Performance criteria of adaptive optical systems for different bases of random wave phase expansion

Yu.N. Isaev and E.V. Zakharova

Institute of Atmospheric Optics,
Siberian Branch of the Russian Academy of Sciences, Tomsk
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We analyze a representation of random wave phase in different bases: the orthogonal Karhunen–Loeve–Obukhov functions, Zernike polynomials, discrete Walsh functions, and Haar wavelets. Performance criteria of an adaptive optical system, which are used for determination of its potential efficiency, are represented in the paper, such as the phase compensation error and Strehl ratio.

When an electromagnetic wave propagates through the atmosphere, phase distortions caused by a passage through a randomly inhomogeneous medium occur. Analysis of phase spatial modulation allows one to extract an information on parameters of the propagation medium. The spatial phase modulation of a wave leads to changes in other parameters of radiation, in particular, in its intensity. Such an effect of the phase may be used effectively to improve the power parameters of the emitted wave. It is used widely in the methods of adaptive compensation for distortions in optical signals. Based on the information on phase distortions one may accomplish the optimization of the performance criteria of operation of an adaptive optical system (AOS).

The phase of a received or formed signal is a complex mathematical object of the study. For convenience of analysis the signal is represented as a series expansion over orthogonal functions, the choice of the latter being determined by the final goal of a problem. The minimum error of a random phase expansion at a given number of modes is obtained using the Karhunen–Loeve–Obukhov basis (KLO)[1] and therefore its use is optimal for representation of a random phase. Unfortunately, the optimal KLO basis has a complex analytical form, therefore, instead of the optimal basis, the Zernike basis, which is close to it, is often used in calculations.[2,3] To reduce the time of data processing, it is convenient to use the bases having the properties of fast transforms such as Walsh and Haar ones.[4] The authors have developed an effective method of transformation of the expansion coefficients of a random wave phase in an arbitrary orthonormal basis into the expansion coefficients in the optimal KLO basis.[5–8] The performance criteria of AOS for various bases of random phase expansion are presented and the results of numerical experiment which allow one to estimate the efficiency of using the given bases are demonstrated in the paper.

For modal compensation of wave front distortions in the systems of adaptive optics, the phase of a wave $S(\rho)$ is presented as an expansion over the orthonormal functions $\varphi_k(\rho)$ ($\rho = \{x, y\} = \{\rho, \theta\}$):

$$S(\rho) = \sum_{k=1}^{\infty} a_k \varphi_k(\rho). \qquad (1)$$

In particular, the phase of a wave passed through a randomly inhomogeneous medium is represented often as an expansion into the Zernike polynomials $Z_k(\rho)$ (Ref. 2). An advantage of this particular basis is the simplicity of analytical representation and relatively simple realization of first modes coinciding with the classical aberrations in the compensating devices of an AOS. It should be noted that the given series expansion is not optimal from the statistical point of view that manifests itself in correlation of the expansion coefficients. Owing to the complexity of analytical calculations of statistical estimations, the correlation of expansion coefficients is often neglected, hereby the accuracy of calculations decreases. It is more convenient in this case to use the statistically optimal KLO basis. For this basis a norm of the error of phase expansion averaged over an ensemble is minimal at a fixed number of terms of the infinite series expansion and the correlation of expansion coefficients is absent.[1] This simplifies considerably the consequent use of the results of expansion and their analysis.

To determine the degree of perfection of the basis used as a compensating device, the residual error of the wave front compensation is used

$$\varepsilon(\rho) = S(\rho) - \sum_{k=1}^{N} a_k \varphi_k(\rho). \qquad (2)$$

It is a random function of coordinates where the first N aberrations are removed successively from a phase distribution. The second moment of the compensation error $\varepsilon(\rho)$ is a distribution of the phase error variance within the aperture $\sigma^2(\rho)$, which is a measure of accuracy of the approximation of a random phase in the orthonormal basis. Supposing that the compensation for the first N modes is complete, we substitute the wave phase $S(\rho)$ (in the form of Eq. (1)) in the expression (2) and, using the operation of the expectation, obtain the expression for the phase error variance $\sigma^2(\rho)$ (Ref. 3)

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$$\sigma^2(\rho) = \sum_{i\geq N} \sum_{k\geq N} \alpha_{ik} \, \varphi_i(\rho) \, \varphi_k(\rho), \qquad (3)$$

where α_{ik} are the second moments of the phase expansion coefficients $< a_i a_k >$ or the coefficients of the Fourier-expansion of a correlation function over the basis $\varphi_k(\rho)$.

