Synthesis of wavelet basis for analysis of optical signals. Part 3. Representation of differential and inverse operators in wavelet bases. Wavelet packages

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Detection of singularities in signals using wavelet decomposition is presented. An algorithm for construction of differential and inverse operators in wavelet bases is described for applications to problems useful from the viewpoint of adaptive optics. Detailed examples of such operators are presented. The construction of adaptive wavelets or the wavelet packages is considered.

Introduction

In restoring optical images, one inevitably faces the need in improving the image sharpness and contrast. These image characteristics are determined by the intensity gradient, that is, the larger are the intensity differences, the higher the contrast and edge sharpness. Consequently, changing the intensity gradient, one can improve blurred images by enhancing their high-frequency components and suppressing the low-frequency background trends.[1]

The problem of reconstruction of the phase of an optical wave from the field of its gradients arises in the problems of adaptive optics and optical control. The problem complicates, if the radiation recorded has passed through a randomly inhomogeneous medium. In this case, singularities arise in the optical wave phase, and these indicate the beginning of beam decomposition into uncorrelated parts.[2]

The wavelet basis is preferable for analysis and synthesis of an optical signal in this case. It allows one to efficiently represent differential and inverse operators, shows the features of fast transformations, the property of locality, and high compression coefficient. The locality makes the basis highly noise-immune, allows presenting signals having a broken speckle structure, and visualizing hidden periodicity of a signal. This paper presents some examples of decomposition of singular signals in terms of wavelets obtained in Refs. 3 and 4, as well as reconstruction of singular signals from their first and second derivatives and examples of operator representation in wavelet bases.

Selection of a basis for representation of functions with singularities

To represent the functions that have singularities, we need a basis similar to generalized functions. The basis function should ignore the low-frequency polynomial trend (regular component) of a signal in the vicinity of a point under consideration, but leave higher-order singularities. Such properties are inherent in wavelet bases. Wavelets of the Mth order are most efficient for representation of singularities; just such wavelet functions I obtained in Refs. 3 and 4. Wavelets of the Mth order obey the following condition:

$$\int \Psi(x)x^n \mathrm{d}x = 0,\ n = 0,\ \ldots,\ M, \qquad (1)$$

where M is the order of the ignored polynomial of the function analyzed in the vicinity of the considered point. The higher the wavelet order, the more efficient is the wavelet representation for high-order singularities of the function analyzed.

Let us exemplify localization of signal singularities using a continuous wavelet decomposition:

$$W(s,\ p) = \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{s}} \Psi\left(\frac{x-p}{s}\right) f(x)\mathrm{d}x. \qquad (2)$$

As a model, let us take a signal

$$f(x) = \sin(2x) + 0.1\mathrm{sign}[\sin(2x^2)], \qquad (3)$$

where

$$\mathrm{sign}(x) = \left\{ \begin{array}{ll} 1, & \text{for } x \geq 0 \\ -1, & \text{for } x < 0 \end{array} \right\} \qquad (4)$$

is the sign function. The graphical form of the signal $f(x)$ and its components $\sin(2x)$ and $\mathrm{sign}(\sin(2x^2))$ are shown in Fig. 1*a*. As wavelet bases, we took symmetric orthogonal wavelets of high orders of different shape (see Fig. 1*d*). One of them resembles delta function, while the other is similar to its derivative. Below, under the plot of the model signal, one can see the patterns of the coefficients of its wavelet decomposition $W(s, p)$ with the successively increasing resolution. The first pattern

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(Fig. 1*b*) of the coefficients of a delta-like wavelet quite clearly demonstrates the signal decomposition into different scales of inhomogeneities, separating the subzones by sharp boundary lines. Signal evolution is seen clearly. As the resolution increases, smaller scales become visible, and the distances between the boundaries in this scale obey the square law [see the second term in Eq. (3)]. And, finally, at the smallest scale, we can see localization of signal singularities, and bright light stripes indicate the coordinate of localization.

