THEORY OF THE VECTOR-FORM OPTICAL TRANSFER OPERATOR OF THE "ATMOSPHERE–OCEAN" SYSTEM

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The vector form of the optical transfer operator (VOTO) of the "atmosphere–ocean" system (AOS) represented as a 1-D, 2-D, or a 3-D plane layer with the reflecting and transmitting inner interface is built up taking into account multiple scattering and polarization of radiation in both media. The perturbation theory series and influence functions of the general boundary problem, in a vector form, of the transfer theory for polarized radiation are being used. The VOTO kernels are tensors of influence functions of both media. The basic models of the vector influence functions are formulated. The structure of the radiation field in the AOS is considered.

INTRODUCTION

In resent years there is observed an increased interest in the numerical models of radiation transfer in the "atmosphere-ocean" system (AOS) as well as in mechanisms of forming the radiation field in AOS that allow for the radiation exchange between the two media.

A comparison among seven one-dimensional models has been carried out under a support from the international science foundations.[1] Five of those models use the Monte Carlo method, one model is based on the imbedding method, and the seventh one on the method of discrete ordinates, or more correctly, the method of spherical harmonics (DISORT program).[2] It is important that the latter two models use a preliminary Fourier expansion over the azimuth angle.

One-dimensional models of the radiation transfer in AOS have been developed using two approaches. According to one of the approaches the calculations are being done by iteration method of characteristics,[3–5] while following the other one the radiation in AOS is described using optical transfer operators (OTO) in terms of the influence functions of the atmosphere and ocean.[6–8] The three-dimensional models of radiation transfer in the AOS with the horizontally inhomogeneous water-air interface also use the OTO-based approach.[9,10]

We have formulated an optical transfer operator (VOTO), in a vector form, for the case of polarized radiation transfer in a system with anisotropically[11,12] and isotropically[13] reflecting underlying surface. In such a model the ocean is considered as the horizontally homogeneous or inhomogeneous base. A sea, lake, or any other water basin can be assumed instead of ocean.

In this paper we develop a VOTO of the AOS using rigorous methods of the perturbation theory and theory of basic solutions. The approach proposed allows for polarization properties of radiation in the AOS, as well as for the multiple scattering effects and radiation exchange between the two media through the water-air interface.[14] Using same methodological basis the VOTOs of the AOS are being constructed for four types of the water-air interface. Those involve isotropic and anisotropic horizontally homogeneous and inhomogeneous reflection and transmission.

STATEMENT OF THE PROBLEM

We shall consider a plane, horizontally infinite $-\infty < x, y < \infty$, while being inhomogeneous and vertically finite ($r_\perp = (x, y)$, $(0 \le z \le H)$), layer composed of two media that scatter, absorb, and polarize the incoming radiation. Here $r_\perp = (x, y)$ is the radius-vector of a point inside the layer. The boundary between the two media that transmits and reflects radiation is at the level $z = h$ inside the layer. It is assumed that the transfer system does not produce multiplication. The set of all directions of radiation propagation, $s = (\mu, \varphi)$, where $\mu = \cos\vartheta$, $\vartheta \in [0, \pi]$ is the zenith angle counted off from the axis z, and $\varphi \in [0, 2\pi]$ is the azimuth angle counted off from the axis x, forms a unit sphere $\Omega = \Omega^+ \cup \Omega^-$ with Ω^+ and Ω^- being the hemispheres of directions with $\mu \in [0, 1]$ and $\mu \in [-1, 0]$, respectively. The projection of the vector s on a horizontal plane is $s_\perp = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi)$. To write the boundary conditions, we introduce the sets labeled with "t" (*top*), "b" (*bottom*), and "d" (*dividing*):

$t = \{z, r_\perp, s: z = 0, s \in \Omega^+\}$;

$b = \{z, r_\perp, s: z = H, s \in \Omega^-\}$;

$d1 = \{z, r_\perp, s: z = h, s \in \Omega^-\}$;

$d2 = \{z, r_\perp, s: z = h, s \in \Omega^+\}$.

Assuming the macroscopically isotropic medium to be in a steady state as well as the constancy of the radiation sources $\mathbf{F}(z, s)$, $\mathbf{F}^0(s^0; r_\perp, s)$, $\mathbf{F}^H(s^H; r_\perp, s)$, $\mathbf{F}^1(s^1; r_\perp, s)$, $\mathbf{F}^2(s^2; r_\perp, s)$, that may depend on the

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parameters s^0, s^H, s^1, s^2, a field of quasi-monochromatic polarized radiation is most completely described by the vector $\mathbf{\Phi}(r, s)$ whose components are the Stokes parameters.[4] The vector of Stokes parameters $\mathbf{\Phi}$ (SVP) is found as a solution of general vector boundary problem (GVBP)

$$\hat{K}\mathbf{\Phi} = \mathbf{F}; \quad \mathbf{\Phi}\big|_t = \mathbf{F}^0, \quad \mathbf{\Phi}\big|_b = \hat{R}\mathbf{\Phi} + \mathbf{F}^H,$$
$$\mathbf{\Phi}\big|_{d1} = \varepsilon\,(\hat{R}_1\mathbf{\Phi} + \hat{T}_{21}\mathbf{\Phi}) + \mathbf{F}^1,$$
$$\mathbf{\Phi}\big|_{d2} = \varepsilon\,(\hat{R}_2\mathbf{\Phi} + \hat{T}_{12}\mathbf{\Phi}) + \mathbf{F}^2 \qquad (1)$$

with the linear operators being as follows:

the transfer operator

$$\hat{D} \equiv (s, \text{grad}) + \sigma(z) = \hat{D}_z + \left(s_\perp, \frac{\partial}{\partial r_\perp}\right),$$
$$\hat{D}_z \equiv \mu\frac{\partial}{\partial z} + \sigma(z);$$

the collision integral

$$\hat{S}\mathbf{\Phi} \equiv \sigma_s(z) \int\limits_{\Omega} \hat{P}(z, s, s')\,\mathbf{\Phi}(z, r_\perp, s')\,\mathrm{d}s';$$
$\mathrm{d}s' = \mathrm{d}\mu'\,\mathrm{d}\varphi'$;

the integro-differential operator $\hat{K} \equiv \hat{D} - \hat{S}$,

the one-dimensional operator $\hat{K}_z \equiv \hat{D}_z - \hat{S}$;

$\hat{P}$ is the scattering phase matrix[4]; $\sigma(z)$ and $\sigma_s(z)$ are the vertical profiles of the extinction and scattering coefficients.

Passage of radiation through the boundary is described by the uniformly finite operators of reflection $\hat{R}_1$ and $\hat{R}_2$ and transmission $\hat{T}_{12}$ and $\hat{T}_{21}$, where the subscript 1 relates to the atmospheric layer ($z \in [0, h]$) and 2 to the ocean layer ($z \in [h, H]$)

$$[\hat{R}_1\,\mathbf{\Phi}]\,(h, r_\perp, s) =$$
$$= \int\limits_{\Omega^+} \hat{q}_1\,(r_\perp, s, s^+)\,\mathbf{\Phi}(h, r_\perp, s^+)\,\mathrm{d}s^+, \; s \in \Omega^-;$$

$$[\hat{R}_2\,\mathbf{\Phi}]\,(h, r_\perp, s) =$$
$$= \int\limits_{\Omega^-} \hat{q}_2\,(r_\perp, s, s^-)\,\mathbf{\Phi}(h, r_\perp, s^-)\,\mathrm{d}s^-, \quad s \in \Omega^+;$$

$$[\hat{T}_{12}\,\mathbf{\Phi}]\,(h, r_\perp, s) =$$
$$= \int\limits_{\Omega^+} \hat{t}_{12}\,(r_\perp, s, s^+)\,\mathbf{\Phi}(h, r_\perp, s^+)\,\mathrm{d}s^+, \quad s \in \Omega^+;$$

$$[\hat{T}_{21}\,\mathbf{\Phi}]\,(h, r_\perp, s) =$$
$$= \int\limits_{\Omega^-} \hat{t}_{21}\,(r_\perp, s, s^-)\,\mathbf{\Phi}(h, r_\perp, s^-)\,\mathrm{d}s^-, \quad s \in \Omega^-.$$

The parameter ε ($0 \le \varepsilon \le 1$) defines the act of interaction between the radiation and the interface at $z = h$; $\hat{q}_1$ and $\hat{q}_2$ are the reflection phase matrixes; $\hat{t}_{12}$ and $\hat{t}_{21}$ are the transmission phase matrixes of the interface.

