MODELING OF DIFFRACTION PATTERNS AT THE OUTPUT OF POORLY FOCUSED OPTICAL SYSTEM WITH AN AXICONE

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Received February 2, 1997

Mathematical techniques is developed for modeling diffraction patterns at the output of a defocused optical system with an axicone. We have derived quite a simple equation for field of a disturbed (defocused) system by solving corresponding equations for a nondisturbed system. Comparison of the solution obtained with the analytical solutions obtained earlier for the same problem is performed for two input actions, i.e., a plane wave and a Gaussian beam.

1. It is a characteristic property of diffraction fields at the output of a properly focused optical system "axicone-lens" or in the far zone (Fraunhofer zone) of a lensless axicone system under the action of a collimated laser radiation is their localization in a narrow ring zone of the formed beam periphery. This salient feature of the fields formed with a focused axicone system attracted much interest in them and their applications in various technologies, from laser engineering[1–3] to space communication.[4] The structure of ring diffraction patterns were considered in several papers[5–9]; thermal effects of their action on substances were studied in Ref. 10. The patterns with a blurred ring structure formed by a defocused (poorly focused) axicone system are studied much less. This is the subject of studies presented this paper. We call these fields "disturbed", in contrast to "undisturbed" fields formed by a focused system.

The models of undisturbed fields are constructed on the basis of Fourier optics equation[11] relating the complex amplitudes of an axial symmetric field at the input $\psi_1(r')$ and at the output $\psi(r)$ of an axicone system in the paraxial approximation of the Kirchhoff scalar diffraction theory

$$\psi(r) = \frac{k}{z}\int\limits_0^R \psi_1(r')\exp(i\omega_0 r')\, I_0(\omega r')\, r'\,\mathrm{d}r', \qquad \omega = kr/z\,, \tag{1}$$

Here $\exp(i\omega_0 r')$ is the transmission function of the axicone which is a linear spatial modulator of the wave phase;

$$\omega_0 = k\phi = kr_0/z \tag{2}$$

is the parameter expressed by the wave number $k = 2\pi/\lambda$ and the angle ϕ of a beam deflection by the axicone; $r_0 = \phi z$ is the radius of the median line of the illumination ring of the output plane of the system; $I_0(\cdot)$ is the zero order "essel function of the first kind; R is the aperture of the axicone. "y setting $z = f$, where f is the focal length, we will consider the focal plane of the lens to be the output plane of the "axicone-lens" system. For a focused lensless system, the output plane is the cross section of the beam in the Fraunhofer zone at a distance z from the axicone. Here, the focusing system is a spatial layer of a sufficiently large extension. Mathematically, both situations are similar and both are described by an equation of the same type (1) with the only difference that the value z corresponding to the Fraunhofer zone of the system considerably exceeds the focal length f of all the lenses that are met in practice.

The equation (1) and models of the ring diffraction pattern that follow from it have been studied quite thoroughly. Much less attention was paid to the equation

$$\psi(r) = \frac{k}{z}\int\limits_0^R \psi_1(r')\exp(i\delta r'^2)\exp(i\omega_0 r')\, I_0(\omega r') r'\mathrm{d}r', \tag{3}$$

describing the same system in a more general case of a weakly focused state that differs from Eq. (1) by the presence of an additional phase term $\exp(i\delta r'^2)$. The latter by analogy with the above-mentioned definition of the axicone can be called the square-law spatial modulator of the wave phase. If the field $\psi(r)$ is considered at a point near the focal plane of the system, the parameter δ can be defined as

$$\delta = \frac{k}{z}\left(\frac{1}{z} - \frac{1}{f}\right) \approx \frac{k(f - z)}{2f^2}\,. \tag{4a}$$

For a lensless system and large z, the parameter δ in Eq. (3) may be written, in the Fresnel approximation, as

$$\delta = k/2z\,. \tag{4b}$$

The defocusing term $\exp(i\delta r'^2)$ in Eq. (3) can be neglected only at $\delta r'^2 \ll 1$ that does not always takes place in practice. For instance, this condition is not satisfied in many laser engineering applications, when a beam formed by an axicone system as a ring concentrator of radiation energy is used for welding or drilling materials of a finite thickness.[1] This condition can also be broken in other applications, in particular, during the operation of a laser beacon with a tubular searchlight beam,[4] when the object is not very far from the beacon. In all these cases, the factor $\exp(i\delta r'^2)$ must be taken into account in Eq. (3). But it complicates the calculations very much.