The second pattern (Fig. 1*c*) of the coefficients is interesting because it shows the signal derivative at different scales. This pattern is observed, when the signal passed through the subaperture (lenslet), whose center moves along the spatial axis of the signal (horizontal coordinate), and the radius changes continuously along the vertical axis. In this case, the boundary lines show signal maxima and minima, while antinodes correspond to signal inflection points. The small-scale coefficients point to localization of the singularity derivative.

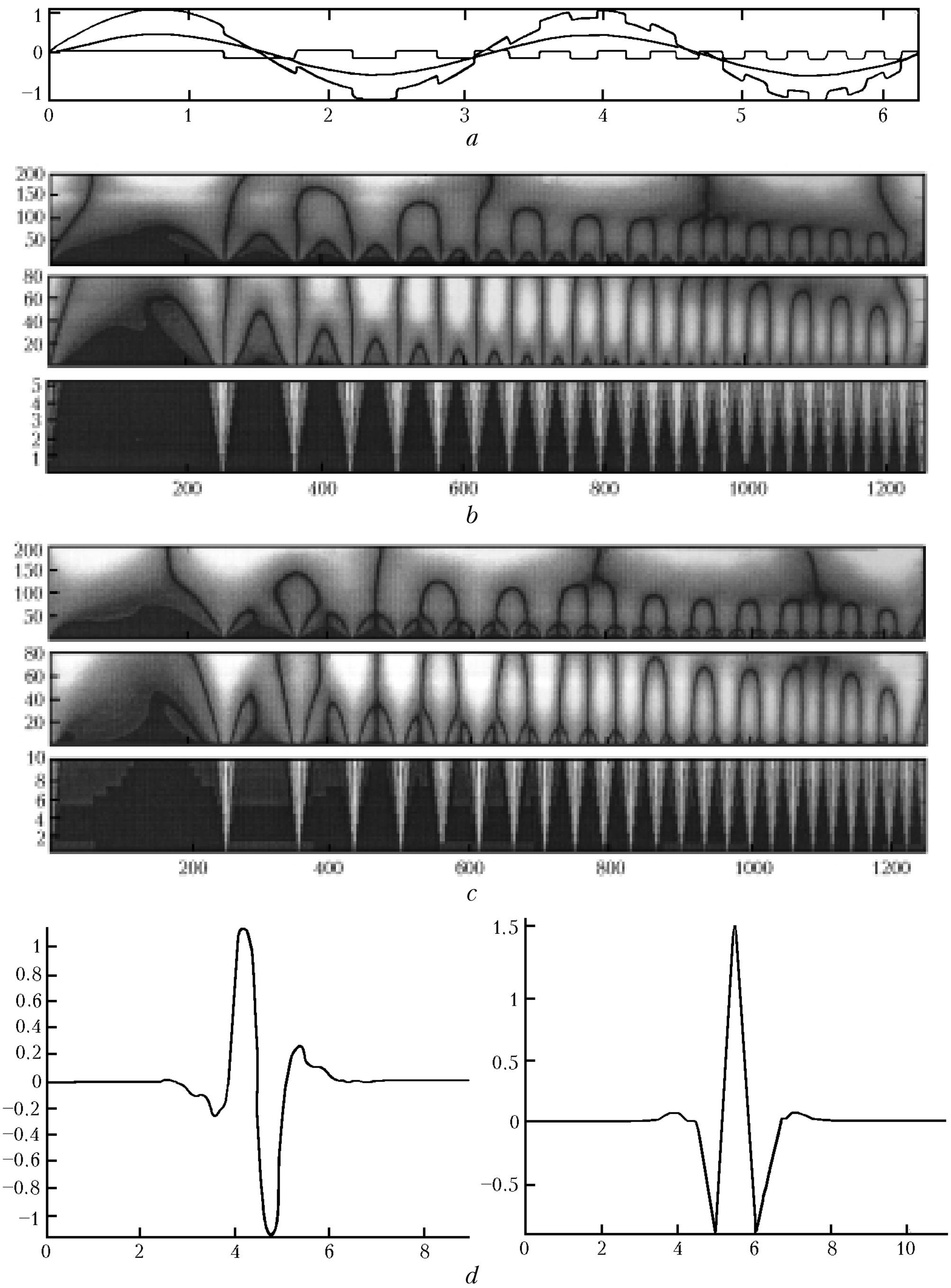

Fig. 1.

