Numerical analysis of the instrumental matrix of a polarization meter

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Optimization of the meter of Stokes vector parameters and 16 elements of the scattering phase matrix is considered. The measurements are made with maximum possible accuracy in the presence of errors in the initial data. The meter is simple in controlling the polarization elements. The measurement accuracy depends on the condition number of the instrumental matrix cond M. Numerical analysis determines the points, at which cond M is minimum as well as the effect is minimum of the deviation from them on the cond M magnitude.

The accuracy of measurements by means of a meter[1] of Stokes parameters **S** and an optimal meter[2,3] of the scattering phase matrix D of a medium is determined by the instrumental matrices M and W. The condition number of the matrix M cond M and that of the matrix W cond W should be minimum.[1–3]

Capabilities of computation technique significantly increased after Ref. 1 was published, that made it possible to numerically analyze the instrumental matrices M and W in a more detail. The results of analysis are presented in this paper. Let us follow the notations of the parameters I, Q, U, and V of the Stokes vector **S** as it was accepted in Refs. 4 and 5. Optical arrangement of the elements comprising the meter of Stokes parameters is shown in Fig. 1.

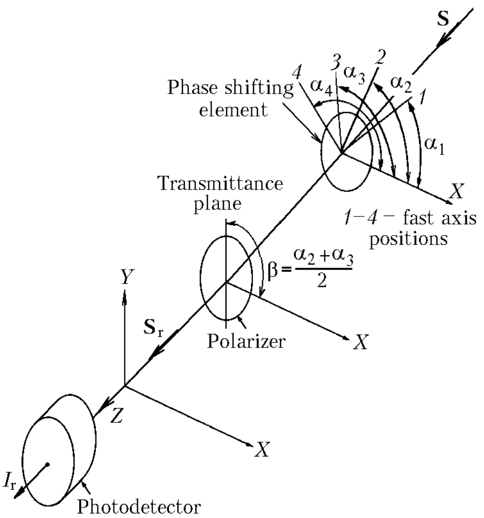

Fig. 1. Measurement of the Stokes vector **S** using four positions of the fast axis of the phase shifting plate.

The Stokes vector $\mathbf{S}_r = (I_r\ Q_r\ U_r\ V_r)^T$, where T is the sign of transposition, of a radiation at the output of the polarization block of a receiver is determined by the following formula:

$$\mathbf{S}_r = P_r\, P'_r\, \mathbf{S}, \qquad (1)$$

where P_r, and P'_r are the Mueller matrices of the polarizer and phase shifting plate, respectively, $\mathbf{S} = (I\ Q\ U\ V)^T$ is the Stokes vector of radiation in the coordinate system of the receiver X, Y.

The signal I_r at the output of the photodetector (taking into account the proportionality coefficient that is equal to the photocurrent of the sensitive element of the photodetector) is derived from Eq. (1) in the form

$$I_r = \frac{1}{2}\{I + Q\,[\cos 2\alpha \cos 2(\alpha - \beta) + \cos\tau \sin 2\alpha \sin 2(\alpha - \beta)] + U\,[\sin 2\alpha \cos 2(\alpha - \beta) - \cos\tau \cos 2\alpha \sin 2(\alpha - \beta)] - V \sin\tau \sin 2(\alpha - \beta)\}, \qquad (2)$$

where β is the angle of orientation of the transmittance plane of the polarizer relative to the X-axis of the coordinate system of the receiver; α is the orientation angle of the fast axis of the phase shifting plate relative to the X-axis; τ is the phase shift of the orthogonal components produced by the phase shifting plate.

Formula (2) at $\beta = \pi/2$ takes the form

$$I_r = \frac{1}{2}\left[I - \frac{Q}{2}(1 + \cos\tau) - \frac{Q}{2}(1 - \cos\tau)\cos 4\alpha - \frac{U}{2}(1 - \cos\tau)\sin 4\alpha + V \sin\tau \sin 2\alpha\right]. \qquad (2a)$$

The functions $f_I(\alpha) = I - Q(1 + \cos\tau)/2$, $f_Q(\alpha) = [(1 - \cos\tau)\cos 4\alpha]/2$, $f_U(\alpha) = [(1 - \cos\tau) \times \sin 4\alpha]/2$, and $f_V(\alpha) = \sin\tau \sin 2\alpha$ belong to the Chebyshev sequence, because the Wronsky determinant

$$W = \left[I - \frac{Q}{2}(1 - \cos\tau)\right]\frac{(1 - \cos\tau)^2}{4} \times \sin\tau \begin{vmatrix} 1 & \cos 4\alpha & \sin 4\alpha & \sin 2\alpha \\ 0 & -4\sin 4\alpha & 4\cos 4\alpha & 2\cos 2\alpha \\ 0 & -16\cos 4\alpha & -16\sin 4\alpha & -4\sin 2\alpha \\ 0 & 64\sin 4\alpha & -64\cos 4\alpha & -8\cos 2\alpha \end{vmatrix} =$$

$$= \left[I - \frac{Q}{2}(1 - \cos\tau) \right] \frac{(1-\cos\tau)^2 \sin\tau}{4} (2048 - 512\cos 2\alpha)$$

is not equal to zero at any α value.[6]

