Estimation of lacunarity of optical spectra

Yu.V. Kistenev

Siberian State Medical University, Tomsk
Institute of Atmospheric Optics,
Siberian Branch of the Russian Academy of Sciences, Tomsk

Received July 11, 2001

The spectra of rotational–vibrational absorption bands of gaseous atmospheric constituents are interpreted in terms of fractal analysis. The lacunarity of an initial spectrum is a basic calculational parameter, from whose variations one can judge the degree of translational and scale invariance of the entire spectrum as a quasistochastic function of frequency.

Introduction

The specifics of calculation of atmospheric radiative fluxes consist in the fact that these calculations must account for the net molecular absorption over a rather wide spectral interval. The extended molecular spectrum of the atmosphere includes thousands of lines and has a rather complex shape reminiscent of a quasirandom process. In this case, it is useful to calculate the net absorption of the medium using parameters characterizing this spectrum "as a whole." This is achieved, for example, by the statistical approach to describing spectra.[1–6]

The procedure of summation of spectral lines over some interval inevitably leads to smoothing of absorption fluctuations, and increasing the summation interval lowers the individuality of neighboring spectral ranges. Obviously, at some point neighboring spectral ranges can be considered as identical with acceptable accuracy. At this point, the spectrum becomes translationally invariant and the net absorption over the entire spectral region can be estimated using a small part of the spectrum.

To estimate the properties of translational symmetry of sets with a complex (in particular, fractal) structure, a parameter referred to as the fluctuation lacunarity or simply lacunarity is used.[7–9] One possible definition of lacunarity Λ for some quasirandom function $s(R)$ is

$$\Lambda = \langle s^2\rangle_s / \langle s\rangle_s^2, \qquad (1)$$

where the averaging is performed over all possible values of the function. Lacunarity characterizes the degree to which values of the function differ from the mean, and $\Lambda = 1$ means that the function is translationally invariant.

In practice, the calculation of lacunarity is connected with averaging of the initial function over some interval of size r. A set of possible values of the function $s(R, r)$ in this case can be obtained for arbitrary changes of positions of the averaging interval inside the domain of definition of the function.[10,11] In this case, the lacunarity depends on the size of the interval r:

$$\Lambda(r) = \langle s^2(R,r)\rangle_s / \langle s(R,r)\rangle_s^2. \qquad (2)$$

Another important property of the lacunarity is that for scale-invariant (self-similar) sets, i.e., fractal and multifractal objects, the dependence $\Lambda(r)$ is linear on a log–log scale.[10,11] For fractals the slope of this straight line is equal to $D-E$, where D is the fractal dimension of the set and E is the dimension of the Euclidian space in which the fractal is nested.

Thus, the lacunarity parameter can serve as an additional tool of fractal analysis along with standard methods for calculating the fractal dimension.

In our case, we can take the density of absorption over some frequency interval[12] or the intensity of the absorption lines averaged over the interval as a density measure of the set.

Calculation of lacunarity of translationally and scale invariant sets

First, let us calculate the lacunarity of sets possessing obvious invariance properties.

Consider an "equidistant" spectrum with nested structure: the spectrum consists of 1000 equidistant lines, and it has periodic (with a period of 40 lines) 20-line gaps (Fig. 1). Here $n(r)$ characterizes the number of lines falling in the averaging interval r. It can be seen that the lacunarity is sensitive to the presence of gaps (lacunae) up to the point at which the averaging interval r exceeds the size of this gap. At that point $\Lambda = 1$ and the set becomes translationally invariant.

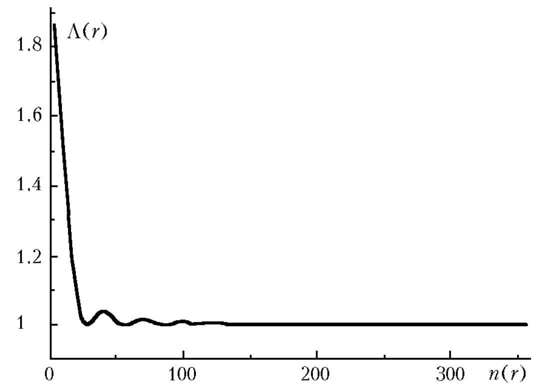

Fig. 1. Calculation of the lacunarity for an equidistant spectrum.

