AVERAGING OF THE RADIATION TRANSFER EQUATION OVER STOCHASTIC PARAMETERS OF A MEDIUM

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Received December 3, 1998
Accepted December 7, 1998

The method of kinetic identities from the statistical physics is applied to the problems of radiation transfer in a stochastically inhomogeneous medium.

1. PREFACE

The statement of the problem considered in this paper has resulted from our discussion with Georgii Aleksandrovich Titov of the well-known problem on violation of the light flux balance in the system "solar radiation − overcast cloudiness." At that time G.A. Titov has been thoroughly studying the hypothesis of "horizontal transfer" in an aerosol stochastic medium. He used for that the Monte Carlo method, in which he was a real master. In his Doctor's Thesis, G.A. Titov gave credit, as well, to a standard technique of the statistical physics − direct averaging of the equations with random parameters, rather than their solutions. In parallel to the Monte Carlo method, he conceived of a detailed study of the "horizontal transfer" hypothesis, which he thought to be a necessary component of the problem. Within the framework of his project, an effort is made by the author of this paper to apply the well-known approach[1−3] to the situation of "radiation transfer equation".

Unfortunately, we had no enough time to combine both these approaches. Hopefully, I look forward to successful solution of this problem in the near future. Here I restrict my consideration to the proof that the method proposed in Ref. 4 is also suitable for the transfer equation.

2. STATEMENT OF THE PROBLEM

Let $I(\mathbf{r}, \mathbf{n} \mid \xi)$ be the spectral function (with frequency ω as an argument) of the intensity of a beam passing through the point $\mathbf{r}$ in the direction of the unit vector $\mathbf{n}$. The medium is characterized by the spatial random functions $\Theta(\mathbf{r})$. These functions can be presented as a standard expansion $\Theta(\mathbf{r}) = \int \mathrm{d}\nu\, \zeta(\nu) \times \exp(i\nu\mathbf{r})$. It is just the parameters ζ that include the statistical parameters ξ specified by the problem content. The function $\Phi(\xi)$ (certainly, multidimensional) describes their distribution. The statistical mean of some $\Theta(\xi)$ is quite a standard relation

$$\Theta_{st} = \int \Theta(\xi)\, \Phi(\xi)\, \mathrm{d}\xi. \tag{1}$$

It is essential that such a description considers ξ only as parameters in the transfer equation. This means that the transfer equation operators effect only $\mathbf{r}$ and $\mathbf{n}$; and if $\hat{T}(\mathbf{r}, \mathbf{n} \mid \xi)$ is such an operator, then the definition given by Eq. (1) holds true for $\hat{T}_{st}$.

The transfer equation has the following form:

$$I(\mathbf{r}, \mathbf{n} \mid \xi) = I^{(0)}\, (\mathbf{r}^{(\sigma)}, \mathbf{n}) \times \exp\left[-\int_0^{|\mathbf{r}-\mathbf{r}^{\sigma}|} \mathrm{d}R'\, \chi\, (\mathbf{r} - R'\mathbf{n} \mid \xi) \right] + \int_0^{|\mathbf{r}-\mathbf{r}^{\sigma}|} \mathrm{d}R\, \exp\left[-\int_0^{R} \mathrm{d}R'\, \chi\, (\mathbf{r} - R'\mathbf{n} \mid \xi) \right] \times \int \mathrm{d}\mathbf{n}'\, \varphi(\mathbf{r} - R\mathbf{n}, \mathbf{n}, \mathbf{n}' \mid \xi)\, I(\mathbf{r} - R\mathbf{n}, \mathbf{n}' \mid \xi). \tag{2}$$

Here χ is the extinction coefficient; φ is the scattering phase function (not normalized to the scattering coefficient); $\mathbf{r}^{(\sigma)}(\mathbf{r}, \mathbf{n})$ is the "boundary point"; $I^{(0)}$ is the radiation coming into the medium from outside.