Measuring I_r at different α values, we obtain the system of four linearly independent equations,[1,2] from which one can determine **S**. Let us write equations of this system in the form

$$I_{ri} = \frac{1}{2}(M_{i1}I + M_{i2}Q + M_{i3}U + M_{i4}V), \; i = \overline{1, 4}, \quad (3)$$

where

$$M_{i1} = 1,$$

$$M_{i2} = \cos 2\alpha_i \cos 2(\alpha_i - \beta) + \cos\tau \sin 2\alpha_i \sin 2(\alpha_i - \beta),$$

$$M_{i3} = \sin 2\alpha_i \cos 2(\alpha_i - \beta) - \cos\tau \cos 2\alpha_i \sin 2(\alpha_i - \beta),$$

$$M_{i4} = -\sin\tau \sin 2(\alpha_i - \beta).$$

The system of equations (3) in the matrix form is as follows

$$\mathbf{I}_r = \frac{1}{2} M\, \mathbf{S}, \quad (4)$$

where $\mathbf{I}_r = (I_{r1}\, I_{r2}\, I_{r3}\, I_{r4})^T$; $\mathbf{S} = (I\, Q\, U\, V)^T$ is the Stokes vector in the coordinate system X, Y; M is the 4×4 matrix with the elements M_{i1}, M_{i2}, M_{i3}, M_{i4}, $i = \overline{1, 4}$.

The determinant of the matrix M is not equal to zero at any values of α_i, $i = \overline{1, 4}$, that is characteristic of the determinant formed by the functions of Chebyshev sequence.

$\mathbf{I}_r$ and the elements of the matrix M are called input data of the problem on determining **S**.

Equation (2) corresponds to the case of an ideal polarizer and phase shifting plate with precisely known α, β, τ, and $\mathbf{I}_r$. The Mueller matrices $\tilde{P}'_r$ and $\tilde{P}_r$ of real phase shifting plate and polarizer are approximately described by P'_r and P_r, and the parameters α, β, and τ differ from real ones $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\tau}$ due to measurement errors. Also $\mathbf{I}_r$ differs from the accurate $\tilde{\mathbf{I}}_r$ due to the errors in I_r measured with the meter.

Thus, we cannot determine accurate value of the Stokes vector **S** from these system of equations

$$\tilde{\mathbf{I}}_r = \frac{1}{2} \tilde{M}\, \tilde{\mathbf{S}}, \quad (5)$$

where $\tilde{M}$ is the 4×4 matrix with the elements $\tilde{M}_{i1} = M_{i1} + \Delta M_{i1}$, $\tilde{M}_{i2} = M_{i2} + \Delta M_{i2}$, $\tilde{M}_{i3} = M_{i3} + \Delta M_{i3}$, and $\tilde{M}_{i4} = M_{i4} + \Delta M_{i4}$; $i = \overline{1, 4}$; $\tilde{\mathbf{I}}_r = \mathbf{I}_r + \Delta\mathbf{I}_r$; ΔM_{i1}; ΔM_{i2}; ΔM_{i3}; ΔM_{i4}; $i = \overline{1, 4}$ are the errors in determining the elements $\tilde{M}$; $\Delta\mathbf{I}_r$ is the vector of measurement errors in $\tilde{\mathbf{I}}_r$.

Let us denote the solution of Eq. (5) as $\tilde{\mathbf{S}}$, which is the accurate value of the Stokes vector.

Then we determine the Stokes vector **S** from Eq. (4). It differs from $\tilde{\mathbf{S}}$ by the error vector $\Delta\mathbf{S}$. Taking into account the introduced errors, Eq. (5) takes the form

$$\mathbf{I}_r + \Delta\mathbf{I}_r = \frac{1}{2}(M + \Delta M)(\mathbf{S} + \Delta\mathbf{S}), \quad (6)$$

where $\Delta M = \tilde{M} - M$; $\Delta\mathbf{S} = \tilde{\mathbf{S}} - \mathbf{S}$.

We cannot know the quantity (5) because of the errors, and we can only solve Eq. (4). It is very important that Eq. (4) has a solution at any α_i, $i = \overline{1, 4}$, τ, β, and the errors ΔM and $\Delta\mathbf{I}_r$ can be estimated. Theoretically, decreasing the errors to any small value by increasing the accuracy of measurement of α, β, τ and improving the technology of manufacturing phase shifting plates and polarizers approaching to the ideal ones, one can make the system (5) to approach to the system of equations (4). The quantities **S** and $\mathbf{I}_r$ are the four-dimensional vectors of real arithmetic Euclidean space.