Let us present the profile $\sigma^2(\rho) = \sigma^2(\rho, 0)$ for the fixed angle $\theta = 0$ in the Zernike basis

$$\sigma^2(\rho) = \sum_{m=0} \sum_{i\geq N} \sum_{k\geq N} \alpha^m_{ik} \, R^m_i(\rho) \, R^m_k(\rho) \qquad (4)$$

($R^m_k(\rho)$ are the radial parts of the Zernike polynomials $Z_j(\rho) = R^m_k(\rho)\exp(im\theta)$).

Coefficients α^m_{ik} can be obtained using the expression[5]

$$\alpha^m_{ik} = \int_0^R \rho \mathrm{d}\rho \int_0^R R^m_i(\rho) \, R^m_k(\rho') \, M_m(\rho, \rho') \, \rho' \mathrm{d}\rho', \quad (5)$$

where R is the radius of an aperture;

$$M_m(\rho, \rho') = -\frac{6.88\pi}{r_0^{5/3}} \int_0^\infty \frac{J_m(\kappa\rho') \, J_m(\kappa\rho) \, \kappa \mathrm{d}\kappa}{\kappa^{11/3}}$$

for the Kolmogorov model of the atmosphere and

$$M_m(\rho, \rho') = -\frac{6.88\pi}{r_0^{5/3}} \int_0^\infty \frac{J_m(\kappa\rho') \, J_m(\kappa\rho) \, \kappa \mathrm{d}\kappa}{(\kappa^2 + 1/L_0^2)^{11/6}}$$

for the Karman one; $J_m(x)$ is the Bessel function of the first kind and mth order; L_0 is the outer scale of turbulence; r_0 is the Fried radius.

To determine the variance of the phase error $\sigma^2(\rho)$ in the basis of Walsh rectangular functions $Wal_k(\rho)$ (Ref. 4), the expression (3) at $\theta = 0$ can be written as

$$\sigma^2(\rho) = \sum_{m=0} \sum_{i\geq N} \sum_{k\geq N} \beta^m_{ik} \, Wal_i(\rho^2) \, Wal_k(\rho^2), \quad (6)$$

where the coefficients $\{\beta^m_{ik}\} = \mathbf{B}$ are the correlation matrix of phase expansion coefficients in the Walsh basis. Moreover, they can be determined from the matrix relation[6]

$$\mathbf{B} = \mathbf{C}^{\mathrm{T}} \, \mathbf{A} \, \mathbf{C} \, , \qquad (7)$$

where $\mathbf{A} = \{\alpha^m_{ik}\}$, $\mathbf{C}$ is the transformation matrix from the Walsh basis to the Zernike basis; T is the symbol of transposition.

Note, that the Walsh functions are calculated by ρ^2 in the expression (6). It is caused by the fact that Walsh functions are determined as one-dimensional ones[4] and for the given problem we construct their spatial form on a circle and take into account that the area of a surface element equals $\rho \mathrm{d}\rho \mathrm{d}\theta$ in the polar coordinate system. This remark should be taken into account also for the Haar wavelet functions presented below.

To determine the phase error variance $\sigma^2(\rho)$ in the basis of Haar wavelets $H_k(\rho)$, it is sufficient to replace, in the expression (6), the functions $Wal_k(\rho)$ by $H_k(\rho)$ and in (7) to replace $\mathbf{C}$ by the transformation matrix from the Haar basis into the Zernike basis.

The most compact form the expression (3) has in the optimal KLO basis. At a fixed angle $\theta = 0$ this expression is as follows:

$$\sigma^2(\rho) = \sum_{m=0} \sum_{k\geq N} \lambda_k \, [K^m_k(\rho)]^2, \qquad (8)$$

where $K^m_k(\rho)$ are the radial parts of the KLO functions $\Psi_s(\rho) = K^m_k(\rho)\exp(im\theta)$; λ_k are the eigenvalues of the Gram matrix $\mathbf{A}$ for a representation of the KLO functions in terms of the Zernike polynomials.[7,8] Eigenvectors of the matrix $\mathbf{A}$ compose the transformation matrix of the coefficients of phase expansion from the Zernike basis to the KLO basis. They allow one to transform the expression (4) into the expression (8).

It should be noted that other statistical performance criteria are the functionals of $\sigma^2(\rho)$. In Refs. 5 and 6 the authors have presented the transformation matrices from an arbitrarily basis into the optimal KLO basis. These matrices allow one to simplify the cumbersome expressions such as (4) and (6), hereby simplifying significantly both the analytical calculations and numerical ones for the performance criteria of an AOS. As an illustration, in Figs. 1 and 2 the profile of $\sigma^2(\rho)$ calculated by the expressions (4) and (8) for the Kolmogorov model of turbulence assuming compensation for the first N aberrations is presented.