Below we use the following designations: x is the set of variables $\mathbf{r}$, $\mathbf{n}$; $Y(x \mid \xi)$ replaces I, $Y^{(0)}_{(x)}$ replaces $I^{(0)}$, $g(x \mid \xi)$ is used for the "Bouguer exponent"; $\hat{T}(x \mid \xi)$ is the integral operator of Eq. (2). Then Eq. (2) takes the form

$$Y(x \mid \xi) = Y^{(0)}(x)\, g(x \mid \xi) + \hat{T}(x \mid \xi)\, Y(x \mid \xi). \tag{3}$$

Next, for brevity, we assume that $Y^{(0)}(x)$ is included in $g(x \mid \xi)$. Let us introduce one more function $Z(x, \xi) = Y(x \mid \xi)\, \Phi(\xi)$, the equation for which

$$Z(x, \xi) = g(x \mid \xi)\, \Phi(\xi) + \hat{T}(x \mid \xi)\, Z(x, \xi) \tag{4}$$

results from simple multiplication of Eq. (3) by $\Phi(\xi)$ (because $\hat{T}$ is not the operator with respect to ξ).

3. APPLICATION OF THE METHOD OF PROJECTION OPERATOR TO THE EQUATION OF THE FORM (4)

The relation $\hat{P}\,\Theta(x, \xi) = \Phi(\xi) \int \mathrm{d}\xi\, \Theta(x, \xi)$ for the integrated (over ξ) function $\Theta(x, \xi)$ introduces the projection operator $\hat{P}$. The function $Z_1(x, \xi) \equiv \hat{P}\, Z(x, \xi) = \Phi(\xi)\, Y_{st}(x)$ in view of Eq. (1)

0235-6880/99/03 247-04 $02.00

and the definitions of Z and $\hat{P}$. One more function $Z_2(x, \xi) = (1 - \hat{P})\, Z$, and certainly $Z_1 + Z_2 = Z$.

By multiplying Eq. (4) by $\hat{P}$ from left, using the expressions presented above, and then integrating over ξ, we obtain the following equation for $Y_{st}(x)$:

$$Y_{st}(x) = g_{st}(x) + \hat{T}_{st}(x)\, Y_{st}(x) + + \int d\xi\, \hat{T}(x \mid \xi)\, Z_2(x, \xi). \tag{5}$$

In order to exclude Z_2 from Eq. (5), we apply the operator $\hat{T}(1 - \hat{P})$ to Eq. (4). As a result, the function $H \equiv \hat{T} Z_2$ arises in the left-hand side, and the first term in the first part turns into $\hat{T}\,[g(x \mid \xi) - g_{st}(x)]\, \Phi(\xi) \equiv \alpha(x, \xi)$, while another one gives the sum $\beta(x, \xi) = \hat{T}(1 - \hat{P})\, \Phi Y_{st}$ and $\hat{T}(1 - \hat{P})\, H$. The new equation $H = \alpha + \beta + \hat{T}(1 - \hat{P})\, H$ relative to H arises, which has the following formal solution:

$$H(x, \xi) = \frac{1}{1 - \hat{T}\,(1 - \hat{P})}\,[\alpha(x, \xi) + \beta(x, \xi)].$$

Here the standard designation $1/\hat{O} = \hat{O}^{-1}$ is used for the operator $\hat{O}$. After integration over ξ we obtain

$$\int d\xi\, \hat{T}\, Z_2 \equiv \Psi(x) = \int d\xi\, \frac{1}{1 - \hat{T}\,(1 - \hat{P})}\,\{\hat{T}(g(x \mid \xi) - - g_{st}(x))\, \Phi(\xi) + \hat{T}\,(1 - \hat{P})\, \hat{T}\, \Phi(\xi)\, Y_{st}(x). \tag{6}$$

Certainly, such transformations of Eq. (6) are needed, after which, first, $\hat{P}$ will not directly enter into the equation and, second, the role of $\hat{T}(x \mid \xi)$ will be "minimal." Some identities for the operators turn out to be useful here.