Then, at $\|\Delta M\|\,\|M^{-1}\| < 1$ (condition of smallness of the errors) the following relationship is true[7]:

$$\frac{\|\Delta\mathbf{S}\|}{\|\mathbf{S}\|} \le \frac{\operatorname{cond} M}{1 - \operatorname{cond} M \dfrac{\|\Delta M\|}{\|M\|}} \left(\frac{\|\Delta M\|}{\|M\|} + \frac{\|\Delta\mathbf{I}_r\|}{\|\mathbf{I}_r\|} \right), \quad (7)$$

where $\|\ldots\|$ is the sign of the norm of the vector (matrix), $\operatorname{cond} M = \|M\|\,\|M^{-1}\|$ is the condition number of the matrix M. The choice of that or another specific norm in practice is determined by the requirements to the accuracy of solution. The choice of the Euclidean norm

$$\|\Delta\mathbf{S}\|_2 = (\Delta I^2 + \Delta Q^2 + \Delta U^2 + \Delta V^2)$$

corresponds to the criterion of smallness of the rms error. The Euclidean space, to which **S** and $\mathbf{I}_r$ belong, is full, so when selecting any norm, the decrease of $\|\Delta\mathbf{S}\|$ leads to the increase of the accuracy of determination of **S**. The system (4) for the optimal meter should be well conditioned, that means that its solution should be weakly sensitive to the errors or uncertainties in the input data.

As follows from Eq. (7) the relative disturbances $\delta M = \dfrac{\|\Delta M\|}{\|M\|}$, $\delta I_r = \dfrac{\|\Delta\mathbf{I}_r\|}{\|\mathbf{I}_r\|}$ are summed linearly, hence, the minimum $\delta S = \dfrac{\|\Delta S\|}{\|S\|}$ is provided at minimum δM, δI_r, and cond M.

Thus, the condition number should be minimum for the optimal meter, the parameters τ, α, and β are measured with high accuracy, and the Mueller matrices of the polarization elements weakly differ from the Mueller matrices of the ideal polarization elements.

Optical arrangement of the elements of the optimal meter of the scattering phase matrix is shown in Fig. 2.

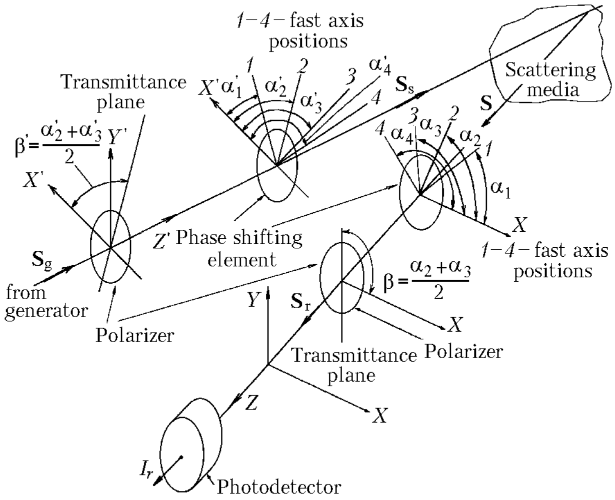

Fig. 2. Optimal meter of the scattering phase matrix.

The scattering phase matrices D relate the Stokes vector of the radiation of a source $\mathbf{S}_s = (I_s\ Q_s\ U_s\ V_s)^T$ and the Stokes vector of the scattered radiation $\mathbf{S} = (I\ Q\ U\ V)^T$ incident on the detector by the following relationship:

$$\mathbf{S} = D\,\mathbf{S}_s. \tag{8}$$

To determine 16 elements of the scattering phase matrix, 16 independent equations are enough, and, if taking into account that every type of polarization of the source radiation contains 4 Stokes parameters, then it is enough for the source to produce 4 types of polarization.[2,3] Let us write this system in the form

$$\mathbf{S}_j = D\,\mathbf{S}_{s_j},\ j = \overline{1,4}\,. \tag{9}$$

Let us form the matrix of the parameters $\mathbf{S}_{sj}$, $j = \overline{1,4}$

$$W = \begin{pmatrix} I_{s1} & Q_{s1} & U_{s1} & V_{s1} \\ I_{s2} & Q_{s2} & U_{s2} & V_{s2} \\ I_{s3} & Q_{s3} & U_{s3} & V_{s3} \\ I_{s4} & Q_{s4} & U_{s4} & V_{s4} \end{pmatrix}. \tag{10}$$