The uniformly finite operator describing the reflection of radiation from the system bottom contains the reflection phase matrix $\hat{q}$

$$[\hat{R}\,\mathbf{\Phi}]\,(H, r_\perp, s) =$$
$$= \int\limits_{\Omega^+} \hat{q}\,(r_\perp, s, s^+)\,\mathbf{\Phi}(H, r_\perp, s^+)\,\mathrm{d}s^+, \quad s \in \Omega^-,$$

The boundary problem (1) is linear, and its solution can be obtained as the superposition $\mathbf{\Phi} = \mathbf{\Phi}_0 + \mathbf{\Phi}_b$. The background radiation $\mathbf{\Phi}_0$ is found as a solution to the first vector boundary problem (FVBP) of the transfer theory assuming the "vacuum" conditions

$$\hat{K}\mathbf{\Phi}_0 = \mathbf{F}, \quad \mathbf{\Phi}_0\big|_t = \mathbf{F}^0,\ \mathbf{\Phi}_0\big|_b = \mathbf{F}^H, \quad \mathbf{\Phi}_0\big|_{d1} = \mathbf{F}^1,$$
$$\mathbf{\Phi}_0\big|_{d2} = \mathbf{F}^2 \qquad (2)$$

in the layer with ideally black (not reflecting and opaque) boundaries, that is having $\hat{R} \equiv 0$, $\hat{R}_1 \equiv 0$, $\hat{R}_2 \equiv 0$, $\hat{T}_{12} \equiv 0$, and $\hat{T}_{21} \equiv 0$ for the given sources of irradiation. It is sufficient that at least one of the functions in the right-hand sides of the system (2) is nonzero. The problem (2), for the layer $z \in [0, H]$, is split into two independent FVBPs: one for the layer $z \in [0, h]$

$$\hat{K}\mathbf{\Phi}_0^1 = \mathbf{F}_1; \quad \mathbf{\Phi}_0^1\big|_t = \mathbf{F}^0, \quad \mathbf{\Phi}_0^1\big|_{d1} = \mathbf{F}^1$$

and the other one for the layer $z \in [0, H]$

$$\hat{K}\mathbf{\Phi}_0^2 = \mathbf{F}_2, \quad \mathbf{\Phi}_0^2\big|_b = \mathbf{F}^H, \quad \mathbf{\Phi}_0^2\big|_{d2} = \mathbf{F}^2,$$

where $\mathbf{F}_1 = \mathbf{F}$ in the first medium; $\mathbf{F}_2 = \mathbf{F}$ in the second medium. Solution of such problems by the method of vector influence functions (VIF) is thoroughly described in Refs. 11 and 12.

The contribution $\mathbf{\Phi}_b$ due to the radiation exchange between the two media and the effect of reflecting bottom is determined as solution to the GVBP

$$\hat{K}\mathbf{\Phi}_b = 0,\ \mathbf{\Phi}_b\big|_t = 0,\ \mathbf{\Phi}_b\big|_b = \hat{R}\mathbf{\Phi}_b + \mathbf{E}^H,$$
$$\mathbf{\Phi}_b\big|_{d1} = \varepsilon\,(\hat{R}_1\mathbf{\Phi}_b + \hat{T}_{21}\,\mathbf{\Phi}_b + \mathbf{E}^1),$$
$$\mathbf{\Phi}_b\big|_{d2} = \varepsilon\,(\hat{R}_2\mathbf{\Phi}_b + \hat{T}_{12}\,\mathbf{\Phi}_b + \mathbf{E}^2) \qquad (3)$$

at a given illumination (irradiance, luminance) of the boundaries

$$\mathbf{E}^H(r_\perp, s) \equiv \hat{R}\mathbf{\Phi}_0,\ \ \mathbf{E}^1(r_\perp, s) \equiv \hat{R}_1\mathbf{\Phi}_0 + \hat{T}_{21}\,\mathbf{\Phi}_0,$$
$$\mathbf{E}^2(r_\perp, s) \equiv \hat{R}_2\mathbf{\Phi}_0 + \hat{T}_{12}\,\mathbf{\Phi}_0\ ,$$

when irradiated by the background radiation.

Without any loss of generality of the results obtained, one may limit the consideration to the GVBP

$$\hat{K}\mathbf{\Phi}_d = 0,\ \ \mathbf{\Phi}_d\big|_t = 0,\ \ \mathbf{\Phi}_d\big|_b = 0,$$

$$\mathbf{\Phi}_d\big|_{d1} = \varepsilon\,(\hat{R}_1\mathbf{\Phi}_d + \hat{T}_{21}\,\mathbf{\Phi}_d + \mathbf{E}^1),$$

$$\mathbf{\Phi}_d\big|_{d2} = \varepsilon\,(\hat{R}_2\mathbf{\Phi}_d + \hat{T}_{12}\,\mathbf{\Phi}_d + \mathbf{E}^2), \qquad (4)$$

that follows from the GVBP (3) for non-reflecting and non-emitting AOS bottom ($\hat{R} \equiv 0$, $\mathbf{F}^H \equiv 0$), and describes the effect of the radiation exchange between the two media through the inner boundary on the formation of the total radiation field of the system $\mathbf{\Phi} = \mathbf{\Phi}_0 + \mathbf{\Phi}_d$.

Solution of the GVBP (4) is sought as a perturbation series for two SVPs

$$\mathbf{\Phi}_d^1 = \sum_{k=1}^{\infty} \varepsilon^k\,\mathbf{\Phi}_k^1\,, \quad \mathbf{\Phi}_d^2 = \sum_{k=1}^{\infty} \varepsilon^k\,\mathbf{\Phi}_k^2\,, \qquad (5)$$

where $\mathbf{\Phi}_d^1$ corresponds to the radiation field in the layer $z \in [0, h]$, and $\mathbf{\Phi}_d^2$ to that in the layer $z \in [h, H]$. Components of the series (5) satisfy a recursion system that is split into the problems for one medium at $z \in [0, h]$

$$k = 1:\ \ \hat{K}\mathbf{\Phi}_1^1 = 0,\ \ \mathbf{\Phi}_1^1\big|_t = 0,\ \ \mathbf{\Phi}_1^1\big|_{d1} = \mathbf{E}^1; \qquad (6)$$

$$k \ge 2:\ \ \hat{K}\mathbf{\Phi}_k^1 = 0,\ \ \mathbf{\Phi}_k^1\big|_t = 0,$$

$$\mathbf{\Phi}_k^1\big|_{d1} = \hat{R}_1\,\mathbf{\Phi}_{k-1}^1 + \hat{T}_{21}\,\mathbf{\Phi}_{k-1}^2 \qquad (7)$$

and for the other one at $z \in [h, H]$

$$k = 1:\ \ \hat{K}\mathbf{\Phi}_1^2 = 0,\ \ \mathbf{\Phi}_1^2\big|_b = 0,\ \ \mathbf{\Phi}_1^2\big|_{d2} = \mathbf{E}^2; \qquad (8)$$

$$k \ge 2:\ \ \hat{K}\mathbf{\Phi}_k^2 = 0,\ \ \mathbf{\Phi}_k^2\big|_b = 0,$$

$$\mathbf{\Phi}_k^2\big|_{d2} = \hat{R}_2\,\mathbf{\Phi}_{k-1}^2 + \hat{T}_{12}\,\mathbf{\Phi}_{k-1}^1. \qquad (9)$$

Each of the problems (6) and (7) is a FVBP of the form

$$\hat{K}\mathbf{\Phi}^1 = 0,\ \mathbf{\Phi}^1\big|_t = 0,\ \mathbf{\Phi}^1\big|_{d1} = \mathbf{f}^1(s^1;\, r_\perp,\, s), \qquad (10)$$

and the problems (8) and (9) are the FVBPs of the form

$$\hat{K}\mathbf{\Phi}^2 = 0,\ \mathbf{\Phi}^2\big|_b = 0,\ \mathbf{\Phi}^2\big|_{d2} = \mathbf{f}^2(s^2;\, r_\perp,\, s). \qquad (11)$$

The parameters $s^1 \in \Omega^-$ and $s^2 \in \Omega^+$ may be absent.