2. Let us call Eq. (3) the equation of a weakly focused optical system with an axicone. Usual approach[6] to analysis of this equation is to expand the disturbing exponential factor $\exp(i\delta r'^2)$ into a power series and replace Eq. (3) by a more cumbersome equation

$$\psi(r) = \frac{k}{z}\sum_{m=0}^{\infty}\frac{(i\delta)^m}{m!}\times$$

$$\times\int\limits_0^R \psi_1(r')\exp(i\omega_0 r')\, I_0(\omega r')\,(r')^{2m+1}\,\mathrm{d}r', \qquad (5)$$

which is, however, simpler in structure of the terms entering into it. In two particular cases, namely, for a plane wave and a Gaussian beam at the system input, the integral can be represented analytically based on the theorem of series expansion over Bessel functions[12]

$$I_0(\lambda t) = \sum_{n=0}^{\infty}\frac{(1-\lambda^2)^n}{n!}\,(t/2)^n\, I_n(t) \qquad (6)$$

and using the formula[12]

$$G_{m,n}(x) = \frac{x^{2(m+n+1)}}{2(m+n+1)\,2^n\,n!}\times$$

$$\times\,{}_2F_2\left(\begin{matrix} n+1/2, & 2m+2n+2 \\ 2n+1, & 2m+2n+3\end{matrix}\;\middle|\; 2ix\right), \qquad (7)$$

that expresses the Luke integral

$$G_{m,n} \triangleq \int\limits_0^x \exp(it)\, I_n(t)\, t^{2m+n+1}\,\mathrm{d}t \qquad (8)$$

in terms of generalized hypergeometric function

$${}_2F_2\left(\begin{matrix} n+1/2, & 2m+2n+2 \\ 2n+1, & 2m+2n+3\end{matrix}\;\middle|\; 2ix\right) =$$

$$= \sum_{k=0}^{\infty}\frac{(n+1/2)_k\,(2m+2n+2)_k}{(2n+1)_k\,(2m+2n+3)_k}\,(2ix)^k,$$

$$(\alpha)_k \triangleq \Gamma(\alpha+k)/\Gamma(\alpha) = \alpha(\alpha+1)\ \ldots\ (\alpha+k+1)\ .$$

The solutions are double series over these functions with the value $x = \omega_0 R$, m and n being the summation indices. The values $(i\tilde{\delta})^m\rho^n/m!$, where

$$\rho = (ka/z)\,(r - r_0)\ , \qquad (9)$$

is the dimensionless, normalized, and displaced by r_0 radial variable r, serve as coefficients at terms of the series. By a we denote here some characteristic size coinciding with the aperture radius R for the case of a plane wave and with the cross size $W/2$ for a Gaussian beam of the type $\psi_1(r') = \sqrt{I_0}\exp(-r'^2/W^2)$. The parameter $i\tilde{\delta}$ is a dimensionless quantity and for the case of a plane wave, it coincides with $i\delta R^2$ while for a Gaussian beam it equals to $(i\delta - 1/W^2)R^2$.

In an asymptotic approximation relative to the parameter x whose values exceed unity by 2–3 orders of magnitude, in all practical cases, the hypergeometric functions are simpler and double series contract, and the solution can be reduced to the form

$$\psi(r) =$$

$$= A\sum_{n=0}^{\infty}\frac{(-i\delta R^2)^n}{n!}\,\frac{3}{3+4n}\,{}_1F_1(2n+3/2,\, 2n+5/2;\, i\rho),$$

$$A = \sqrt{I_0\,\frac{4\,R^3}{9\,zr_0\,\lambda}} \qquad (10)$$

for a plane wave and

$$\psi(r) = \tilde{B}\Big[\,{}_1F_1(3/4,\ 1/2; -\tilde{\rho}^2) -$$

$$- 2i\tilde{\rho}\,\frac{\Gamma(5/4)}{\Gamma(3/4)}\,{}_1F_1(5/4,\ 3/2; -\tilde{\rho}^2)\Big], \qquad (11)$$

$$\tilde{B} = B/(1 + i\delta W^2)^{3/4}, \qquad \tilde{\rho} = \rho/(1 + i\delta W^2)^{1/2}$$

for a Gaussian beam. By ${}_1F_1(a, b; z)$, we denote confluent Kummer functions.