0235-6880/01/09 762-03 $02.00

An example of a scale-invariant set is the so-called Cantor dust. The Cantor set can be obtained by dividing a straight-line segment into a triad with rejection of the central part (Fig. 2*a*). In the limit, repetition of this procedure gives a self-similar set (fractal) with fractal dimension $D = \log2/\log3 = 0.63$ (Ref. 13). Figure 2*b* depicts the calculation of the lacunarity for the Cantor set at the stage of 14 steps of the fractal generation process described above (in this case, we speak of a pre-fractal of 14th generation). It is seen that the dependence of the lacunarity on the averaging interval is close to linear on a log–log scale, and the slope of this straight line corresponds to $D - E = -0.3548$; this is rather close to the fractal dimension of the limiting set.

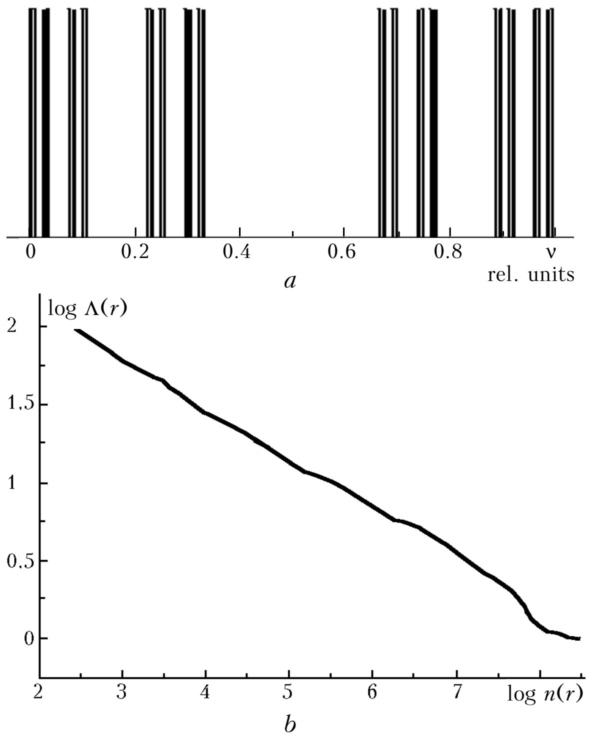

Fig. 2. Structure of the Cantor set (*a*) and calculation of the lacunarity of the Cantor set (*b*).

Calculation of the lacunarity of optical spectra

Let us analyze peculiarities of the lacunarity for the spectra of the CO_2 and O_3 molecules. This choice is motivated by the important role played by these gases in the dynamics of atmospheric processes. Thus, carbon dioxide is one of the most important greenhouse gases, and the ozone layer protects the Earth against the harmful effects of UV radiation.

We consider, as an example, the 01101–00001 band of the main isotopic species of CO_2 (Fig. 3*a*). The calculated lacunarity data (Fig. 3*b*) show that the dependence $\log\Lambda(r)$ is markedly nonlinear. However, the situation changes for $^{17}O^{12}C^{18}O$ (Fig. 4).

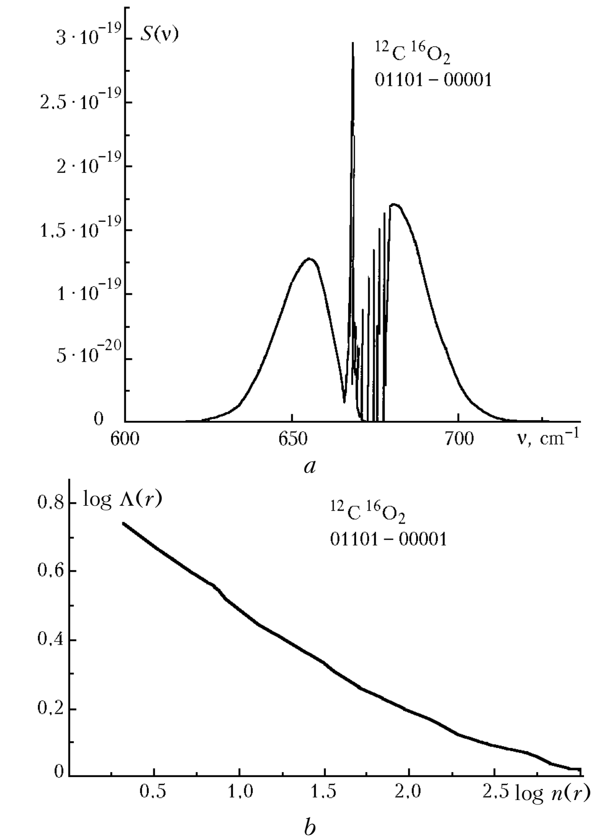

Fig. 3. Distribution of line intensities in the absorption band (*a*) and calculation of lacunarity (*b*).