Then let us group the system (9) in four systems, each of which determines the row of the matrix D:

$$W\mathbf{D}_1 = \mathbf{I}_W;\ W\,\mathbf{D}_2 = \mathbf{Q}_W;\ W\,\mathbf{D}_3 = \mathbf{U}_W;\ W\mathbf{D}_4 = \mathbf{V}_W, \tag{11}$$

where

$$\mathbf{D}_m^T = (D_{m1}\ D_{m2}\ D_{m3}\ D_{m4}),\ m = \overline{1,4}$$

is the mth row of the matrix D;

$$\mathbf{I}_W = (I_1\ I_2\ I_3\ I_4)^T;\ \mathbf{Q}_W = (Q_1\ Q_2\ Q_3\ Q_4)^T;$$

$$\mathbf{U}_W = (U_1\ U_2\ U_3\ U_4)^T;\ \mathbf{V}_W = (V_1\ V_2\ V_3\ V_4)^T$$

are the vectors composed of the parameters of the vectors

$$\mathbf{S}_j = (I_j\ Q_j\ U_j\ V_j)^T,\ j = \overline{1,4}\,.$$

W, $\mathbf{I}_W$, $\mathbf{Q}_W$, $\mathbf{U}_W$, and $\mathbf{V}_W$ are the input parameters of the system (11). All the above-said about the system (4) also refers to the system (11). Thus, the condition number of the matrix W is minimum for the optimal meter, and the parameters τ', α', and β' are measured with high accuracy, and the Mueller matrices of the polarization elements weakly differ from the Mueller matrices of the ideal polarization elements.

Taking into account that

$$\mathbf{S}_{s_j} = (I_{s_j}\ Q_{s_j}\ U_{s_j}\ V_{s_j})^T =$$

$$= \begin{pmatrix} 1 \\ \cos 2\alpha'_j \cos 2(\alpha'_j - \beta') + \cos\tau' \sin 2\alpha'_j \sin 2(\alpha'_j - \beta') \\ \sin 2\alpha'_j \cos 2(\alpha'_j - \beta') - \cos\tau' \cos 2\alpha'_j \sin 2(\alpha'_j - \beta') \\ \sin\tau' \sin 2(\alpha'_j - \beta') \end{pmatrix}, \tag{12}$$

where β' is the angle of orientation of the transmittance plane of the polarizer relative to the X'-axis, α'_j is the angle of the fast axis orientation relative to the X'-axis and comparing expressions (12) and (3), one can see that the matrices W and M differ by the sign in the fourth column at $\alpha'_j = \alpha_i$, $\beta' = \beta$, and $\tau' = \tau$.

The matrix W also is not degenerate at any values α'_j, $j = \overline{1,4}$, τ', β'.

The matrix inverse to M is defined in the form

$$M^{-1} = \frac{1}{|M|}\begin{pmatrix} A_{11} & A_{21} & A_{31} & A_{41} \\ A_{12} & A_{22} & A_{32} & A_{42} \\ A_{13} & A_{23} & A_{33} & A_{43} \\ A_{14} & A_{24} & A_{34} & A_{44} \end{pmatrix},$$

where A_{mn} is the algebraic complement to the element M_{mn} of the matrix M, $|M|$ is the determinant of the matrix M.

Then, at $\alpha'_j = \alpha_i$, $\beta' = \beta$, and $\tau' = \tau$

$$W = \begin{pmatrix} M_{11} & M_{12} & M_{13} & -M_{14} \\ M_{21} & M_{22} & M_{23} & -M_{24} \\ M_{31} & M_{32} & M_{33} & -M_{34} \\ M_{41} & M_{42} & M_{43} & -M_{44} \end{pmatrix},\ |W| = -|M|,$$

$$W^{-1} = \frac{1}{|M|}\begin{pmatrix} A_{11} & A_{21} & A_{31} & A_{41} \\ A_{12} & A_{22} & A_{32} & A_{42} \\ A_{13} & A_{23} & A_{33} & A_{43} \\ -A_{14} & -A_{24} & -A_{34} & -A_{44} \end{pmatrix}. \tag{13}$$

It is difficult to calculate the norm $\|M\|_2$ subordinated to the Euclidean norm of the vector, because one needs to determine the eigenvalues of the matrix M^TM.

The Euclidean norm of the matrix $\|M\|_E$ can be calculated in a more simple way:

$$\|M\|_E = \sqrt{\sum_{m,\,n}^{4} M_{mn}^2}\,. \qquad (14)$$

Using Eq. (14), we obtain

$$\|M\|_E = \|W\|_E,\ \|M^{-1}\|_E = \|W^{-1}\|_E, \text{ and}$$

$$\text{cond}\, M = \|M\|_E\ \|M^{-1}\|_E = \|W\|_E\ \|W^{-1}\|_E = \text{cond}\, W,$$

so, let us analyze only cond M.

Cond M is a function of the parameters $\alpha_1, \ldots, \alpha_4$, β, τ, i.e., cond $M = f(\alpha_1, \ldots, \alpha_4, \beta, \tau)$. The purpose of numerical analysis of the instrumental matrix M is to determine the parameters $\alpha_1, \ldots, \alpha_4$, β, τ, providing for the minimum of cond M.