Representation of differential and inverse operators in the basis of scaling functions

Describe now the algorithm for representation of differential operators $\frac{d^n}{dx^n}$ in the basis of scaling functions obeying the property of multiresolution analysis:

$$\varphi(x) = \sum_{k=0}^{N} p_k \varphi(2x - k). \quad (5)$$

Determine the elements of the matrix of operators $\frac{d^n}{dx^n}$:

$$D_k^{(n)} = \langle \varphi(x) \left| \frac{d^n}{dx^n} \right| \varphi(x - k) \rangle. \quad (6)$$

Here $\langle f_1(x) | f_2(x) \rangle = \int_{-\infty}^{\infty} f_1(x) f_2(x) dx$ is the scalar product.

After substitution of Eq. (5) into Eq. (6) we obtain

$$D_k^{(n)} = \sum_{s,m=0}^{N} p_s p_m \langle \varphi(2x - s) \left| \frac{d^n}{dx^n} \right| \varphi(2x - 2k - m) \rangle =$$

$$= 2^{n-1} \sum_{s,m=0}^{N} p_s p_m D_{2k-s+m}^{(n)}. \quad (7)$$

By use of simple transformations, we can obtain Eq. (7) in a more detail form

$$D_k^{(n)} = 2^{n-1} \sum_{sm=0}^{N} p_s p_m D_{2k-s+m}^{(n)} =$$

$$= 2^{n-1} \sum_{m=-N+1}^{N-1} \sum_{s=0}^{N} p_s p_{m-2k+s} D_m^{(n)}. \quad (8)$$

Introduce the following designation:

$$A_{k,m}^{(n)} = 2^{n-1} \sum_{s=0}^{N} p_s p_{m-2k+s} = 2^{n-1} a_{m-2k},$$

$$-N + 1 \le k,\ m \le N - 1, \quad (9)$$

where $a_s = \sum_{m=0}^{N} p_m p_{m+s}$, then system (8) can be written as an algebraic equation for eigenvalues:

$$\mathbf{D}^{(n)} = A^{(n)} \mathbf{D}^{(n)} = 2^n A^{(0)} \mathbf{D}^{(n)}. \quad (10)$$

Once having the matrix elements of differential operators determined, we can find the operators inverse to the differential ones by inverting the obtained matrices. However, it should be kept in mind that the diagonal elements of the matrix of the first derivative are zero and the matrix inverse to it becomes singular after the inversion. Therefore, it is necessary to perform regularization (pseudoinversion of the matrix). As was mentioned above, the local basis is stable, and therefore it is sufficient to make a slight shift of the spectrum.

Present an example of representation of the considered operators for the trapezoidal scaling function $\varphi(x)$ (Fig. 2) and restoration of the model function from its first and second derivatives using the function $\varphi(x)$. Write the double-resolution relationship:

$$\varphi(x) = \frac{1}{2} \sum_{k=0}^{3} \varphi(2x - k). \quad (11)$$

In this relationship all the coefficients p are equal, that is, $p_0 = p_1 = p_2 = p_3 = \frac{1}{2}$.

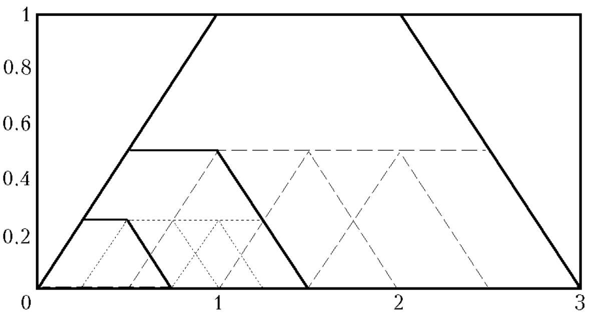

Fig. 2.