INFLUENCE FUNCTIONS OF THE "ATMOSPHERE–OCEANB SYSTEM

Various possible states of polarization of a plane electromagnetic wave case are, in the general case, represented by the vector $\mathbf{\Phi}$ composed of four real values Φ_m, $m = 1, \ldots, M$, $M = 4$, which have the dimensionality of radiation intensity being the expansion coefficients of the vector $\mathbf{\Phi}$ over the unit vectors $\mathbf{i}_m$ of some coordinate system, $\mathbf{\Phi} = \mathbf{I}_1\,\Phi_1 + \mathbf{i}_2\,\Phi_2 + \mathbf{i}_3\,\Phi_3 + \mathbf{i}_4\,\Phi_4$, that depends on the way in which a polarized radiation is described.[4] Polarization states of radiation coming from a source, $\mathbf{f} = \{f_n\}$, $n = 1,\ldots, N$, $N \le 4$, and of radiation $\mathbf{\Phi}$ in the system may be different. Depending on optical properties of a medium that can scatter, absorb, and polarize the incoming radiation, the radiation in the layer may become polarized, as a result of the transfer process, even if the source emits unpolarized radiation. The state and (or) the degree of polarization of a polarized incident radiation may also be modified when propagated in the layer. Starting from some order of scattering the number of nonzero components of the radiation SVP may change and the situations are possible where $N \le M$ as well as $N \ge M$.

In the general case when components of the source radiation SVP, $\mathbf{f}^1 = \{f_n^1\}$, $n = 1,\ldots, N_1$, $N_1 \le 4$ are noncoinciding anisotropic horizontally inhomogeneous parameters $f_n^1(s^1;\, r_\perp,\, s)$, a solution to the linear FVBP (10) may be represented as a superposition

$$\mathbf{\Phi}^1(s^1;\, r,\, s) = \sum_{n=1}^{N_1} \mathbf{\Phi}_n^1(s^1;\, r,\, s)\,,$$

whose terms are solutions of a set of FVBPs

$$\hat{K}\mathbf{\Phi}_n^1 = 0,\quad \mathbf{\Phi}_n^1\big|_t = 0,\quad \mathbf{\Phi}_n^1\big|_{d1} = \mathbf{t}_n\,f_n^1 \qquad (12)$$

with the vectors $\mathbf{t}_n = \{\delta_{mn}\}$, $m = 1, \ldots, M_1$, $n = 1, \ldots, N_1$. Here δ_{mn} is the Kronecker symbol. By analogy with a scalar problem of the transfer theory,[15,16] a solution of the FVBP (12), at a fixed n, is obtained as a linear vector functional

$$\mathbf{\Phi}_n^1 = (\mathbf{\Theta}_n^1,\, f_n^1) = \frac{1}{2\pi}\int_{\Omega^-} \mathrm{d}s^- \times$$

$$\times \int_{-\infty}^{\infty} \Theta(s^-; z,\, r_\perp - r'_\perp,\, s)\, f_n^1(s^1;\, r'_\perp,\, s^-)\,\mathrm{d}r'_\perp\,.$$

The vector influence functions of the atmosphere $\mathbf{\Theta}_n^1 = \{\Theta_{mn}^1\}$, $n = 1,\ldots, N_1$, with the components being the Stokes parameters $\Theta_{mn}^1(s^-;\, z,\, r_\perp,\, s)$, $m = 1,\ldots, M_1$, are sought as a solution to the set of FVBP for the layer $z \in [0, h]$

$$\hat{K}\mathbf{\Theta}_n^1 = 0;\quad \mathbf{\Theta}_n^1\big|_t = 0,\quad \mathbf{\Theta}_n^1\big|_{d1} = \mathbf{t}_n\,f_\delta^1 \qquad (13)$$

with the function of source $f_\delta^1(s^-;\, r_\perp,\, s) = \delta(r_\perp) \times \delta(s - s^-)$ and parameter $s^- \in \Omega^-$. Components of the SVP, $\mathbf{\Phi}_n^1 = \{\Phi_{mn}^1\}$, are calculated as scalar functionals

$$\Phi_{mn}^1(s^1;\, z,\, r_\perp,\, s) = (\Theta_{mn}^1,\, f_n^1) = \frac{1}{2\pi}\int_{\Omega^-} \mathrm{d}s^- \times$$

$$\times \int_{-\infty}^{\infty} \Theta_{mn}^1(s^-;\, z,\, r_\perp - r'_\perp,\, s)\, f_n^1(s^1;\, r'_\perp,\, s^-)\,\mathrm{d}r'_\perp. \qquad (14)$$

Following our previous papers,[11,12] where we have proposed to do this for the first time, let us introduce the tensor of influence functions (IFT) of the atmosphere determined in terms of N_1 of SVPs, $\mathbf{\Theta}_n^1$,

$$\hat{\Pi}^1 = \begin{bmatrix} \Theta^1_{11} & \ldots\ \Theta^1_{1n} & \ldots\ \Theta^1_{1N_1} \\ \ldots & \ldots\ \ldots & \ldots\ \ldots \\ \Theta^1_{m1} & \ldots\ \Theta^1_{mn} & \ldots\ \Theta^1_{mN_1} \\ \ldots & \ldots\ \ldots & \ldots\ \ldots \\ \Theta^1_{M_1 1} & \ldots\ \Theta^1_{M_1 n} & \ldots\ \Theta^1_{M_1 N_1} \end{bmatrix}. \quad (15)$$

The first index, $m = 1, \ldots, M_1$, $M_1 \le 4$, of the element Θ^1_{mn} of the tensor $\hat{\Pi}^1$ corresponds to an ordinal number of the Stokes parameter of VIF $\mathbf{\Theta}^1_n$, and the second one, $n = 1, \ldots, N_1$, $N_1 \le 4$, to the index of the source vector $\mathbf{t}_n$ in the set of problems (13) describing the model used to calculate VIF $\mathbf{\Theta}^1_n$.

Let us introduce a linear vector functional of the vector $\mathbf{f}^1$

$$\mathbf{\Phi}^1 = (\hat{\Pi}^1, \mathbf{f}^1) = \{\Phi^1_m\},\ m = 1,\ldots,M_1,\ M_1 \le 4, \quad (16)$$

where the Stokes parameters are solutions of the FVBP (10);

$$\Phi^1_m = \sum_{n=1}^{N_1} (\Theta^1_{mn}, f^1_n) = \sum_{n=1}^{N_1} \Phi^1_{mn}$$

are linear combinations of the linear scalar functionals (14).

If a source is isotropic and horizontally inhomogeneous, then a solution to the FVBP (10) is determined by the vector linear functionals

$$\mathbf{\Phi}^1_n(z, r_\perp, s) = (\mathbf{\Theta}^1_{rn}, f^1_n) =$$

$$= \int_{-\infty}^{\infty} \mathbf{\Theta}^1_{rn}(z, r_\perp - r'_\perp, s)\, f^1_n(r'_\perp)\, \mathrm{d}r'_\perp ,$$

with the kernels that are VIF of the atmosphere

$$\mathbf{\Theta}^1_{rn}(z, r_\perp, s) = \frac{1}{2\pi} \int_{\Omega^-} \mathbf{\Theta}^1_n(s^-; z, r_\perp, s)\, \mathrm{d}s^- \quad (17)$$

which satisfy the FVBP

$$\hat{K}\mathbf{\Theta}^1_{rn} = 0,\ \ \mathbf{\Theta}^1_{rn}\big|_t = 0,\ \ \mathbf{\Theta}^1_{rn}\big|_{d1} = \mathbf{t}_n\, \delta(r_\perp). \quad (18)$$

In the case of an anisotropic and horizontally homogeneous source a solution to the problem (10) is sought as a linear functional

$$\mathbf{\Phi}^1_n(s^1; z, s) = (\mathbf{\Theta}^1_{zn}, f^1_n) =$$

$$= \frac{1}{2\pi} \int_{\Omega^-} \mathbf{\Theta}^1_{zn}(s'; z, s)\, f^1_n(s^1; s')\, \mathrm{d}s'$$

with the kernel that is VIF of the atmosphere

$$\mathbf{\Theta}^1_{zn}(s^-; z, s) = \int_{-\infty}^{\infty} \mathbf{\Theta}^1_n(s^-; z, r_\perp, s)\, \mathrm{d}r_\perp, \quad (19)$$

which is a solution of a one-dimensional FVBP

$$\hat{K}_z\, \mathbf{\Theta}^1_{zn} = 0,\ \mathbf{\Theta}^1_{zn}\big|_t = 0,$$

$$\mathbf{\Theta}^1_{zn}\big|_{d1} = \mathbf{t}_n\, \delta(s - s^-);\ s^- \in \Omega^-. \quad (20)$$