Thus for the operator

$$\hat{A}\,(x, \xi)\, \frac{1}{1 + \hat{T}\,\hat{P}}\, \hat{A}\, \Phi = = \hat{A}\, \Phi - \hat{T}\, \Phi\, \frac{1}{1 + \hat{T}_{st}} \int \hat{A}\, \Phi\, d\xi. \tag{7}$$

Really, the series

$$\frac{1}{1 + \hat{T}\,\hat{P}} = 1 - \hat{T}\,\hat{P} + \hat{T}\,\hat{P}\,\hat{T}\,\hat{P} - \hat{T}\,\hat{P}\,\hat{T}\,\hat{P}\,\hat{T}\,\hat{P} + \ldots$$

exists, because in our case with the transfer equation $\|\hat{T}\| < 1$, and the eigenvalue of the operator $\hat{P}$ is ± 1. The definition of $\hat{P}$ implies the equality $(\hat{T}\,\hat{P})^m\, \hat{A}\, \Phi = = \hat{T}\, \Phi\, \hat{T}_{st}^{m-1} \int \hat{A}\, \Phi\, d\xi$ for integer $m \geq 1$; and the following summation of the series gives Eq. (7). After integration of Eq. (7) over ξ, the integral $\int \hat{A}\, \Phi\, d\xi$ could be written in the right-hand side with the factor $1 - [\hat{T}_{st}/(1 + \hat{T}_{st})] = 1/(1 + \hat{T}_{st})$, because only one operator $\hat{T}_{st}$ works here. Therefore,

$$\int \frac{1}{1 + \hat{T}\,\hat{P}}\, \hat{A}\, \Phi(\xi)\, d\xi = \frac{1}{1 + \hat{T}_{st}} \int \hat{A}\, \Phi\, d\xi. \tag{8}$$

Then, it can be noted that there exists the operator M being a solution of the equivalent equations

$$\hat{M} = \hat{T} + \hat{T}\, \frac{1}{1 + \hat{T}\,\hat{P}}\, \hat{M}, \quad \hat{M} = \hat{T} + \hat{M}\, \frac{1}{1 + \hat{T}\,\hat{P}}\, \hat{T} \tag{9}$$

that enters (as $\hat{M}_E$) into the general definition

$$\frac{1}{\hat{C} - \hat{E}} = \frac{1}{\hat{C}}\left(1 + \hat{M}_E\, \frac{1}{\hat{C}}\right). \tag{10}$$

Similar identities are very popular in transformations of operators of the "resolvent" type (see, for example, Ref. 3).

Direct substitution indicates that Eqs. (9) and (10) (with our operators $\hat{C}$ and $\hat{E}$) satisfy the relation

$$\hat{M} = \hat{T} + \hat{T}\, \frac{1}{1 + \hat{T}\,\hat{P} - \hat{T}}\, \hat{T}.$$

The latter expression, Eqs. (7) and (8), and the definition of $\hat{P}$ give the following chain for the operator acting (under the $\int d\xi\ldots$ sign) upon Y_{st} from Eq. (6):

$$\frac{1}{1 - \hat{T}\,(1 - \hat{P})}\, \hat{T}\,(1 - \hat{P})\, \hat{T}\, \Phi = \hat{T}\,(1 - \hat{P}) \times$$

$$\times \frac{1}{1 - \hat{T}\,(1 - \hat{P})}\, \hat{T}\, \Phi = T\, \frac{1}{1 + \hat{T}\,\hat{P} - \hat{T}}\, \hat{T}\, \Phi -$$

$$- T\Phi \int d\xi\, \frac{1}{1 + \hat{T}\,\hat{P} - \hat{T}}\, \hat{T}\, \Phi = (\hat{M} - \hat{T})\, \Phi -$$

$$- \hat{T}\, \Phi \int d\xi\, (\hat{T}^{-1}\hat{M} - 1)\, \Phi = \hat{M}\, \Phi -$$

$$- \hat{T}\Phi \int d\xi \left(\Phi + \frac{1}{1 + \hat{T}\,\hat{P}}\, \hat{M}\, \Phi\right) =$$

$$= \hat{M}\, \Phi - \hat{T}\, \Phi\, \frac{1}{1 + \hat{T}_{st}} \int \hat{M}\, \Phi\, d\xi - \hat{T}\, \Phi.$$

After integration over ξ the corresponding term of Ψ takes the form

$$\frac{1}{1 + \hat{T}_{st}} \int \hat{M}\, \Phi\, d\xi\, Y_{st} - \hat{T}_{st}\, Y_{st}. \tag{11}$$

The integral from Eq. (11) transforms with involving the solution (9) in the form of the series

$$\hat{M}\, \Phi = (\hat{A}_0 + \hat{A}_1 + \hat{A}_2 + \ldots)\, \Phi; \quad \hat{A}_0 \equiv \hat{T},$$

$$\hat{A}_1 = \hat{T}\, \frac{1}{1 + \hat{T}\,\hat{P}}\, \hat{A}_0, \quad \hat{A}_2 = \hat{T}\, \frac{1}{1 + \hat{T}\,\hat{P}}\, \hat{A}_1, \ \ldots\ .$$