It is known that the Euclidean norm of the matrix

$$\|M\|_E \geq \|M\|_2, \qquad (15)$$

so

$$\|M\|_E\ \|M^{-1}\|_E \geq \|M\|_2\ \|M^{-1}\|_2. \qquad (16)$$

To find the minimum of cond $M = f(\alpha_1, \ldots, \alpha_4, \beta, \tau)$, let us use the Euclidean norm of the matrix $\|M\|_E$, because it is simpler, and the determined value of the minimum of cond M is greater or equal to cond M determined using $\|M\|_2$. According to Eq. (7), it leads to the increase of the expected errors $\sqrt{\Delta I^2 + \Delta Q^2 + \Delta U^2 + \Delta V^2}$ and does not worsen the calculation of the accuracy characteristics of the device.

One can calculate the matrix D using one system of equations, if introduce the matrix K of the 16×16 size.

Let us denote

$$\mathbf{I}_{rij} = (I_{r11} \ldots I_{r14} I_{r21} \ldots I_{r24} I_{r31} \ldots I_{r34} I_{r41} \ldots I_{r44})^{\mathrm{T}},$$

where I_{rij} is the value of the signal I_r at the ith position of the phase shifting plate of the receiver and jth position of the phase shifting plate of the source.

$$\mathbf{D}_{mn} = (D_{11} \ldots D_{14} D_{21} \ldots D_{24} D_{31} \ldots D_{34} D_{41} \ldots D_{44})^{\mathrm{T}},$$

where D_{mn} is the element at the cross of mth row and nth column of the matrix D.

Then $K\,\mathbf{D}_{mn} = \mathbf{I}_{rij}$, where $K = M \otimes W$ is the Kronecker (direct) product of M and W. Using the relationships[7]

$$\|M\|_E = \sqrt{\mathrm{Sp} M^{\mathrm{T}} M},$$

where $\mathrm{Sp} M^{\mathrm{T}} M$ is the trace of the matrix $M^{\mathrm{T}} M$; $K^{\mathrm{T}} = M^{\mathrm{T}} \otimes W^{\mathrm{T}}$; $K^{-1} = M^{-1} \otimes W^{-1}$; $(M^{\mathrm{T}} \otimes W^{\mathrm{T}})(M \otimes W) = (M^{\mathrm{T}} M) \otimes (W^{\mathrm{T}} W)$, $\mathrm{Sp}[(M^{\mathrm{T}} M) \otimes (W^{\mathrm{T}} W)] = \mathrm{Sp} M^{\mathrm{T}} M\ \mathrm{Sp} W^{\mathrm{T}} W$, then one can write

$$\text{cond}\, K = \|K\|_E \|K^{-1}\|_E = \sqrt{\mathrm{Sp} K^{\mathrm{T}} K\ \mathrm{Sp}(K^{-1})^{\mathrm{T}} K^{-1}} =$$

$$= \sqrt{\mathrm{Sp} M^{\mathrm{T}} M\ \mathrm{Sp} W^{\mathrm{T}} W\ \mathrm{Sp}(M^{-1})^{\mathrm{T}} M^{-1}\ \mathrm{Sp}(W^{-1})^{\mathrm{T}} W^{-1}} =$$

$$= \sqrt{\mathrm{Sp} M^{\mathrm{T}} M\ \mathrm{Sp}(M^{-1})^{\mathrm{T}} M^{-1}} \sqrt{\mathrm{Sp} W^{\mathrm{T}} W\ \mathrm{Sp}(W^{-1})^{\mathrm{T}} W^{-1}} =$$

$$= \text{cond}\, M\ \text{cond}\, W.$$

Hence, cond K reaches minimum when cond M and cond W have reached minimum.

The rigorous minimum of the function cond $M = f(\alpha_1, \ldots, \alpha_4, \beta, \tau)$ was determined using the Euclidean norm of the matrix for all aforementioned variables with the preset step in the range $0 \leq \alpha_1, \ldots, \alpha_4 \leq 180°$, $0 < \tau < 180°$ at $\beta = 90°$. After finding the minimum value of cond M among the nodes of the network of the values of the variables at the set step of the change of their values, the step of the change of the variables was decreased, and alternation of the values of all aforementioned parameters was applied again in the small range around the determined node of the network of the values of the variables, and the node was determined again, at which the value of cond M is minimum. The step was decreased until the condition fulfilled at the determined node $x^{(0)} = \{\alpha_1 = 38.54°;\ \alpha_2 = 75.14°;\ \alpha_3 = 105.38°;\ \alpha_4 = 141.857°;\ \beta = 90°;\ \tau = 131.795°\}$ with sufficient accuracy

$$\frac{\partial f(x^{(0)})}{\partial \alpha_i} = \frac{\partial f(x^{(0)})}{\partial \beta} = \frac{\partial f(x^{(0)})}{\partial \tau} = 0,\ i = \overline{1,\,4}\,.$$

To do this, the plots were constructed cond $M = f(\alpha_1, \ldots, \alpha_4, \beta, \tau)$ for each variable (Fig. 3) at the values of other variables taken from $x^{(0)} = \{\alpha_1 = 38.54;\ \alpha_2 = 75.14;\ \alpha_3 = 105.38;\ \alpha_4 = 141.857°;\ \beta = 90°;\ \tau = 131.795°\}$.