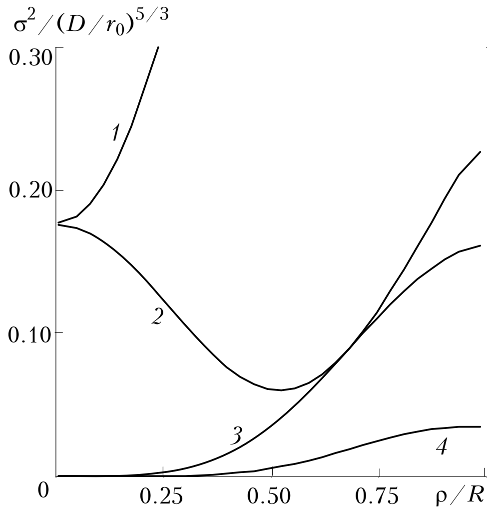

Fig. 1. The distribution of the phase error variance along the aperture radius for the KLO basis: correction of the average phase (*1*); correction of the average phase and the tilts (*2*); correction of the first four modes (*3*); correction of the first six modes (*4*); D is the aperture diameter.

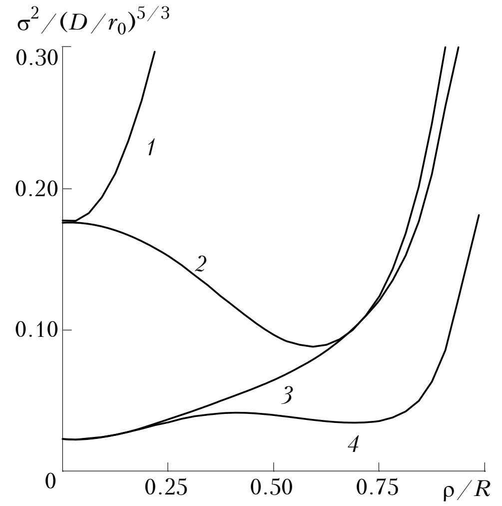

Fig. 2. The distribution of the phase error variance along the aperture radius for the Zernike basis: correction of the averaged phase (*1*); correction of the averaged phase and tilts (*2*); the same and correction for defocusing (*3*); the same plus correction for astigmatism (*4*).

The complexity of an analytical form of the KLO basis and difficulties of its realization as a compensating device may be considered as disadvantages of the basis. In this case it is most convenient to use the rectangular functions such as Walsh discrete functions or Haar wavelets, which belong to the fast transform type. Usually the phase expansion coefficients are determined from the set of local tilts of a phase that is preceded by bulky numerical calculations. To reduce the time of convergence of the linear estimation algorithm, the fast transform algorithms are preferred, what allows one to use specialized processors. Figures 3 and 4 present the profiles of $\sigma^2(\rho)$ calculated by the expression (6) using the Walsh basis and analogous expression for the Haar basis and Kolmogorov model of turbulence when the first N aberrations are compensated for. Since the considered bases are not optimal, it is necessary to take into account a large number of modes.

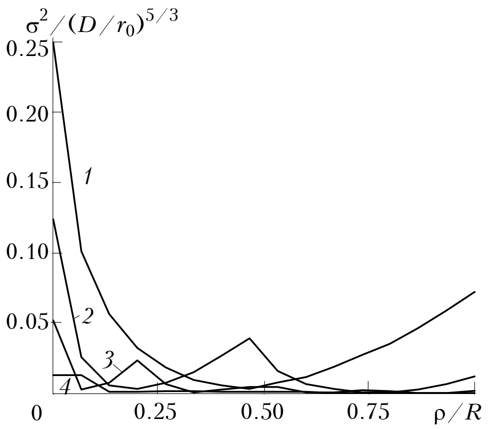

Fig. 3. The distribution of the phase error variance along the aperture radius for the Haar basis when the compensation is carried out for all angle indices: correction of the first mode (*1*); correction of two first modes (*2*); correction of the first four modes (*3*); correction of the first eight modes (*4*).

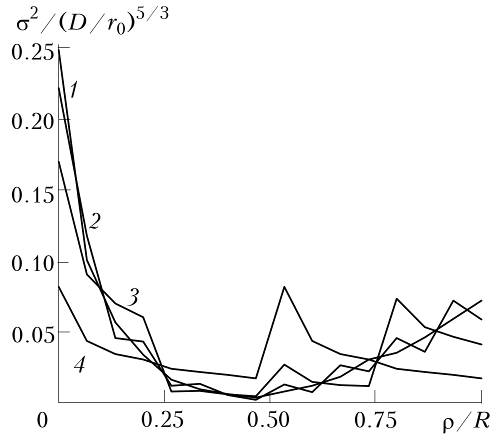

Fig. 4. The distribution of the phase error variance along the aperture radius for the Walsh basis when the compensation is carried out for all angle indices: correction of the first mode (*1*); correction of the first two modes (*2*); correction of the first four modes (*3*); correction of the first eight modes (*4*).