3. Analysis of equation (3) is based on the transform of Eq. (3) to the form (5) and uses Eqs. (6) and (7) and summation of the series over the generalized hypergeometric functions.[6] It is quite rigorous but very cumbersome and too “specialized”. It is applicable only to two particular cases of input action upon the system considered. Below, we present a more general approach which enables one to obtain, and relatively simply, a solution of the “disturbed” problem (3) using the solution of the corresponding "undisturbed" problem (1). The solution of the latter can easily be obtained for a wide class of axial symmetric input actions described by smooth functions of the radial coordinate, if not analytically, then by numerical or semianalytical methods (see, for instance, Ref. 4). As applied to the particular cases of a plane wave and a Gaussian beam considered above, the proposed approach yields the results coinciding with Eqs. (10) and (11).

4. Let us consider the equations (1) and (3) as Fourier transforms with respect to frequency ω_0 of the corresponding integrands. The possibility of making such a representation follows from the very form of these equations. Then the function defined by the disturbed equation (3) (let us denote it by $\psi_d(r) \equiv \psi_d(\omega_0)$) can be described by the convolution integral over the frequency region

$$\psi_d(r) = \psi(\omega_0) * f(\omega_0) = \int\limits_{-\infty}^{\infty} \tilde{f}(\tau)\, \psi(\omega_0 - \tau)\, d\tau . \qquad (12)$$

Here $\psi(\omega_0) \equiv \psi(r)$ is the solution of the undisturbed problem (1) and

$$\tilde{f}(\omega_0) = F_{\omega_0} \{\exp(i\delta r'^2)\} = \sqrt{\pi / i\delta}\, \exp(-i\, \omega_0^2 / 4\delta)$$

is the Fourier transform of the disturbation term $\exp(i\delta r'^2)$. The latter equality may be derived from the obvious transformations

$$F_{\omega_0} \{\exp(i\delta r'^2)\} \triangleq \int\limits_{-\infty}^{\infty} \exp(i\delta r'^2)\, \exp(i\omega_0 r')\, dr' =$$

$$= \exp(-i\, \omega_0^2 / 4\delta) \int\limits_{-\infty}^{\infty} \exp(i\delta x^2)\, dx =$$

$$= \sqrt{i\pi / \delta}\, \exp(-i\, \omega_0^2 / 4\delta) .$$

"y expanding $\psi_0(\omega_0 - \tau)$ into the Taylor series over τ (here, in contrast to Ref. 6, we expand not the disturbing function $\exp(i\delta r'^2)$, but the solution of the undisturbed problem (1) convoluted with its Fourier transform) and performing term by term integration in Eq. (12), we obtain

$$\psi_d(r) = (2\pi)^{-1} \sum_{p=0}^{\infty} (-i)^p \frac{m_p}{p!} \frac{d^p \psi(\omega_0)}{d\omega_0{}^p} . \qquad (13)$$

The coefficients of the series are the moments of the function $\tilde{f}(\omega_0)$

$$m_p = \int\limits_{-\infty}^{\infty} \tau^p\, \tilde{f}(\tau)\, d\tau , \quad p = 0, 1, 2, \ldots ,$$

that may simply be expressed by the defocusing factor δ. Using well-known formulas for m_p [13] and taking into account the identity

$$(2n - 1)!!\, 2^n = (2n)! \,/\, n!$$

we obtain

$$m_p = \begin{cases} 2\pi \dfrac{p!}{(p/2)!} (i\delta)^{p/2} , & \text{if } p \text{ is even} \\ 0, & \text{if } p \text{ is odd} . \end{cases}$$

Substitution of these formulas into Eq. (13) yields a relatively simple solution to the problem that presents the function $\psi_d(r)$ disturbed by the defocusing term $\exp(i\delta r'^2)$ in terms of undisturbed function $\psi(r) \equiv \psi(\omega_0)$ and its derivatives:

$$\psi_d(r) = \sum_{n=0}^{\infty} \frac{(-i\delta)^n}{n!} \frac{d^{2n} \psi(\omega_0)}{d\omega_0{}^{2n}} . \qquad (14)$$

The equality

$$\frac{d^{2n} \psi(\omega_0)}{d\omega_0{}^{2n}} \exp(i\,\omega_0\, r') = (-\, r'^2)^n \exp(i\,\omega_0\, r')$$

obviously implies the equivalence of the expressions (3) and (12) for the cases when $\psi(\omega_0) \equiv \psi(r)$ exactly corresponds to the equation (1).