In the case of an isotropic and horizontally homogeneous source a solution of the problem (10)

$$\mathbf{\Phi}^1_n(z, s) = f^1_n\, \mathbf{W}^1_n(z, s),\quad f^1_n = \mathrm{const},$$

is calculated in terms the VIF of the atmosphere

$$\mathbf{W}^1_n(z, s) = \frac{1}{2\pi} \int_{\Omega^-} \mathrm{d}s^- \int_{-\infty}^{\infty} \mathbf{\Theta}^1_n(s^-; z, r_\perp, s)\, \mathrm{d}r_\perp =$$

$$= \int_{-\infty}^{\infty} \mathbf{\Theta}^1_{rn}(z, r_\perp, s)\, \mathrm{d}r_\perp = \frac{1}{2\pi} \int_{\Omega^-} \mathbf{\Theta}^1_{zn}(s^-; z, s)\, \mathrm{d}s^{-1}, \quad (21)$$

It is also called a vector transmission function that allows for multiple scattering and is determined as a solution of a one-dimensional FVBP[4,13]

$$\hat{K}_z\, \mathbf{W}^1_n = 0,\ \ \mathbf{W}^1_n\big|_t = 0,\ \ \mathbf{W}^1_n\big|_{d1} = \mathbf{t}_n. \quad (22)$$

The relations (17), (19), and (21) can be used as criteria for estimating the accuracy of the VIF $\mathbf{\Theta}^1_n$, $\mathbf{\Theta}^1_{rn}$, and $\mathbf{\Theta}^1_{zn}$ calculation in terms of solutions of less complicated problems (18), (20), and (22). Actually the influence function tensor $\hat{\Pi}^1$ (15) determined by VIF $\mathbf{\Theta}^1_n(s^-; z, r_\perp, s)$ describes the field of polarized radiation in a layer with non-reflecting boundaries. This field is caused by the processes of multiple scattering of a stationary, elliptically polarized radiation in a narrow beam along the direction $s^- \in \Omega^-$, whose source is at the boundary $z = h$ at the center of a horizontal coordinate system x, y. The parameter of VIF, $\mathbf{\Theta}^1_{rn}(z, r_\perp, s)$, corresponding to the radiation intensity coincides with the point spread function and its Fourier image over $r_\perp$ in the nadir direction when $s = (\mu = -1, \varphi = 0)$, coincides with the modulation transfer function. The $\hat{\Pi}^1$ tensor determined by the VIF $\mathbf{\Theta}^1_{zn}(s^-; z, s)$ describes the field of a polarized radiation formed in the layer when a parallel beam of elliptically polarized radiation is incident from outside the layer boundary, at $z = h$, along the direction $s^- \in \Omega^-$. The vector influence functions $\mathbf{\Theta}^1_n$, $\mathbf{\Theta}^1_{rn}$, $\mathbf{\Theta}^1_{zn}$, and $\mathbf{W}^1_n$ compose a complete set of basic models of influence functions of FVBP (10).

In the general case, when the components of the Stokes vector parameter of a source $\mathbf{f}^2 = \{f^2_n\}$, $n = 1, \ldots, N_2$, $N_2 \le 4$, are anisotropic horizontally inhomogeneous parameters $f^2_n(s^2; r_\perp, s)$, a solution to the linear FVBP (11) can be represented as a superposition

$$\mathbf{\Phi}^2(s^2; r, s) = \sum_{n=1}^{N_2} \mathbf{\Phi}^2_n(s^2; r, s)\ ,$$

whose terms are a solution of a set of the FVBPs

$$\hat{K}\,\mathbf{\Phi}_n^2 = 0, \quad \mathbf{\Phi}_n^2\big|_b = 0, \quad \mathbf{\Phi}_n^2\big|_{d2} = \mathbf{t}_n\, f_n^2. \tag{23}$$

A solution of FVBP (23) at a fixed n is obtained as a vector linear functional

$$\mathbf{\Phi}_n^2 = (\mathbf{\Theta}_n^2, f_n^2) = \frac{1}{2\pi}\int_{\Omega^+} \mathrm{d}s^+ \times$$

$$\times \int_{-\infty}^{\infty} \mathbf{\Theta}_n^2(s^+;\, z,\, r_\perp - r'_\perp,\, s)\, f_n^2(s^2;\, r'_\perp,\, s^+)\mathrm{d}r'_\perp .$$

The vector influence functions of the ocean $\mathbf{\Theta}_n^2 = \{\Theta_{mn}^2\}$, $n = 1,\dots, N_2$, whose components are the Stokes parameters $\Theta_{mn}^2(s^+;\, z,\, r_\perp,\, s)$, $m = 1,\dots, M_2$, are sought as a solution to the set of FVBPs for the layer $z \in [h, H]$

$$\hat{K}\mathbf{\Theta}_n^2 = 0, \quad \mathbf{\Theta}_n^2\big|_b = 0, \quad \mathbf{\Theta}_n^2\big|_{d2} = \mathbf{t}_n\, f_\delta^2 \tag{24}$$

with the source $f_\delta^2(s^+;\, r_\perp,\, s) = \delta(r_\perp)\, \delta(s - s^+)$ and the parameter $s^+ \in \Omega^+$. The components of SVP $\mathbf{\Phi}_n^2 = \{\Phi_{mn}^2\}$ are calculated as scalar functionals

$$\Phi_{mn}^2(s^2;\, z,\, r_\perp,\, s) = (\Theta_{mn}^2, f_n^2) =$$

$$= \frac{1}{2\pi}\int_{\Omega^+} \mathrm{d}s^+ \int_{-\infty}^{\infty} \Theta_{mn}^2(s^+;\, z,\, r_\perp - r'_\perp,\, s) \times$$

$$\times f_n^2(s^2;\, r'_\perp,\, s^+)\mathrm{d}r'_\perp . \tag{25}$$

Let us introduce the IFT of the ocean determined by N_2 of SVPs, $\mathbf{\Theta}_n^2$

$$\hat{\Pi}^2 = \begin{bmatrix} \Theta_{11}^2 & \dots & \Theta_{1n}^2 & \dots & \Theta_{1N_2}^2 \\ \dots & \dots & \dots & \dots & \dots \\ \Theta_{m1}^2 & \dots & \Theta_{mn}^2 & \dots & \Theta_{mN_2}^2 \\ \dots & \dots & \dots & \dots & \dots \\ \Theta_{M_2 1}^2 & \dots & \Theta_{M_2 n}^2 & \dots & \Theta_{M_2 N_2}^2 \end{bmatrix} \tag{26}$$

and linear vector functional of the vector $\mathbf{f}^2$ in the form

$$\mathbf{\Phi}^2 = (\hat{\Pi}^2, \mathbf{f}^2) = \{\Phi_m^2\},\ m = 1, \dots, M_2, \quad M_2 \le 4, \tag{27}$$

where the Stokes parameters are solutions of the FVBP (11) and

$$\Phi_m^2 = \sum_{n=1}^{N_2} (\Theta_{mn}^2, f_n^2) = \sum_{n=1}^{N_2} \Phi_{mn}^2$$

are the linear combinations of the linear scalar functionals (25).