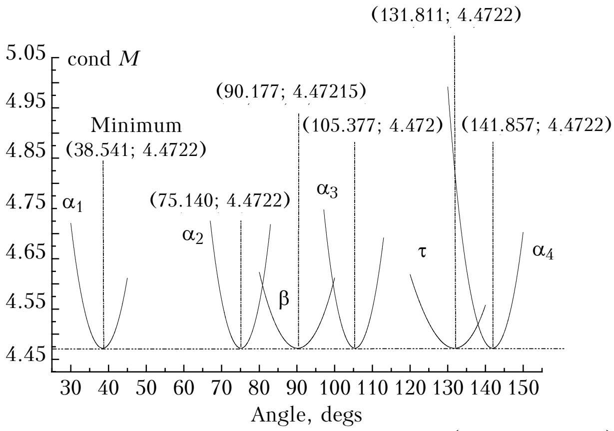

Fig. 3. Numerical investigation of cond $M = f(\alpha_1, \ldots, \alpha_4, \beta, \tau)$ for extremes with respect to each variable at the point $x^{(0)}$.

Figure 3 shows good closeness of $x^{(0)}$ to the point of rigorous minimum of cond M, because the values of the extremes with respect to each variable and the values of the variables at the points of extremes are close to $x^{(0)}$. The values $\alpha_1, \ldots, \alpha_4$, β, and τ can be determined close to their values at $x^{(0)}$ with one or another accuracy. Investigation of the sensitivity of cond M to the deviation of α_1, ..., α_4, β, and τ from the optimal values will enable us to state the requirements to the accuracy of their determination relative to $x^{(0)}$. The accuracy of determination is considered to mean the accuracy of performing the mechanical operation, then the determined value of the parameter can be measured with any preset accuracy and, hence, cond M can be determined with the same accuracy.

Curve *1* (Fig. 4) shows the maximum possible value of cond M at β = 90°, τ = 131.795° and absolute error Δ in setting α_1, ..., α_4 from their values at the point of rigorous minimum. It corresponds to the case when the parameters β and τ are set with very high accuracy, and the parameters α_1, ..., α_4 are successively set during measurements with a less accuracy. Let us ascertain, what maximum value of cond M can be realized. In calculating, the maximum values of cond M was determined among the nodes of the network of the values of the variables at β = 90° and τ = 131.795° at alternation with the preset step of the change of the variables in the range α_1 = 38.54° ± Δ; α_2 = 75.14° ± Δ; α_3 = 105.38° ± Δ; α_4 = 141.857° ± Δ.

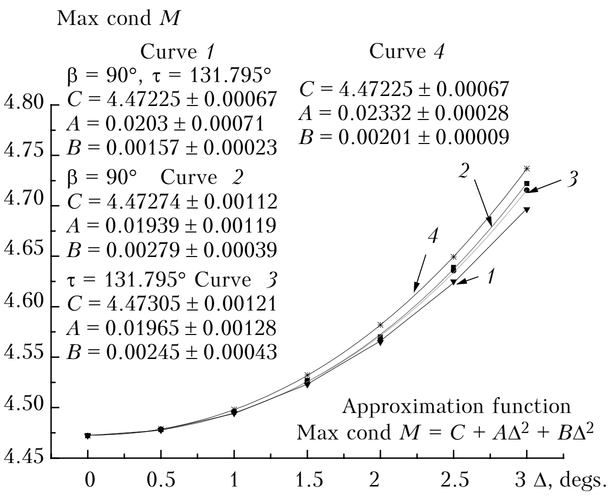

Fig. 4. Sensitivity of cond M to the absolute deviations of α_1, ..., α_4, β, τ from their values at the point of rigorous minimum $x^{(0)}$, min cond M = 4.472195 at the point $x^{(0)}$.

Curve *2* (Fig. 4) shows the maximum possible value of cond M at β = 90° and absolute error Δ in setting α_1, ..., α_4 and τ from their values at the point of rigorous minimum.

Curve *3* (Fig. 4) shows the maximum possible value of cond M at τ = 131.795° and absolute error Δ in setting α_1, ..., α_4 and β from their values at the point of rigorous minimum.

Curve *3* (Fig. 4) shows the maximum possible value of cond M at absolute error Δ in setting α_1, ..., α_4, τ, and β from their values at the point of rigorous minimum. It corresponds to the case when all parameters have beet set with deviations from their values at the point of rigorous minimum, then the maximum value of cond M can be greater than in other cases. Comparison of the curves *1–4* (Fig. 4) shows that accuracy of setting τ stronger affects the values of the maximum possible value of cond M than the accuracy of setting β, but it is not decisive for the value of the maximum cond M.