To represent the matrix elements of the operators $\frac{d^n}{dx^n}$, it is necessary to determine the elements of the matrix $A^{(0)}$ [see Eq. (8)]. In the case under consideration, they are

$$a_0 = \sum_{m=0}^{3} p_m^2 = 1, \quad a_1 = \sum_{m=0}^{3} p_m p_{m+1} = 0.75,$$

$$a_2 = \sum_{m=0}^{3} p_m p_{m+2} = 0.5,\ a_3 = \sum_{m=0}^{3} p_m p_{m+3} = 0.25,$$

$$a_{-1} = \sum_{m=0}^{3} p_m p_{m-1} = 0.75,\ a_{-2} = \sum_{m=0}^{3} p_m p_{m-2} = 0.5,$$

$$a_{-3} = \sum_{m=0}^{3} p_m p_{m-3} = 0.25. \quad (12)$$

Write the matrix $\mathbf{A}^{(0)}$ in the expanded form

$$\mathbf{A}^{(0)} = \frac{1}{2} \begin{pmatrix} a_2 & a_3 & 0 & 0 & 0 \\ a_0 & a_1 & a_2 & a_3 & 0 \\ a_{-2} & a_{-1} & a_0 & a_1 & a_2 \\ 0 & a_{-3} & a_{-2} & a_{-1} & a_0 \\ 0 & 0 & 0 & a_{-3} & a_{-2} \end{pmatrix} =$$

$$= \begin{pmatrix} 0.25 & 0.125 & 0 & 0 & 0 \\ 0.5 & 0.375 & 0.25 & 0.125 & 0 \\ 0.25 & 0.375 & 0.5 & 0.375 & 0.25 \\ 0 & 0.125 & 0.25 & 0.375 & 0.5 \\ 0 & 0 & 0 & 0.125 & 0.25 \end{pmatrix}. \quad (13)$$

Find the elements of the matrices $D_k^{(0)} = \langle \varphi(x) | \varphi(x-k) \rangle$, $D_k^{(1)} = \langle \varphi(x) \left| \frac{\mathrm{d}}{\mathrm{d}x} \right| \varphi(x-k) \rangle$, $D_k^{(2)} = \langle \varphi(x) \left| \frac{\mathrm{d}^2}{\mathrm{d}x^2} \right| \varphi(x-k) \rangle$ (k varies within $-2 \le k \le 2$) by solving the equations $\mathbf{D}^{(0)} = A^{(0)}\,\mathbf{D}^{(0)}$, $\mathbf{D}^{(1)} = 2A^{(0)}\,\mathbf{D}^{(1)}$, $\mathbf{D}^{(2)} = 4A^{(0)}\,\mathbf{D}^{(2)}$.

As a result, we get the following solutions:

$$\mathbf{D}^{(0)} = \left(\frac{1}{6}, 1, \frac{10}{6}, 1, \frac{1}{6}\right) = (0.167, 1, 1.67, 1, 0.167),$$

$$\mathbf{D}^{(1)} = \left(-\frac{1}{2}, -1, 0, 1, \frac{1}{2}\right) = (-0.5, -1, 0, 1, 0.5),$$

$$\mathbf{D}^{(2)} = \left(\frac{1}{2}, 0, -1, 0, \frac{1}{2}\right) = (0.5, 0, -1, 0, 0.5).$$