If the radiation source is isotropic and horizontally inhomogeneous, then a solution to the FVBP (11) is determined by the vector linear functionals

$$\mathbf{\Phi}_n^2(z,\, r_\perp,\, s) = (\mathbf{\Theta}_{rn}^2, f_n^2) =$$

$$= \int_{-\infty}^{\infty} \mathbf{\Theta}_{rn}^2(z,\, r_\perp - r'_\perp,\, s)\, f_n^2(r'_\perp)\mathrm{d}r'_\perp,$$

whose kernels are the VIF of the ocean

$$\mathbf{\Theta}_{rn}^2(z,\, r_\perp,\, s) = \frac{1}{2\pi}\int_{\Omega^+} \mathbf{\Theta}_n^2(s^+;\, z,\, r_\perp,\, s)\, \mathrm{d}s^+ \tag{28}$$

that satisfy the conditions of the FVBP

$$\hat{K}\mathbf{\Theta}_{rn}^2 = 0, \quad \mathbf{\Theta}_{rn}^2\big|_b = 0, \quad \mathbf{\Theta}_{rn}^2\big|_{d2} = \mathbf{t}_n\, \delta(r_\perp). \tag{29}$$

For an anisotropic horizontally homogeneous source a solution to the problem (11) is sought by the vector linear functional

$$\mathbf{\Phi}_n^2(s^2;\, z,\, s) = (\mathbf{\Theta}_{zn}^2, f_n^2) =$$

$$= \frac{1}{2\pi}\int_{\Omega^+} \mathbf{\Theta}_{zn}^2(s';\, z,\, s)\, f_n^2(s^2;\, s')\, \mathrm{d}s'$$

with the kernel that is a VIF of the ocean

$$\mathbf{\Theta}_{zn}^2(s^+;\, z,\, s) = \int_{-\infty}^{\infty} \mathbf{\Theta}_n^2(s^+;\, z,\, r_\perp,\, s)\, \mathrm{d}r_\perp, \tag{30}$$

which is a solution of the one-dimensional FVBP

$$\hat{K}_z\, \mathbf{\Theta}_{zn}^2 = 0,\ \mathbf{\Theta}_{zn}^2\big|_b = 0,$$

$$\mathbf{\Theta}_{zn}^2\big|_{d2} = \mathbf{t}_n\, \delta(s - s^+);\ s^+ \in \Omega^+. \tag{31}$$

For isotropic horizontally homogeneous source a solution of the problem (11)

$$\mathbf{\Phi}_n^2(z,\, s) = f_n^2\, \mathbf{W}_n^2(z,\, s), \quad f_n^2 = \mathrm{const},$$

is calculated using the VIF of the ocean

$$\mathbf{W}_n^2(z,\, s) = \frac{1}{2\pi}\int_{\Omega^+} \mathrm{d}s^+ \int_{-\infty}^{\infty} \mathbf{\Theta}_n^2(s^+;\, z,\, r_\perp,\, s)\, \mathrm{d}r_\perp =$$

$$= \int_{-\infty}^{\infty} \mathbf{\Theta}_{rn}^2(z,\, r_\perp,\, s)\, \mathrm{d}r_\perp = \frac{1}{2\pi}\int_{\Omega^+} \mathbf{\Theta}_{zn}^2(s^+;\, z,\, s)\, \mathrm{d}s^+, \tag{32}$$

which is determined as a solution of a one-dimensional FVBP in the layer $z \in [h, H]$

$$\hat{K}_z\mathbf{W}_n^2 = 0, \quad \mathbf{W}_n^2\big|_b = 0, \quad \mathbf{W}_n^2\big|_{d2} = \mathbf{t}_n. \tag{33}$$

The vector influence functions of the ocean $\mathbf{\Theta}_n^2$, $\mathbf{\Theta}_{rn}^2$, $\mathbf{\Theta}_{zn}^2$, and $\mathbf{W}_n^2$ which are the solutions of the FVBPs (24), (29), (31), and (33) related by the expressions (28), (30), and (32) compose a complete set of basic models of influence functions of the FVBP (11).

VECTOR OPTICAL TRANSFER OPERATOR OF THE AOS

Let us use the models of VIF formulated above and representations of solutions to the FVBPs (10) and (11) in the form of vector linear functionals (16) and (27) whose kernels are the IFT (15) and (26) to construct a solution of the GVBP (4). If a source in the GVBP (4) is determined by a single interaction of the background radiation with the boundary, then a power of the parameter ε corresponds to the power in the dependence of the solution of the problem (4) on the characteristics of the operators of reflection $\hat{q}_1$ and $\hat{q}_2$ and transmission $\hat{t}_{12}$ and $\hat{t}_{21}$.

Let us introduce algebraic vectors as columns

$$\mathbf{\Phi}_d = \begin{bmatrix} \mathbf{\Phi}_d^1 \\ \mathbf{\Phi}_d^2 \end{bmatrix}, \ \mathbf{\Phi}_k = \begin{bmatrix} \mathbf{\Phi}_k^1 \\ \mathbf{\Phi}_k^2 \end{bmatrix}, \ \mathbf{E} = \begin{bmatrix} \mathbf{E}^1 \\ \mathbf{E}^2 \end{bmatrix},$$

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \end{bmatrix}, \ \mathbf{Z} = \begin{bmatrix} \mathbf{Z}_a \\ \mathbf{Z}_{oc} \end{bmatrix},$$

$$\mathbf{\Theta} = \begin{bmatrix} \mathbf{\Theta}^1 \\ \mathbf{\Theta}^2 \end{bmatrix}, \ \hat{\Pi} = \begin{bmatrix} \hat{\Pi}^1 \\ \hat{\Pi}^2 \end{bmatrix}, \ (\hat{\Pi}, \mathbf{f}) = \begin{bmatrix} (\hat{\Pi}^1, \ \mathbf{f}^1) \\ (\hat{\Pi}^2, \ \mathbf{f}^2) \end{bmatrix}$$

and define the matrix operation describing the radiation passage through the boundary by IFT taking into account multiple scattering and polarization of radiation in the two media

$$[\hat{G}\mathbf{f}] \equiv \hat{P}(\hat{\Pi}, \mathbf{f}) = \begin{bmatrix} \hat{R}_1(\hat{\Pi}^1,\mathbf{f}^1) + \hat{T}_{21}(\hat{\Pi}^2,\mathbf{f}^2) \\ \hat{R}_2(\hat{\Pi}^2,\mathbf{f}^2) + \hat{T}_{12}(\hat{\Pi}^1,\mathbf{f}^1) \end{bmatrix}, \tag{34}$$

where P is the matrix composed of the reflection and transmission operators:

$$\hat{P} \equiv \begin{bmatrix} \hat{R}_1 & \hat{T}_{21} \\ \hat{T}_{12} & \hat{R}_2 \end{bmatrix}.$$

The boundary problems (6) and (8) for a linear approximation are solved using the vector linear functionals (16) and (27):

$$\mathbf{\Phi}_1 = \begin{bmatrix} \mathbf{\Phi}_1^1 \\ \mathbf{\Phi}_1^2 \end{bmatrix} = \begin{bmatrix} (\hat{\Pi}^1, \ \mathbf{E}^1) \\ (\hat{\Pi}^2, \ \mathbf{E}^2) \end{bmatrix} = (\hat{\Pi}, \mathbf{E}) .$$

It can be shown, by the induction method, that two successive k-approximations are connected by the recursion relation

$$\mathbf{\Phi}_k = (\hat{\Pi}, \hat{P}\,\mathbf{\Phi}_{k-1})$$

and for $k \geq 1$ (assume $\mathbf{F}_0 \equiv \mathbf{E}$) an algebraic vector of a source is

$$\mathbf{F}_k = \hat{P}\,\mathbf{\Phi}_k = \hat{G}\,\mathbf{F}_{k-1} = \hat{G}^k\,\mathbf{E} ,$$

and the algebraic vector of the k-approximation of the solution to FVBP (7) and (9) is

$$\mathbf{\Phi}_k = (\hat{\Pi}, \mathbf{F}_{k-1}) = (\hat{\Pi}, \hat{G}^{k-1}\,\mathbf{E}).$$

As a result we obtain an asymptotically accurate solution of the GVBP (4)

$$\mathbf{\Phi}_d = (\hat{\Pi}, \mathbf{Z}), \tag{35}$$

where the two-component vector of "scenario" at the boundary

$$\mathbf{Z} \equiv \hat{Z}\mathbf{E} \equiv \sum_{k=0}^{\infty} \hat{G}^k\,\mathbf{E} \tag{36}$$

is the sum of Neumann series over the orders of radiation passage through the boundary taking into account the contribution from multiple scattering as well as the polarization of radiation in two media with IFT of the view (15) and (26).

The representation of a solution to the GVBP (4) as a linear vector functional (35) establishing an explicit relation of the radiation recorded to the "scenario" (36) at two sides of the boundary we call the optical vector transfer operator of the transfer system in two media. In its turn, the "scenario" is described explicitly by the characteristics of reflection and transmission of the boundary for given its illumination. The Neumann series (36) determines the "scenario" of optical image formed as a result of multiple scattering of radiation in the two media and passage of the boundary taking into account mechanisms of polarization and depolarization both in the layer and at the boundary. Naturally, the universal representation of VOTO (35) is extended to all cases of the spatial and angular dependence of characteristics of the boundary and the sources considered above.

STRUCTURE OF RADIATION FIELD

Proposed approach allows one to study mechanisms of formation of field of optical and millimeter-wave polarized radiation in AOS in detail and to obtain different approximations of the VOTO.