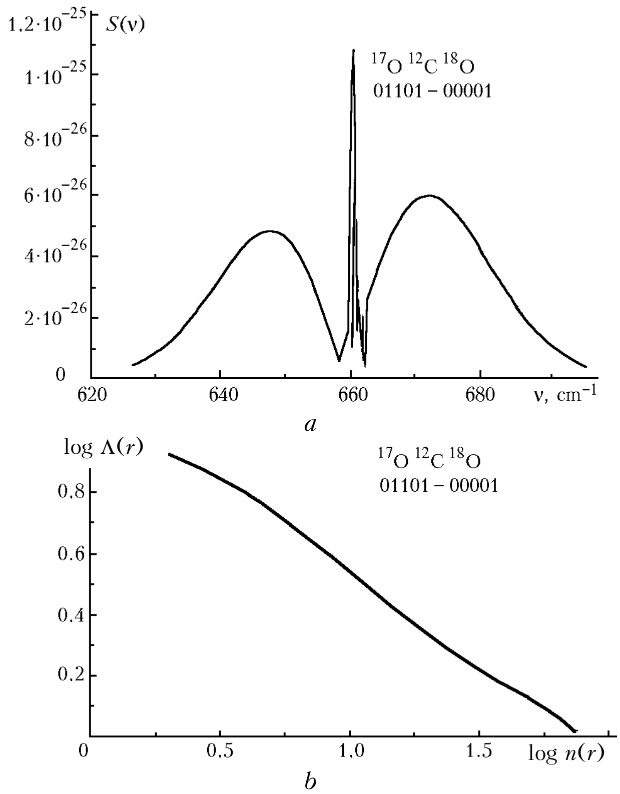

Fig. 4. Distribution of line intensities in the absorption band (*a*) and calculation of lacunarity (*b*).

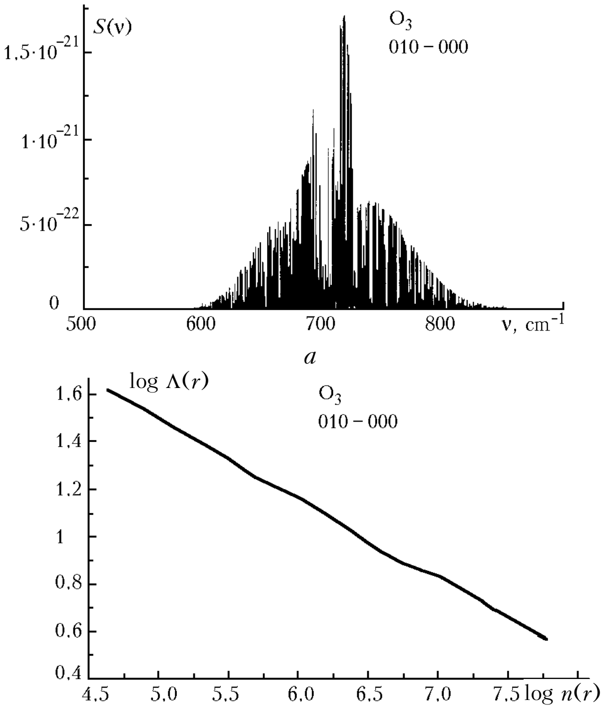

Fig. 5. Distribution of line intensities in absorption band (a) and calculation of lacunarity (b).