The phase plates with τ = 90° are widely used now. The point $x^{(1)}$ = {α_1 = 38.137°; α_2 = 75.541°; α_3 = 104.379°; α_4 = 141.830°; β = 90°; τ = 90°}, at which the following condition is fulfilled with a sufficient accuracy, was found using the program described above at τ = 90° and β = 90° with the successive decrease of the step of the change of the variables α_1, ..., α_4

$$\frac{\partial f(x^{(1)})}{\partial \alpha_i} = \frac{\partial f(x^{(1)})}{\partial \beta} = 0,\ i = \overline{1,\ 4}\ .$$

Figure 5 shows good closeness of $x^{(1)}$ to the point of the minimum of cond $M = f(\alpha_1, ..., \alpha_4, \beta, \tau = 90°)$, because the values of extremes with respect to each variable and the values of the variables at the points of extremes are close to $x^{(1)}$.

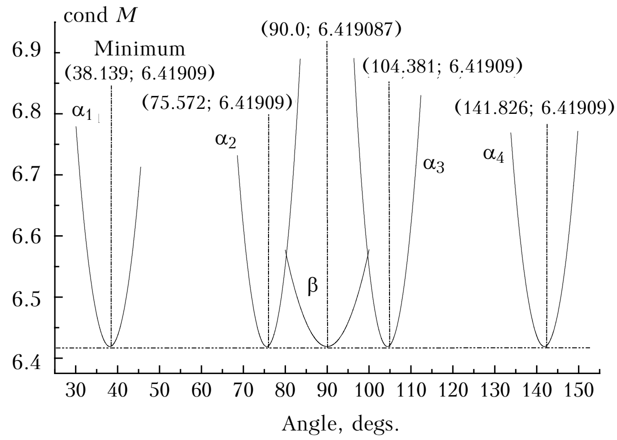

Fig. 5. Numerical investigation of cond $M = f(\alpha_1, ..., \alpha_4, \beta, \tau = 90°)$ for extreme with respect to each variable at the point $x^{(1)}$.

Figure 6 shows the effect of the absolute error Δ in setting α_1, ..., α_4, β, and τ near their values at the point $x^{(1)}$ on the value of cond M. Comparison of Fig. 6 and Fig. 4 shows that the value of cond M at the point $x^{(1)}$ is greater than that at the point $x^{(0)}$. Comparison of the curves *1–4* (Fig. 6) shows that the absolute error in setting τ relative to its value at the point $x^{(1)}$ stronger affects the value of the maximum cond M than the absolute error in setting β relative to its value at the point $x^{(1)}$, and it is decisive for the maximum value of cond M.

If the condition of constancy of the step of the change α is included in the algorithm of the control of the polarization block, i.e., $\alpha_4 - \alpha_3 = \alpha_3 - \alpha_2 = \alpha_2 - \alpha_1$, then the point of the conditional minimum of cond M at τ = 131.795° $x^{(2)}$ = {α_1 = 39.3°; α_2 = 73.1°; α_3 = 106.9°; α_4 = 140.7°; β = 90°; τ = 131.795°}.

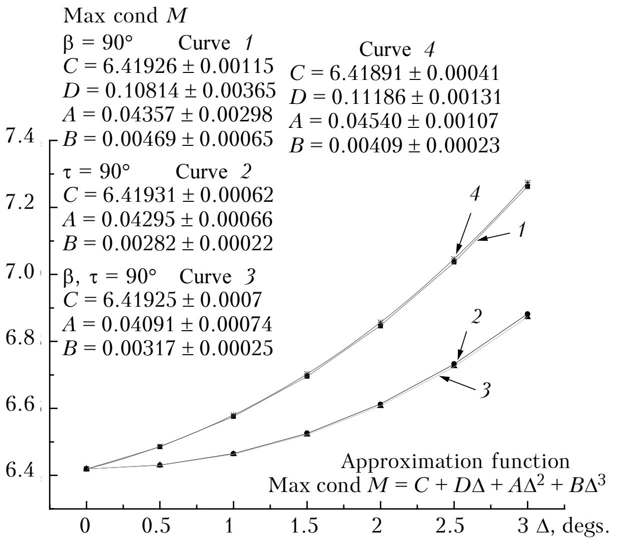

Fig. 6. Sensitivity of cond M to the absolute deviations of α_1, ..., α_4, β, τ from their values at the point $x^{(1)}$, min cond M = 6.419086 at the point $x^{(1)}$.