Let us consider in a more detail solution of the problem (4) when sources are the singular direct flux $\mathbf{\Phi}^0$ incident from the atmosphere along the direction $s_0 = (\mu_0, \varphi_0) \in \Omega^+$ and the down going diffuse background radiation $\mathbf{\Phi}_\alpha$ multiply scattered in the atmosphere. We shall do this for directions $s^+ = (\mu^+, \varphi^+) \in \Omega^+$: $\mathbf{\Phi}_0 = \mathbf{\Phi}^0 + \mathbf{\Phi}_\alpha$. For the sake of clarity the label "1" used for the atmospheric layer we replace by the label "a" and the label "2" for the ocean layer for the label "oc", and also the symbols $\tilde{z} = z - h$, $\tilde{\mu}$, and $\tilde{\varphi}$ are introduced for coordinates in the ocean. In this case the functions of sources located from the side of the atmosphere and ocean, with respect to the interface, at the height $z = h$, between them are as follows:

$$\mathbf{E}^1 = \mathbf{E}_a = \mathbf{E}_a^s + \mathbf{E}_a^d, \ \ \mathbf{E}^2 = \mathbf{E}_{oc} = \mathbf{E}_{oc}^s + \mathbf{E}_{oc}^d$$

and can contain the singular components:

$$\mathbf{E}_a^s(\mu_0, \varphi_0; h, \mu^-, \varphi^-) = \hat{R}_1\,\mathbf{\Phi}^0 =$$
$$= \mathbf{E}_a^s(\mu_0, \varphi_0; h, -\mu_0, \varphi_0)\,\delta(\mu^- + \mu_0)\,\delta(\varphi^- - \varphi_0)$$

which is the direct flux reflected into the atmosphere from the interface along the direction $s_0^- = (-\mu_0, \varphi_0) \in \Omega^-$,

$$\mathbf{E}_{oc}^s(\mu_0, \varphi_0; h, \bar{\mu}^+, \varphi^+) = \hat{T}_{12}\,\mathbf{\Phi}^0 =$$

$= \mathbf{E}^s_{oc}(\mu_0, \varphi_0; h, \bar\mu_0, \varphi_0)\, \delta(\bar\mu^+ - \bar\mu_0)\, \delta(\varphi^+ - \varphi_0)$

being the direct flux refracted into the ocean through the interface along the direction $\bar s_0 = (\bar\mu_0, \varphi_0) \in \Omega^+_{crit}$, $\bar\mu_0 \geq \bar\mu_{crit} > 0$, and smooth diffusion components:

$\mathbf{E}^d_a(\mu_0, \varphi_0; h, \mu^-, \varphi^-) = \hat R_1\, \mathbf{\Phi}^+_a$

that is the background radiation of the atmosphere reflected from the interface into the atmosphere along the directions $s^- = (\mu^-, \varphi^-) \in \Omega^-$; $\mu^- = -\mu^+$; and $\varphi^- = \varphi^+$, and

$\mathbf{E}^d_{oc}(\mu_0, \varphi_0; h, \bar\mu^+, \varphi^+) = \hat T_{12}\, \mathbf{\Phi}^+_a$

or the background radiation of the atmosphere refracted into the ocean through the boundary along the directions $\bar s^+ = (\bar\mu^+, \varphi^+) \in \Omega^+_{crit}$; $\bar\mu^+ \in [\bar\mu_{crit}, 1]$, where $\bar\mu_{crit}$ corresponds to the direction of the interface shadow in the ocean.

Components of the algebraic source vector $\mathbf{F}_k$ for the problems (6)–(9) are pairs of Stokes vector-parameters

$\mathbf{F}_{a,k} \equiv \hat R_1\, \mathbf{\Phi}^+_{a,k} + \hat T_{21}\, \mathbf{\Phi}^-_{oc,k}, \quad \mathbf{F}_{a,0} \equiv \mathbf{E}_a,$

$\mathbf{F}_{oc,k} \equiv \hat R_2\, \mathbf{\Phi}^-_{oc,k} + \hat T_{12}\, \mathbf{\Phi}^+_{a,k}, \quad \mathbf{F}_{oc,0} \equiv \mathbf{E}_{oc}.$

Solutions of the problems (6) and (7) are defined in terms of IFT of the atmosphere (15), that is, $\hat\Pi_a = \hat\Pi^s_a + \hat\Pi^d_a$ with the *mn*-elements being

$\Theta_a(\mu^-_h, \varphi^-_h; z, \mu, \varphi) = \Theta^s_a + \Theta^d_a$

that are solutions of the problems (13) with the parameter $s^-_h = (\mu^-_h, \varphi^-_h) \in \Omega^-$. In these solutions the singular components are being separated out

$$\Theta^s_a(\mu^-_h, \varphi^-_h; z, \mu, \varphi) =$$
$$= f_a \exp\left[-\frac{\tau(h) - \tau(z)}{|\mu^-_h|}\right] \delta(\mu - \mu^-_h)\, \delta(\varphi - \varphi^-_h)$$

and the diffusion components are smooth functions $\Theta^d_a(\mu^-_h, \varphi^-_h; z, \mu, \varphi)$ with the parameters $\mu^-_h \in [-1, 0)$ and $\varphi^-_h = 0$. In this case the linear functionals (16) are calculated as sums of four linear functionals

$$\mathbf{\Phi}_{a,1} = (\hat\Pi_a, \mathbf{E}_a) = (\hat\Pi^s_a, \mathbf{E}^s_a) + (\hat\Pi^d_a, \mathbf{E}^s_a) +$$
$$+ (\hat\Pi^s_a, \mathbf{E}^d_a) + (\hat\Pi^d_a, \mathbf{E}^d_a)\ .$$

Actually, the latter expression is a superposition

$$\mathbf{\Phi}_{a,1} = \mathbf{\Phi}^0_{a,1} + \mathbf{\Phi}^d_{a,1}\ ,$$

where the direct radiation from the interface is being determined only for the directions $s^- \in \Omega^-$

$$\mathbf{\Phi}^0_{a,1} = \mathbf{\Phi}^{0,s}_{a,1} + \mathbf{\Phi}^{0,d}_{a,1}$$

and contains the singular part along the directions $s^-_0 = (-\mu_0, \varphi_0) \in \Omega^-$ only

$$\mathbf{\Phi}^{0,s}_{a,1}(\mu_0, \varphi_0; z, \mu^-, \varphi^-) = (\hat\Pi^s_a, \mathbf{E}^s_a) \neq 0$$

and the smooth part

$$\mathbf{\Phi}^{0,d}_{a,1}(\mu_0, \varphi_0; z, \mu^-, \varphi^-) = (\hat\Pi^s_a, \mathbf{E}^d_a)\ ,$$

that are calculated explicitly; while the contribution due to diffusion, being determined for all directions $s \in \Omega$

$$\mathbf{\Phi}^d_{a,1} = \mathbf{\Phi}^{d,s}_{a,1} + \mathbf{\Phi}^{d,d}_{a,1}$$

contains the component caused by multiple scattering in the atmosphere of the direct flux reflected from the interface which is calculated explicitly in terms of the diffusion component of the atmospheric IFT

$$\mathbf{\Phi}^{d,s}_{a,1}(\mu_0, \varphi_0; z, \mu, \varphi) = (\hat\Pi^d_a, \mathbf{E}^s_a)$$

and the component caused by multiple scattering in the atmosphere of the diffuse background radiation reflected from the interface and calculated for every *mn*-component through the functional with the IFT of the atmosphere by the quadrature method

$$\Phi^{d,d}_{a,1}(\mu_0, \varphi_0; z, \mu, \varphi) = (\Theta^d_a, E^d_a) =$$
$$= \frac{1}{2\pi}\int\limits_0^\pi d\varphi' \int\limits_{-1}^0 [\Theta^d_a(\mu', 0; z, \mu, \varphi - \varphi') +$$
$$+ \Theta^d_a(\mu', 0; z, \mu, \varphi + \varphi')]\, E^d_a(\mu_0, \varphi_0; h, \mu', \varphi')\, d\mu'.$$

Solutions of the problems (8) and (9) are determined by the IFT of the ocean (26), $\hat\Pi_{oc} = \hat\Pi^s_{oc} + \hat\Pi^d_{oc}$, with the *mn*-elements

