results of comparing $F_v(k)$ with the Poisson distribution are presented in Table II. For $\bar{k} = 10$ ($\beta = 4.5$), the continuous distribution quite satisfactorily approximates the Poisson distribution. With the decrease of $\bar{k}$, for $\bar{k} = 1.0$ ($\beta = 1.6$) we see a significant, reaching 0.2 for $\bar{k} = 1$, differences between the discrete and continuous distributions. With the further decrease of the average number of particles, e.g. the case of $\bar{k} = 0.1$ ($\beta = 0.97$), a good coincidence is observed again. As a matter of fact this case is close to the limiting one and is not of practical interest because the density of the continuous distribution degenerates into the delta function and the discrete distribution tends to a degenerate distribution.[2]

TABLE II. Comparison of the Poisson distribution $P(k)$ with its continuous approximation $F_v(k)$ for some values of the average number of particles $\bar{k}$ in a volume V.

k	$F_v(k)$	$P(k)$	k	$F_v(k)$	$P(k)$
$\bar{k}$ = 10; β = 4.48			$\bar{k}$ = 1; β = 1.57		
0	0.002	< 10^{-4}	0	0.368	0.368
1	0.003	0.001	1	0.536	0.736
2	0.006	0.003	2	0.820	0.920
5	0.057	0.067	3	0.965	0.981
6	0.104	0.130	4	0.996	0.996
7	0.172	0.220	5	0.9997	0.9994
8	0.264	0.330	$\bar{k}$ = 0.1; β = 0.97		
9	0.376	0.458			
10	0.500	0.583			
12	0.736	0.792	0	0.884	0.904
15	0.943	0.951	1	0.959	0.995
16	0.971	0.973	2	0.998	0.999
20	0.999	0.998			

Let us also note a satisfactory coincidence of the probability to observe zero number of particles in a volume. It describes the concentration intermittence effect.[1]

Now let us turn to the criterion enabling one to establish the moment for “switchingB from the continuous statistics to its discrete analog. The nature of continuous and discrete distribution laws is fundamentally different, and we cannot pass smoothly from one distribution type to another one. At the same time, there is a possibility to assess when the discrete law can be used instead of the continuous one for dispersal of aerosol admixtures in the atmosphere. It is clear that the average number of particles in a volume V must tend to zero.

If $\bar{k}$ tends to zero in Eq. (5), the variance of particles' number in a given volume also tends to zero so that

$$\sigma_k^2 \to \left(\frac{2}{\pi^{1/2}}\,\beta\right)\bar{k}\ . \qquad (7)$$

This relation shows the order of the value for the parameter β so that the variance of the number of particles would be equal to their mathematical expectation when the average number of the particles in a volume tends to zero, in correspondence with the Poisson distribution law. This takes place if and only if the value enclosed in round brackets in Eq. (7) is close to unity in its magnitude. The data from Table I demonstrate that the magnitude of $\bar{k}$ must be less than a few tenths. In correspondence with the relation $\beta = V\alpha$, we can also obtain the connection between α and V.

Let us consider the dispersal process for a monodisperse aerosol in the boundary layer of the atmosphere. Let the aerosol be emitted by a stationary point source of power 10^5 g/s at the height 100 m, with the diameter of particles being $2\cdot10^{-5}$ m. To calculate the field of wind velocity, we used a numerical-analytical model.[6] The coefficients of turbulent diffusion were determined based on the hypothesis that they are proportional to the corresponding components of the Reynolds viscous stress tensor.[5] They and some accompanying parameters were specified using an algebraic model.[5,7]

TABLE III. Calculated values of the parameters of the continuous and discrete distributions.