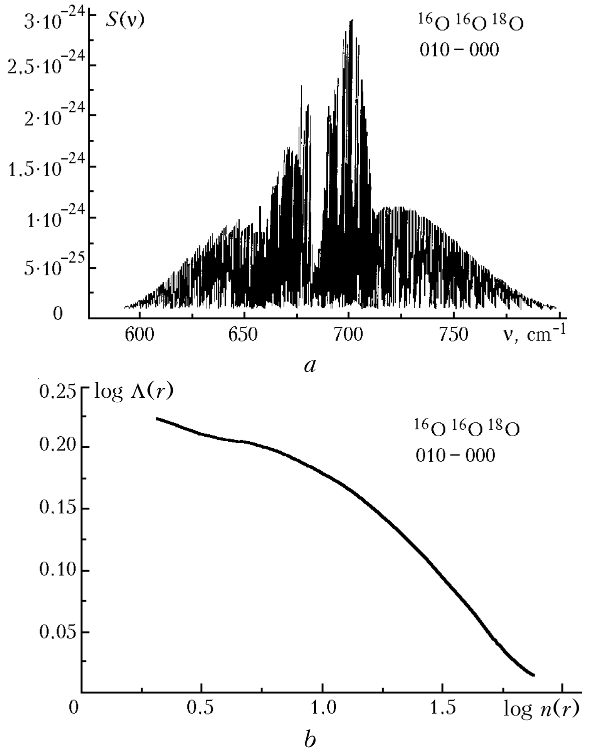

Fig. 6. Distribution of line intensities in the absorption band (a) and calculation of lacunarity (b).

A feature of the spectrum of the fundamental band of atmospheric ozone is that it contains many more rotational lines (more than 7000) as compared to CO_2. Results of similar calculations are shown in Figs. 5 and 6.

It can be seen that the lacunarity here is also sensitive to variations of isotopic composition of the molecule, and the dependence $\log\Lambda(r)$ for the main isotopic species is nearly linear. This is an indirect confirmation of the presence of scale-invariance in this spectrum.[12]

Conclusion

The lacunarity parameter $\log\Lambda(r)$ carries information about the translational invariance of regular and quasirandom distributions. Calculation of the lacunarity for rotational–vibrational absorption bands of various molecules show that this parameter is sensitive to variations of the molecular and isotopic composition of absorbing atmospheric constituents.

The obtained dependence $\log\Lambda(r)$ allows us to estimate the spectral averaging interval for which the spectrum can be considered to be translationally invariant with acceptable accuracy.

The calculated results show that the dependence $\log\Lambda(r)$ is nearly linear on a log–log scale for some rotational–vibrational absorption bands of gaseous atmospheric constituents. This is an indirect confirmation of the presence of scale invariance in these spectra.

Acknowledgments

This work was partly supported by the Russian Foundation for Basic Research (Grant No. 01–05–65152a).

References

1. G.M. Zaslavskii and N.N. Filonenko, Zh. Eksp. Teor. Fiz. **65**, Issue 2(8), 643–656 (1973).
2. I.C. Persival, J. Phys. Ser. B **6**, No. 9, L229–L232 (1973).
3. G.M. Zaslavskii, Usp. Fiz. Nauk **129**, 129 (1979).
4. T.A. Brody, J. Flores, J.B. French, et al., Rev. Mod. Phys. **53**, 385 (1981).
5. P.V. Elyutin, Usp. Fiz. Nauk **155**, No. 3, 397–442 (1988).
6. L.E. Reichl, *The Transition to Chaos: in Conservative Classical Systems: Quantum Manifestations* (Springer-Verlag, New York, Berlin, Heidelberg, London, 1992).
7. R. Blumenfeld and B.B. Mandelbrot, Phys. Rev. E **56**, No. 1, 112–118 (1997).
8. R. Blumenfeld and R. Ball, Phys. Rev. E. **47**, No. 4, 2298–2302 (1993).
9. Y. Gefen, Y. Meir, B.B. Mandelbrot, and A. Aharony, Phys. Rev. Lett. **50**, No. 3, 145–148 (1983).
10. C. Allain and M. Cloitre, Phys. Rev. A **44**, No. 6, 3552–3558 (1991).
11. R.E. Plotnick, R.H. Gardner, W.W. Hargrove, K. Prestegaard, and M. Perlmutter, Phys. Rev. E **53**, No. 5, 5461–5468 (1996).
12. Yu.V. Kistenev and Yu.N. Ponomarev, Opt. Spektrosk. **90**, No. 3, 419–423 (2001).
13. J. Feder, *Fractals* (Plenum Press, New York, 1988).