$\Theta_{oc}(\tilde\mu^+_h, \tilde\varphi^+_h; \tilde z, \tilde\mu, \tilde\varphi) = \Theta^s_{oc} + \Theta^d_{oc}$

that are solutions of the problems (24) with the parameter $\tilde s^+_h = (\tilde\mu^+_h, \tilde\varphi^+_h) \in \Omega^+$. In these solutions the singular components are separated out

$$\Theta^s_{oc}(\tilde\mu^+_h, \tilde\varphi^+_h; \tilde z, \tilde\mu^+, \tilde\varphi^+) =$$
$$= f_{oc} \exp\left[-\frac{\tau(\tilde z)}{\tilde\mu^+_h}\right] \delta(\tilde\mu^+ - \tilde\mu^+_h)\, \delta(\tilde\varphi^+ - \tilde\varphi^+_h)$$

and the diffusion components, are smooth functions $\Theta^d_{oc}(\tilde\mu^+_h, \tilde\varphi^+_h; \tilde z, \tilde\mu, \tilde\varphi)$ with the parameters $\tilde\mu^+_h \in [0, 1)$ and $\tilde\varphi^+_h = 0$. The linear functionals (27) are calculated assuming four terms

$$\mathbf{\Phi}_{oc,1} = (\hat\Pi_{oc}, \mathbf{E}_{oc}) = (\hat\Pi^s_{oc}, \mathbf{E}^s_{oc}) +$$
$$+ (\hat\Pi^d_{oc}, \mathbf{E}^s_{oc}) + (\hat\Pi^s_{oc}, \mathbf{E}^d_{oc}) + (\hat\Pi^d_{oc}, \mathbf{E}^d_{oc}).$$

The latter expression can be represented as a superposition

$$\mathbf{\Phi}_{oc,1} = \mathbf{\Phi}^0_{oc,1} + \mathbf{\Phi}^d_{oc,1}\ ,$$

where the direct radiation from the interface being determined only for the directions $\tilde s^+ \in \Omega^+$

$$\mathbf{\Phi}^0_{oc,1} = \mathbf{\Phi}^{0,s}_{oc,1} + \mathbf{\Phi}^{0,d}_{oc,1}$$

contains the singular part along the directions $\bar{s}_0^+ = (\bar{\mu}_0, \varphi_0) \in \Omega^+_{crit}$ only

$$\mathbf{\Phi}^{0,s}_{oc,1}(\bar{\mu}_0, \varphi_0; \tilde{z}, \tilde{\mu}^+, \tilde{\varphi}^+) = (\hat{\Pi}^s_{oc}, \mathbf{E}^s_{oc}) \neq 0$$

and the smooth one

$$\mathbf{\Phi}^{0,d}_{oc,1}(\mu_0, \varphi_0; \tilde{z}, \tilde{\mu}^+, \tilde{\varphi}^+) = (\hat{\Pi}^s_{oc}, \mathbf{E}^d_{oc}) ,$$

which are calculated explicitly; while the diffusion contribution determined for all directions $\tilde{s} \in \Omega$

$$\mathbf{\Phi}^d_{oc,1} = \mathbf{\Phi}^{d,s}_{oc,1} + \mathbf{\Phi}^{d,d}_{oc,1}$$

contains the component due to multiple scattering in the ocean of direct radiation from the atmosphere refracted through the interface and explicitly calculated using the IFT of the ocean

$$\mathbf{\Phi}^{d,s}_{oc,1}(\bar{\mu}_0, \varphi_0; \tilde{z}, \tilde{\mu}, \tilde{\varphi}) = (\hat{\Pi}^d_{oc}, \mathbf{E}^s_{oc}).$$

The component caused by multiple scattering in ocean of the background radiation of the atmosphere refracted through the interface and calculated for every *mn*-component using the IFT of the ocean by the quadrature method

$$\mathbf{\Phi}^{d,d}_{oc,1}(\mu_0, \varphi_0; \tilde{z}, \tilde{\mu}, \tilde{\varphi}) = (\Theta^d_{oc}, E^d_{oc}) =$$

$$= \frac{1}{2\pi} \int\limits_0^{\pi} d\varphi' \int\limits_{-1}^{0} [\Theta^d_{oc}(\mu', 0; \tilde{z}, \tilde{\mu}, \tilde{\varphi} - \varphi') +$$

$$+ \Theta^d_{oc}(\mu', 0 ;\tilde{z}, \tilde{\mu}, \tilde{\varphi} + \varphi')] \, E^d_{oc}(\mu_0, \varphi_0; h, \mu', \varphi') \, d\mu'.$$

In each approximation at $n \geq 2$ in the iteration cycle over the orders of radiation interaction with the interface only the diffusion sources enter the problems (7) and (9). Along the direction from the atmosphere it is

$$\mathbf{F}_{a,k} = \mathbf{F}^d_{a,k}(h, \mu^-, \varphi^-) = \mathbf{F}^{d,a}_{a,k} + \mathbf{F}^{d,oc}_{a,k},$$

where the first term corresponds to the influence of radiation coming from the atmosphere

$$\mathbf{F}^{d,a}_{a,k}(h, \mu^-, \varphi^-) = \hat{R}_1 \, \mathbf{\Phi}^{d+}_{a,k},$$

and the second term corresponds to the influence of radiation coming from the ocean

$$\mathbf{F}^{d,oc}_{a,k}(h, \mu^-, \varphi^-) = \hat{T}_{21} \, \mathbf{\Phi}^{d-}_{oc,k}.$$

Along the direction from the ocean it is

$$\mathbf{F}_{oc,k} = \mathbf{F}^d_{oc,k}(h, \tilde{\mu}^+, \tilde{\varphi}^+) = \mathbf{F}^{d,a}_{oc,k} + \mathbf{F}^{d,oc}_{oc,k} ,$$

where the first term describes the influence of radiation coming from the atmosphere

$$\mathbf{F}^{d,a}_{oc,k}(h, \bar{\mu}^+, \bar{\varphi}^+) = \hat{T}_{12} \, \mathbf{\Phi}^{d+}_{a,k},$$

and the second one describes the influence of radiation coming from the ocean

$$\mathbf{F}^{d,oc}_{oc,k} (h, \tilde{\mu}^+, \tilde{\varphi}^+) = \hat{R}_2 \mathbf{\Phi}^{d-}_{oc,k}.$$

Note that in order to calculate sources, only two angular distributions of the diffuse radiation are needed. The one for radiation incident along the direction from the atmosphere $\mathbf{\Phi}^{d+}_{a,k}(h, \mu^+, \varphi^+)$ and the other for that incident on the interface along the direction from ocean $\mathbf{\Phi}^{d-}_{oc,k}(h, \tilde{\mu}^-, \tilde{\varphi}^-)$.

Solution of the problem (7) in every iteration is determined as a functional with the IFT of the atmosphere (15)

$$\mathbf{\Phi}_{a,k}(z, \mu, \varphi) = (\hat{\Pi}_a, \mathbf{F}_{a,k-1}) = \mathbf{\Phi}^0_{a,k} + \mathbf{\Phi}^d_{a,k} ,$$

in which two types of radiation are separated out: the direct diffuse radiation from the boundary for the directions of ascending radiation $s^- = (\mu^-, \varphi^-) \in \Omega^-$ calculated explicitly by the singular component of IFT of the atmosphere

$$\mathbf{\Phi}^0_{a,k} = \mathbf{\Phi}^{0,d}_{a,k}(z, \mu^-, \varphi^-) = (\hat{\Pi}^s_a, \mathbf{F}^d_{a,k-1})$$

and diffusion radiation multiple scattered in the atmosphere for all directions $s \in \Omega$ calculated by the method of quadrature

$$\mathbf{\Phi}^d_{a,k} = \mathbf{\Phi}^{d,d}_{a,k}(z, \mu, \varphi) = (\hat{\Pi}^d_a, \mathbf{F}^d_{a,k-1}) .$$