Curves *1–4* (Fig. 7) show the effect of the absolute errors in setting α_1, ..., α_4, β, τ relative to their values at the point of the conditional minimum $x^{(2)}$. Comparison of Fig. 7 and Fig. 4 shows insignificant increase of cond M at the point $x^{(2)}$ as compared with $x^{(0)}$. Comparison of curves *2* and *3* (Fig. 7) shows that the absolute error in setting τ relative to its value at the point $x^{(2)}$ stronger affects the value of cond M than the absolute error in setting β relative to its value at the point $x^{(2)}$, but it is not decisive.

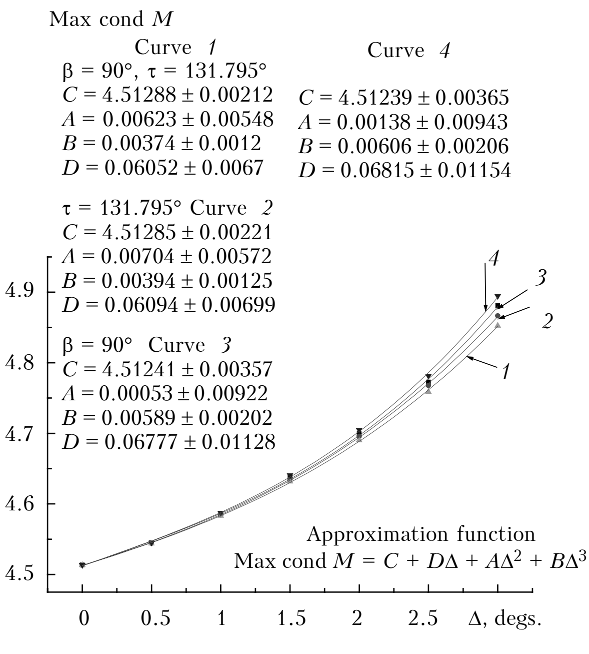

Fig. 7. Sensitivity of cond M to the absolute deviations of α_1, ..., α_4, β, τ from their values at the point of conditional minimum $x^{(2)}$, min cond M = 4.513566 at the point $x^{(2)}$.

If the condition of constancy of the step of the change α is included in the algorithm of control of the polarization block, but at τ = 90°, the point of the conditional minimum of cond M $x^{(3)}$ = = {α_1 = 39.9°; α_2 = 73.3°; α_3 = 106.7°; α_4 = 140.1°; β = 90°; τ = 90°}.

Curves *1–4* (Fig. 8) show the effect of the absolute errors in setting α_1, ..., α_4, β, τ relative to their values at the point of the conditional minimum $x^{(3)}$. The value of cond M at the point $x^{(3)}$ is greater than the value of cond M at the points $x^{(2)}$, $x^{(1)}$, and $x^{(0)}$, but it insignificantly differs from the value of cond M at the point $x^{(1)}$. Comparison of curves *2* and *3* (Fig. 8) shows that the absolute error in setting τ relative to its value at the point $x^{(3)}$ stronger affects the value of cond M than the absolute error in setting β relative to its value at the point $x^{(3)}$, and it is decisive for the maximum possible value of cond M. Curves in Figs. 4 and 6–8 were calculated with the step Δ = 0.5. The function that approximates the calculated tables and the coefficients at the approximation function for each curve are presented in each figure.

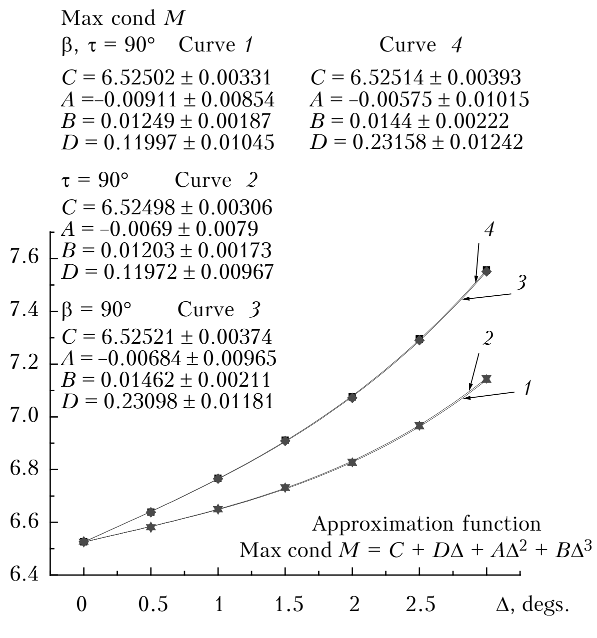

Fig. 8. Sensitivity of cond M to the absolute deviations of α_1, ..., α_4, β, τ from their values at the point of conditional minimum $x^{(3)}$, min cond M = 6.526094 at the point $x^{(3)}$.

The calculations have shown that the maximum value of cond M for the considered absolute errors Δ lies at a node point of the network of values of the variables, which contains the boundaries of the ranges of their possible values determined by Δ. Hence, the increase of the accuracy of setting any parameter always leads only to a decrease of cond M, i.e., to the increase of the measurement accuracy.

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