Expression (7) gives rise to the chain $\hat{A}_0\, \Phi = \hat{T}\, \Phi$:

$$\hat{A}_1\, \Phi = \hat{T}\, \hat{A}_0\, \Phi - \hat{T}^2\, \Phi\, \frac{1}{1 + \hat{T}_{st}}\, g_0,\ g_0 = \int \hat{A}_0\, \Phi\, d\xi = \hat{T}_{st},$$

$$\hat{A}_2\, \Phi = \hat{T}\, \hat{A}_1\, \Phi - \hat{T}^2\, \Phi\, \frac{1}{1 + \hat{T}_{st}}\, g_1,\ \ g_1 = \int \hat{A}_1\, \Phi\, d\xi,$$

with the obvious consequence that

$$\hat{M}\,\Phi - \hat{T}\,\Phi = \hat{T}\,\hat{M}\,\Phi - \hat{T}^2\,\Phi\,\frac{1}{1+\hat{T}_{st}}\,\hat{X}. \qquad (12)$$

Let us introduce $\hat{X}_m = \int d\xi\,\hat{T}^m\,\hat{M}\,\Phi$ with the integer m, and then the sought parameter is $X_0 \equiv X$.

It is clear that integration of Eq. (12) over ξ will not give the equation for X, because the unknown X_1 will also enter into it. Therefore, the standard procedure is necessary: equation (12) is multiplied by $\hat{T}$ to the corresponding power (formally, starting from the first power) from the left and is integrated over ξ; the result will be the following system:

$$\hat{X} - \hat{T}_{st} = \hat{X}_1 - (\hat{T}^2)_{st}\,\frac{1}{1+\hat{T}_{st}}\,\hat{X},$$

$$\hat{X}_1 - (\hat{T}^2)_{st} = \hat{X}_2 - (\hat{T}^3)_{st}\,\frac{1}{1+\hat{T}_{st}}\,\hat{X},$$

$$\hat{X}_2 - (\hat{T}^3)_{st} = \hat{X}_3 - (\hat{T}^4)_{st}\,\frac{1}{1+\hat{T}_{st}}\,\hat{X}.$$

After term-by-term summation the operators $\hat{X}_m$ with $m \geq 1$ disappear, thus yielding the following expression:

$$\hat{X} - [\hat{T}_{st} + (\hat{T}^2)_{st} + (\hat{T}^3)_{st} + \ldots] =$$

$$= -\,[(\hat{T}^2)_{st} + (\hat{T}^3)_{st} + (\hat{T}^4)_{st} + \ldots]\,\frac{1}{1+\hat{T}_{st}}\,\hat{X}.$$

The expressions in the square brackets can be transformed to

$$\left(\hat{T}\,\frac{1}{1-\hat{T}}\right)_{st},\quad \left(\hat{T}^2\,\frac{1}{1-\hat{T}}\right)_{st}.$$

As a result, we have the following equation for X with the formal solution

$$\hat{X} \equiv \int \hat{M}\,\Phi\,d\xi = \frac{1}{1+\left(\hat{T}^2\,\frac{1}{1-\hat{T}}\right)_{st}\frac{1}{1+\hat{T}_{st}}} \times$$

$$\times \left(\hat{T}\,\frac{1}{1-\hat{T}}\right)_{st}.$$

Now the operator acting upon Y_{st} in Eq. (11) is as follows:

$$\frac{1}{1+\hat{T}_{st}}\,\frac{1}{1+\left(\hat{T}^2\,\frac{1}{1-\hat{T}}\right)_{st}\frac{1}{1+\hat{T}_{st}}}\left(\hat{T}\,\frac{1}{1-\hat{T}}\right)_{st} - \hat{T}_{st}.$$

Using c and $\hat{e}$ for denominators in the first two expressions, we obtain $\hat{c}^{-1}\,\hat{e}^{-1} = (ec)^{-1}$, which simplifies, to a certain extent, the latter expression

$$\frac{1}{1+\hat{T}_{st}+\left(\hat{T}^2\,\frac{1}{1-\hat{T}}\right)_{st}}\left(\hat{T}\,\frac{1}{1-\hat{T}}\right)_{st} - \hat{T}_{st}.$$