Solution of the problem (9) at each iteration is determined as a functional with the IFT characteristic of the ocean (26)

$$\mathbf{\Phi}_{oc,k}(\tilde{z}, \tilde{\mu}, \tilde{\varphi}) = (\hat{\Pi}_{oc}, \mathbf{F}_{oc,k-1}) = \mathbf{\Phi}^0_{oc,k} + \mathbf{\Phi}^d_{oc,k},$$

in which two types of radiation are separated out: the direct diffuse radiation reflected from the interface along the directions of down going radiation $\tilde{s}^+ = (\tilde{\mu}^+, \tilde{\varphi}^+) \in \Omega^+$ calculated explicitly using the singular component of the IFT of ocean

$$\mathbf{\Phi}^0_{oc,k} = \mathbf{\Phi}^{0,d}_{oc,k}(\tilde{z}, \tilde{\mu}^+, \tilde{\varphi}^+) = (\hat{\Pi}^s_{oc} ,\mathbf{F}^d_{oc,k-1})$$

and diffuse radiation multiply scattered in the ocean along all directions $\tilde{s} \in \Omega$ calculated by the quadrature method

$$\mathbf{\Phi}^d_{oc,k} = \mathbf{\Phi}^d_{oc,k}(\tilde{z}, \tilde{\mu}, \tilde{\varphi}) = (\hat{\Pi}^d_{oc}, \mathbf{F}^d_{oc,k-1}) .$$

The asymptotically accurate solution of the problem (4) for the atmospheric layer $z \in [0, h]$ that completely allows for the contribution due to the influence of the ocean in the calculation structuring model considered can be represented as a superposition of following functionals:

$$\mathbf{\Phi}^a(z, \mu, \varphi) \equiv \sum_{k=1}^{\infty} \mathbf{\Phi}_{a,k} = \mathbf{\Phi}^1_d = (\hat{\Pi}^s_a, \mathbf{E}^s_a) +$$

$$+ (\hat{\Pi}^d_a, \mathbf{E}^s_a) + (\hat{\Pi}^s_a, \mathbf{Z}_a) + (\hat{\Pi}^d_a, \mathbf{Z}_a). \qquad (37)$$

The diffusion "scenario" at the interface from the side of the atmosphere being caused by the radiation exchange between the ocean and the atmosphere is

$$\mathbf{Z}_a(h, \mu^-, \varphi^-) \equiv \sum_{k=1}^{\infty} \mathbf{F}^d_{a,k-1} = \hat{R}_1 \, \mathbf{Y}_a +$$

$$+ \hat{T}_{21}\,\mathbf{Y}_{oc} = \hat{R}_1\,\mathbf{\Phi}_a^+ + \hat{R}_1\,\tilde{\mathbf{Y}}_a + \hat{T}_{21}\,\mathbf{Y}_{oc},$$

where the total diffuse irradiance of the interface by radiation coming form the atmosphere is

$$\mathbf{Y}_a(h,\, \mu^+,\, \varphi^+) = \mathbf{\Phi}_a^+ + \tilde{\mathbf{Y}}_a, \quad \tilde{\mathbf{Y}}_a \equiv \sum_{k=1}^{\infty} \mathbf{\Phi}_{a,k}^{d+},$$

and by the radiation from the ocean

$$\mathbf{Y}_{oc}(h,\, \tilde{\mu}^-,\, \tilde{\varphi}^-) \equiv \sum_{k=1}^{\infty} \mathbf{\Phi}_{oc,k}^{d-}.$$

The asymptotically exact and complete solution of the problem (4) for the ocean layer $z \in [0,\, H]$ taking into account the radiation exchange between the ocean and the atmosphere is presented as the following superposition of functionals:

$$\mathbf{\Phi}^{oc}(\tilde{z},\, \tilde{\mu},\, \tilde{\varphi}) \equiv \sum_{k=1}^{\infty} \mathbf{\Phi}_{oc,k} = \mathbf{\Phi}_d^2 = (\hat{\Pi}_{oc}^s,\, \mathbf{E}_{oc}^s) +$$

$$+ (\hat{\Pi}_{oc}^d,\, \mathbf{E}_{oc}^s) + (\hat{\Pi}_{oc}^s,\, \mathbf{Z}_{oc}) + (\hat{\Pi}_{oc}^d,\, \mathbf{Z}_{oc}). \qquad (38)$$

The diffusion “scenario” at the interface from the ocean side caused due to the radiation exchange between the ocean and the atmosphere is

$$\mathbf{Z}_{oc}(h,\, \tilde{\mu}^+,\, \tilde{\varphi}^+) \equiv \sum_{k=1}^{\infty} \mathbf{F}_{oc,k-1}^d = \hat{T}_{12}\,\mathbf{Y}_a +$$

$$+ \hat{R}_2\,\mathbf{Y}_{oc} = \hat{T}_{12}\,\mathbf{\Phi}_a^+ + \hat{T}_{12}\,\tilde{\mathbf{Y}}_a + \hat{R}_2\,\mathbf{Y}_{oc}.$$

Let us write the representation (37) by separating the linear approximation

$$\mathbf{\Phi}^a = \mathbf{\Phi}_{a,1} + (\hat{\Pi}_a^s,\, \tilde{\mathbf{Z}}_a) + (\hat{\Pi}_a^d,\, \tilde{\mathbf{Z}}_a).$$

Here the diffusion “scenario” from the side of the atmosphere at the interface, being at the height $z = h$, due to the nonlinear orders of radiation exchange between the ocean and the atmosphere,

$$\tilde{\mathbf{Z}}_a(h,\, \mu^-,\, \varphi^-) \equiv \sum_{k=1}^{\infty} \mathbf{F}_{a,k}^d = \hat{R}_1\,\tilde{\mathbf{Y}}_a + \hat{T}_{21}\,\mathbf{Y}_{oc},$$

is determined by the total irradiance of the interface by radiation coming from the atmosphere $\mathbf{Y}_a$ and by the complete irradiance by radiation coming from the ocean $\mathbf{Y}_{oc}$.

Let us now separate out the linear approximation in the representation (38)

$$\mathbf{\Phi}^{oc}(\tilde{z},\, \tilde{\mu},\, \tilde{\varphi}) = \mathbf{\Phi}_{oc,1} + (\hat{\Pi}_{oc}^s,\, \tilde{\mathbf{Z}}_{oc}) + (\hat{\Pi}_{oc}^d,\, \tilde{\mathbf{Z}}_{oc}).$$

Here the diffusion “scenario” at the interface $z = h$ for radiation coming from the ocean that is caused by nonlinear orders of radiation exchange between the ocean and the atmosphere

$$\tilde{\mathbf{Z}}_{oc}(h,\, \tilde{\mu}^+,\, \tilde{\varphi}^+) \equiv \sum_{k=1}^{\infty} \mathbf{F}_{oc,k}^d = \hat{T}_{12}\,\tilde{\mathbf{Y}}_a + \hat{R}_2\,\mathbf{Y}_{oc}$$

is determined by the partial irradiance of the interface with radiation coming from the atmosphere $\mathbf{Y}_a$ and total irradiance by the radiation coming from the ocean $\mathbf{Y}_{oc}$.

CONCLUSION

The vector optical transfer operator (35) constructed using rigorous mathematical methods is a new model for the transfer of polarized radiation in a two-media system that is adequate to GVBP (4). New results obtained using the approach proposed are the reduction of the initial GVBP (4) with a complex nonlinear dependence on the interface properties to the solution of a FVBP with “vacuum” boundary conditions for each of two media separately and formulation of the VOTO (35) in a matrix form with the kernel being a two-component algebraic vector IFT, Π. The universal functions of the horizontal variations, and angular dependences of the boundary conditions and sources of GVBP (1) and (4) are separated out, that are invariant relative to the polarization characteristics of radiation.

Having a set of such invariant VIFs that are solutions to one in the pairs of FVBPs (13) and (24), or (17) and (28), or (19) and (31), or (22) and (33), one can obtain, using the Neumann series (35), solutions of the problems with different spatial and angular structures of sources and kernels of reflection and transmission operators in any order approximation of the radiation exchange between the media taking into account multiple scattering and polarization in both media by the IFT for every passage of radiation through the interface.

The operator recursion relation obtained for the Neumann series (35) terms increases the efficiency of calculations by nonlinear approximations. The method of splitting the GVBP (4) for a two-media system into the FVBPs for each medium separately allows one, using the VOTO (35), to obtain full fields of polarized radiation in the transfer systems combined from media represented by different optical and physical models and (or) having different properties of the interface.

The basic mathematical models of VIF constructed ((13), (18), (20), (22), (24), (29), (31), (33)), IFT (15), (26), and VOTO (35) allows one to develop new algorithms for numerical simulations of the transfer of polarized optical and millimeter-wave (in quasi-optical approximation) radiation in two-media systems like “atmosphere–ocean”, “atmosphere–hydrometeors”, “atmosphere–vegetation”, and also to calculate the radiation corrections in the methods of remote sensing, vision theory, and theory of image transfer through turbid polarizing media.

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