Now let us show how do the obtained representations work. Take a model signal, whose graphical form is depicted in Fig. 3*a* and the first and second derivatives are shown in Figs. 3*b* and *c*, respectively (dashed curves):

$$f(x) = \sin(x^3\,0.4)\exp[-(x-2)^4\,0.7],\ 0 \le x \le 4.8. \quad (14)$$

Form a basis of scaling functions $\varphi(2^j x - k)$, $j = 6$, $0 \le k \le M = 320$. Find the coefficients b_k of signal resolution in the basis

$$f(x) = \sum_{k=0}^{M} b_k\,\varphi(2^j x - k), \quad (15)$$

and for this purpose form a five-diagonal matrix

$$d_{m,s}^{(0)} = D_{m-s}^{(0)} = \langle \varphi(x2^j - m) \,|\, \varphi(x2^j - s) \rangle,$$

$$0 \le m,\ s \le M$$

$$\mathbf{d}^{(0)} =$$

$$= \begin{pmatrix} 1.67 & 1 & 0.167 & 0 & 0 & 0 & \dots \\ 1 & 1.67 & 1 & 0.167 & 0 & 0 & \dots \\ 0.167 & 1 & 1.67 & 1 & 0.167 & 0 & \dots \\ 0 & 0.167 & 1 & 1.67 & 1 & 0.167 & \dots \\ 0 & 0 & 0.167 & 1 & 1.67 & 1 & \dots \end{pmatrix} \quad (16)$$

and the column vector $c_s = \langle f(x) \,|\, \varphi(x2^j - s) \rangle$. The coefficients b_k of resolution of the signal (15) in this case are determined by the matrix equation:

$$\mathbf{b} = \mathbf{d}^{(0)-1} \cdot \mathbf{c}. \quad (17)$$

The resulting resolution is shown in Fig. 3*a* (solid curve).

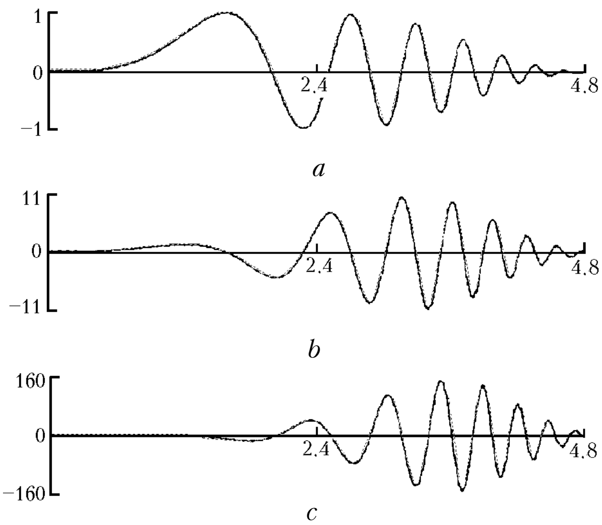

Fig. 3. Model function $f(x)$ and its wavelet restoration (*a*); $\mathrm{d}f(x)/\mathrm{d}x$ and its wavelet resolution (*b*); $\mathrm{d}^2 f(x)/\mathrm{d}x^2$ and its wavelet resolution (*c*).

Now determine the coefficients $b1_k$ of resolution of the model function derivative

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \sum_{k=0}^{M} b1_k\,\varphi(2^j x - k). \quad (18)$$

Form a five-diagonal matrix of the operator of the first derivative

$$d_{m,s}^{(1)} = D_{m-s}^{(1)} = \langle \varphi(x2^j - m) \left| \frac{\mathrm{d}}{\mathrm{d}x} \right| \varphi(x2^j - s) \rangle,$$

$$0 \le m,\ s \le M,$$

$$\mathbf{d}^{(1)} =$$

$$\begin{pmatrix} 0 & 1 & 0.5 & 0 & 0 & 0 & \dots \\ -1 & 0 & 1 & 0.5 & 0 & 0 & \dots \\ -0.5 & -1 & 0 & 1 & 0.5 & 0 & \dots \\ 0 & -0.5 & -1 & 0 & 1 & 0.5 & \dots \\ 0 & 0 & -0.5 & -1 & 0 & 1 & \dots \end{pmatrix}, \quad (19)$$

find the coefficients $b1_k$ (18) using the matrix product of the vector b_k in Eq. (15) by the matrix (19)

$$\mathbf{b1} = -\mathbf{d}^{(1)} \cdot \mathbf{b}. \quad (20)$$

The restored by Eq. (18) result on the derivative of the function $f(x)$ is depicted in Fig. 3*b* (dashed curve).