5. If $\psi(r)$ is represented by Kummer functions, the expression (12) may be written in a different form with the allowance for the well-known identity[12]

$$\frac{d^p}{dz^p}\, {}_1F_1(a, b; z) = \frac{(a)_p}{(b)_p}\, {}_1F_1(a + p, b + p; z) .$$

For instance, if we have a plane wave at the input ($R_1 = 0$), the expression (14) and the equality

$$\frac{d^2}{d\omega_0{}^2} = R^2 \frac{d^2}{d\rho^2}$$

yield

$$\psi(r) = A \sum_{n=0}^{\infty} \frac{(-i\delta R_2^2)^n}{n!} \times$$

$$\times \frac{3}{3 + 4n}\, {}_1F_1(2n + 3/2, 2n + 5/2; i\rho) ,$$

$$\rho = R(\omega - \omega_0) , \qquad (15)$$

what coincides with Eq. (10).

In the case of a Gaussian beam, $\psi(r)$ may be written in terms of the function $u_p(z) = \exp(z^2/4) D_p$ ($p = -3/2$, $z = -i\sqrt{2}\rho$, $\rho = (W/2)(\omega - \omega_0)$) satisfying the equalities

$$\frac{d^2 u_p}{d\omega_0{}^2} = -\frac{W^2}{2} \frac{d^2 u_p}{dz^2} ,$$

$$\frac{d^2 u_p}{dz^2} = \frac{W^2}{2} (p u_p - \rho\, du_p / d\rho) . \qquad (16)$$

The first equality follows from the definition, while the second one can easily be derived from the classic differential equation for the parabolic cylinder functions[12]:

$$\frac{d^2}{dz^2} D_p + (p + 1/2 - z^2/4)\, D_p = 0 .$$

Taking into account Eq. (16), we obtain, from Eq. (14), the following expression for the case of a Gaussian beam at the input of the system in the first order approximation relative to the small parameter δ:

$$\psi(r) = 2^{-3/4}\, \Gamma(3/2)\, \tilde{B}\, u_{-3/2}(-\, i \sqrt{2}\, \tilde{\rho}) =$$
$$= \tilde{B} \left[{}_1F_1(3/4,\, 1/2; -\tilde{\rho}^2) - \right.$$
$$\left. - 2i\tilde{\rho}\, \frac{\Gamma(5/4)}{\Gamma(3/4)}\, {}_1F_1(5/4,\, 3/2; -\tilde{\rho}^2) \right] , \qquad (17)$$

where

$$\tilde{B} \approx B[1 - i(3/4)\delta W^2] , \quad \tilde{\rho} = \rho(1 - i\delta W^2/2) .$$

This agrees with Eq. (11) for small δ. When deriving Eq. (17), we used the following relationship:

$$\psi(\rho) - i(\delta W^2/2)\, \rho\, \frac{d\psi(\rho)}{d\rho} \approx \psi(\rho) .$$

6. Consideration of the optical system with an axicone in a spatial frequency region (with respect to the parameter ω_0) leads to the equation (14) that models the diffraction patterns $\psi_d(r)$ at the output of a disturbed (weakly focused) system (3) in the approximation that enables one to obtain the solution of the corresponding undisturbed problem (1) $\psi(r)$ entering into Eq. (14). The existing methods (see, for instance, Ref. 4) allow one to calculate the latter for any form of the input action with the accuracy sufficient for all technological applications; the fields at the output of a weakly focused system are described by equation (14) with the same accuracy if the obtained solution $\psi(r)$ is substituted into it. This conclusion is verified by comparing corollaries of the Eq. (14) with the analytical results obtained earlier for two particular cases, i.e., a plane wave and a Gaussian beam at the input of the system.

The solution (14) is represented as a series. The series converges rapidly at a slightly defocused system (and this is just the case that is interesting for practical use of ring diffraction patterns formed with such a system). So it is sufficient to use only two or three first terms of the series.

APPENDIX

The equation (3) can be considered as a Fourier transform of the product of two summable (Lebesgue integrable) functions $f \in L_1$ and $g \in L_1$ from the x-plane to the ω_0-plane

$$\psi_d(\omega_0) = \int_{-\infty}^{\infty} f(x)\, g(x)\, \exp(i\omega_0 x)\, dx ,$$

One of the functions, namely,

$$f(x) = \frac{k}{z}\, \psi_1(x)\, I_0(\omega x)\, [U(x - R) - U(x)] ,$$

is a finite function. Here $U(\cdot)$ is the Heaviside function. One can apply the convolution theorem to the functions (more generally, to all distributions with a bounded support or tempered distributions[15])

$$F[f\, g] = F[f] * F[g] .$$

In our case, $\psi_d(\omega_0)$ and $F[f]$ are output functions of the disturbed (3) and undisturbed (1) equations. It is just this fact that validates formula (12).

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