Writing finally the denominator of the first factor under the common sign $(\ldots)_{st}$ and reconstructing Y_{st}, we reduce the considered term of Eq. (6) to the form

$$\frac{1}{\left(\frac{1}{1-\hat{T}}\right)_{st}}\left(\hat{T}\,\frac{1}{1-\hat{T}}\right)_{st} Y_{st} - \hat{T}_{st}\,Y_{st}.$$

In a similar way we can transform the another term of Eq. (6) and finally obtain

$$\Psi(x) = \frac{1}{\left(\frac{1}{1-\hat{T}}\right)_{st}}\left(\hat{T}\,\frac{1}{1-\hat{T}}\right)_{st} Y_{st} +$$

$$+\frac{1}{\left(\frac{1}{1-\hat{T}}\right)_{st}}\left(\hat{T}\,\frac{1}{1-\hat{T}}\,(g-g_{st})\right)_{st} - \hat{T}_{st}\,Y_{st}. \qquad (13)$$

Equations (5), (6), and (13) give the expression

$$Y_{st} = \tilde{Y} + \hat{T}_{st}\,Y_{st} + \hat{L}\,Y_{st}, \qquad (14)$$

in which

$$\hat{L} = \frac{1}{\left(\frac{1}{1-\hat{T}}\right)_{st}}\left(\hat{T}\,\frac{1}{1-\hat{T}}\right)_{st} - \hat{T}_{st} \equiv \hat{K} - \hat{T}_{st}, \qquad (15)$$

$$\tilde{Y} = g_{st} + \frac{1}{\left(\frac{1}{1-\hat{T}}\right)_{st}}\left(\hat{T}\,\frac{1}{1-\hat{T}}\,(g-g_{st})\right)_{st}. \qquad (16)$$

4. DISCUSSION

Equations (14) – (16), which are exact in their mathematical structure, are the identities having only the form of equations. This can be easily checked by involving the formal solution $Y = (1/(1-\hat{T}))\,g$ to equation (3). As usually, due to this circumstance, we should be very careful when doing identical simplifications such as Eqs. (14) – (16). Otherwise we may finally obtain a trivial equality (for example, such as $Y_{st} = Y_{st}$). This is generally typical for the situation, when an equation is constructed from a formally known solution. However, the aim of this seemingly "backward step" is also obvious: it creates the prerequisites for approximations based on the corresponding physical and mathematical grounds. Here they must be oriented, to a certain degree, to nullifying the operator given by Eq. (15) and the second term in Eq. (16), when the medium is not stochastic. Therefore, the problem is to find the form (14), which is most rational for such an action. The central idea is quite standard: approximations in the additional terms arising after averaging – compare Eqs. (3) and (14) – will prove to be the exact identities "partially corrected" by iterations.

Let us describe briefly these quite clear reasonings, which can be treated as prerequisites for the following approximations.

Consider first the condition $\|\hat{T}\| < 1$. As known, this inequality implies the principle fact of existence of absorption (though very small) at any frequency. It guarantees the convergence of the Neumann series for the transfer equation.

Second, we can, as usually, orient to separation of correlations, especially as the power increases in the expressions such as the "central moment". Therefore, for the expansion parameter ε, which is the characteristic of statistical fluctuations, we can write the following estimate: $\varepsilon < \|\hat{T} - \hat{T}_{st}\|$. It also holds true for other similar constructions, for example $|g - g_{st}| = 0(\varepsilon)$.

Third, by applying the "statistics of photons" (using the terminology of the Monte Carlo method) we undoubtedly can strengthen the above-considered possibility of a simplification. This is possible because the "statistics of photons" smooths out the stochastic properties of the medium. One simply should keep in mind that powers of $\hat{T}$ describe the multiple scattering.

The factor, for which we will use the term "double protection", should act as an additional "smallness parameter". The meaning of this term is very simple – in expressions like $\hat{A}\,\nu$ (where $\hat{A}$ is an operator and ν is a function), both $\hat{A}$ and ν vanish in the absence of stochastic properties of a medium. It is easy to check that all components of Eq. (14) have such a "double protection".

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