One can see from Figs. 1–4 that the phase error distribution along the radius for bases of rectangular functions is higher at the center of an aperture in contrast to the KLO and Zernike bases. It is explained by the dependence of the Haar and Walsh functions on ρ^2. Note, that for the Haar basis the phase error variance is a little bit better than for the Walsh one because the Haar basis is a local one and has a larger number of degrees of freedom. Note also, that the optimal KLO basis has the least variance of the phase error.

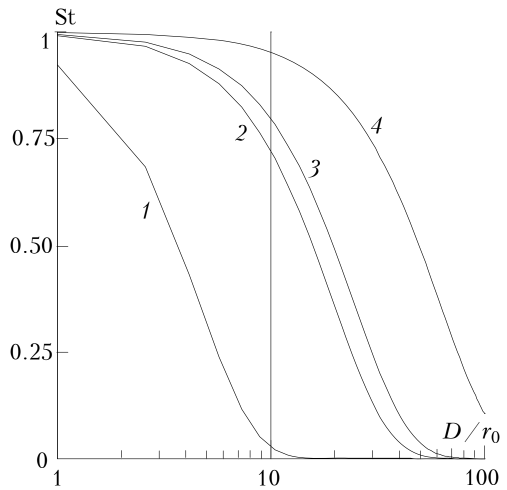

Fig. 5. The Strehl ratio as a function of corrector aperture diameter for the KLO basis: is correction of the average phase (*1*); correction of the average phase and tilts (*2*); correction of the first four modes (*3*); correction of the first six modes (*4*).

A convenient performance criterion for many optical systems is the Strehl parameter St that is the ratio of the radiation intensity in the focus of a real system to the intensity in a system without distorsions.[9] Figures 5 and 6 show St as a function of the normalized aperture diameter when the various number of modes of the KLO and Haar bases are corrected, respectively. The estimating calculations have been performed by the formula[3]

$$\mathrm{St} \approx \exp(-\varepsilon^2),$$

where $\varepsilon^2 = S^{-1}\int_S \sigma^2(\rho)\, d^2\rho$ is the compensation square error averaged over the aperture.

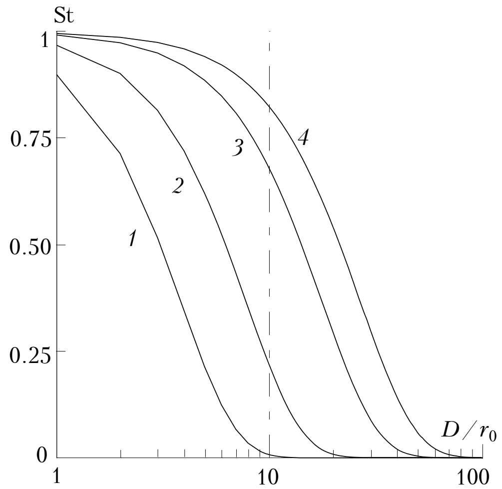

Fig. 6. The Strehl ratio as a function of corrector aperture diameter for the Haar basis when the compensation is carried out for all angle indices: correction of the first mode (*1*); correction of the first two modes (*2*); correction of the first four modes (*3*); correction of the first eight modes (*4*).

It follows from Figs. 1–6 that in the expansions (4), (6), and (8) for the given number of corrected modes the KLO basis is the best. As for the time intervals characteristic of the calculating algorithms, here the authors give a preference to the Walsh and Haar bases, because the transforms by these bases are executed in a more economic and quick way even if we take a relatively large number of expansion terms.

From the researches carried out one can draw the conclusion that the considered bases have the mutually exclusive advantages, therefore we need a possibility of transforming from the expansion of a random wave phase in any basis to the expansion in another basis in order to use different properties of these bases for analysis and adaptive control of a wave front. In particular, if the wave front reconstruction is carried out in an arbitrary basis requiring a large number of modes then it is sufficient to multiply the vector of the phase expansion coefficients in the given basis by the transformation matrix for the KLO basis. In this case the compensating system is optimized, i.e., the combinations of the group of modes take a shape of the KLO functions. It prepares potentially the system to receive a signal and in future the group of modes is controlled by one signal. Thus, the number of control signals is reduced and it is not necessary to design a compensating device in the KLO basis, which, as ought to be noted, is changed when a state of the atmosphere is changed.

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