m	$\bar{S}$, g/m^3	$\bar{C}$, pieces/m^3	$\bar{k}$	β, pieces/m^3	α^{-1}, cm^3
1	$0.58\cdot10^{-15}$	$0.14\cdot10^{-6}$	$0.26\cdot10^{-5}$	$0.22\cdot10^{-9}$	$0.19\cdot10^{8}$
2	$0.54\cdot10^{-13}$	$0.13\cdot10^{-4}$	$0.11\cdot10^{-3}$	$0.47\cdot10^{-9}$	$0.89\cdot10^{7}$
3	$0.40\cdot10^{-11}$	$0.95\cdot10^{-3}$	$0.46\cdot10^{-2}$	$0.87\cdot10^{-9}$	$0.48\cdot10^{7}$
4	$0.23\cdot10^{-9}$	$0.55\cdot10^{-1}$	$0.11\cdot10^{0}$	$0.21\cdot10^{-8}$	$\underline{0.20\cdot10^{7}}$
5	$0.10\cdot10^{-7}$	$0.23\cdot10^{1}$	$\underline{0.70\cdot10^{0}}$	$0.15\cdot10^{-7}$	$0.28\cdot10^{6}$
6	$0.33\cdot10^{-6}$	$0.79\cdot10^{2}$	$0.13\cdot10^{1}$	$0.27\cdot10^{-6}$	$0.15\cdot10^{5}$
7	$0.97\cdot10^{-5}$	$0.23\cdot10^{4}$	$0.15\cdot10^{1}$	$0.64\cdot10^{-5}$	$0.65\cdot10^{3}$
8	$0.23\cdot10^{-3}$	$0.55\cdot10^{5}$	$0.16\cdot10^{1}$	$0.14\cdot10^{-3}$	$0.29\cdot10^{2}$
9	$0.72\cdot10^{-2}$	$0.17\cdot10^{7}$	$0.92\cdot10^{1}$	$0.78\cdot10^{-3}$	$0.54\cdot10^{1}$

Some results of the calculation are presented in Table III where m is the number of the point on a line perpendicular to the wind direction at the distance 14 km from the source. The distance between the points is 2 km. The point with $m = 9$ is on the symmetry axis of the aerosol plume, and the point with $m = 1$ is at its periphery. In the table, $\bar{S}$ and $\bar{C}$ are mathematical expectations of the aerosol concentration. These are calculated using a model at the height of 2 m from the underlying surface and expressed in g/m^3 and in m^{-3}, respectively; $\bar{k}$ is the mathematical expectation of the concentration in the case of Poisson distribution; β is the second parameter of the continuous distribution function of concentration.

Note that mathematical expectation and variance in the Poisson distribution are dimensionless. However, when describing a physical process, we mean that the values are referred to a certain unit volume.

Let V_0 be a unit volume such that mathematical expectations for concentration of the discrete and continuous distribution functions coincide in it. The value V_0 calculated in this way turned to be similar to the parameter α^{-1} which can be interpreted as a unit volume for the Poisson distribution. The corresponding $\bar{k}$ and α^{-1} are underlined in the table. These values indicate the spatial boundary where the conditions $\bar{k} \le 1$ and $V \le \alpha^{-1}$ are violated and one should pass from the continuous concentration distribution to its discrete analog. Since $\bar{k}$ does not depend on the source power and β depends on it linearly, α^{-1} is inversely proportional to it. Therefore, the boundary indicated in the sixth column of the table will be displaced into the domain of small concentration values when the source power increases.

REFERENCES

1. A.I. Borodulin, G.M. Maistrenko, and B.M. Chaldin, *Statistical Description of the Aerosol Dispersal in the* Atmosphere (State University, Novosibirsk, 1992), 124 pp.

2. G.A. Korn and T.M. Korn, *Mathematical Handbook for Scientists and Engineers* (McGraw-Hill, New York, 1968).

3. V.I. Tikhonov and M.A. Mironov, *Markovian Processes* (Sov. Radio, Moscow, 1977), 488 pp.

4. A.S. Monin and A.M. Yaglom, *Statistical Hydromechanics. Mechanics of Turbulence* (Nauka, Moscow, 1965), 640 pp.

5. V. Rodi, in: W. Kollman ed., *Prediction Methods for Turbulent Flows,* (A von Karman Institute Book Hemisphere Publishing Corporation, 1980).

6. B.M. Desyatkov, Trudy Zap.-Sib. Nauchno-Issled. Gidrometeorol. Inst., No. 77, 68–75 (1986).

7. E.N. Teverovskii and E.S. Dmitriev, *Transport of Aerosol Particles by Turbulent Flows* (Energoatomizdat, Moscow, 1988), 160 pp.