Represent the second derivative as a series in terms of the scaling functions $\varphi(x)$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) = \sum_{k=0}^{M} b2_k \, \varphi(2^j x - k) \tag{21}$$

and determine the coefficients $b2_k$ in Eq. (21) using the matrix product of the vector b_k in Eq. (15) by the five-diagonal matrix of the operator of the second derivative

$$d^{(2)}_{m,s} = D^{(2)}_{m-s} = \langle \varphi(x2^j - m) \left| \frac{\mathrm{d}^2}{\mathrm{d}x^2} \right| \varphi(x2^j - s) \rangle,$$

$$0 \le m,\ s \le M$$

$$\mathbf{d}^{(2)} = \begin{pmatrix} -1 & 0 & 0.5 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0.5 & 0 & 0 & \dots \\ 0.5 & 0 & -1 & 0 & 0.5 & 0 & \dots \\ 0 & 0.5 & 0 & -1 & 0 & 0.5 & \dots \\ 0 & 0 & 0.5 & 0 & -1 & 0 & \dots \end{pmatrix}, \tag{22}$$

and obtain

$$\mathbf{b}2 = -\mathbf{d}^{(2)} \cdot \mathbf{b}. \tag{23}$$

The resolution of the second derivative (21) with the coefficients obtained by Eq. (23) is depicted in Fig. 3*c* (solid curve). The plots demonstrate quite good restoration of the model function and its first and second derivatives.

For solution of the inverse problem, restore the model function $f(x)$ from its first $\frac{\mathrm{d}f(x)}{\mathrm{d}x}$ and second $\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}$ derivatives. Toward this end, it is sufficient to invert Eqs. (20) and (23), that is, to find the coefficients $\mathbf{b}$ by the equations

$$\mathbf{b} = -\mathbf{d}^{(1)-1} \cdot \mathbf{b}1, \tag{24}$$

$$\mathbf{b} = -\mathbf{d}^{(2)-1} \cdot \mathbf{b}2, \tag{25}$$

and then substitute the determined coefficients into Eq. (15). As was noted above it is necessary to make, in Eq. (24), a slight spectral shift, since the matrix $\mathbf{d}^{(1)}$ has zero diagonal elements. Equation (24) should be written as

$$\mathbf{b} = -(\mathbf{d}^{(1)} - \mathbf{I} \cdot \lambda)^{-1} \cdot \mathbf{b}1, \tag{26}$$

where $\mathbf{I}$ is the unit matrix; $\lambda = 10^{-8}$. Here we omit the plots of $f(x)$ restored from the coefficients determined by Eqs. (24) and (26), since they are identical to the initial model plot (see Fig. 3*a*). Note only that the maximum square deviation of the restored functions from the model one was $\varepsilon = 10^{-7}$. It should be emphasized that when the model signal $f(x)$ is restored from its derivatives, no information about the constant and linear components of $f(x)$ is lost as in the case of the polynomial factoring. The wavelet basis works in the generalized meaning.

For demonstration, let us present the derivatives of the model signal (3) (see Fig. 4*a*). Figures 4*b* and *c* depict the first and second derivatives of the signal, respectively, that were obtained through wavelet resolution by the algorithm described above. Figure 4*c* shows the same plot but on different scales. The inverse problem, namely, restoration of a signal from its first and second derivatives was solved by Eqs. (24) and (26) and the error of signal restoration did not exceed $\varepsilon = 10^{-6}$. The coefficients of the matrices $\mathbf{D}^{(n)}$, $n = 0, 1, 2$, for the wavelets shown in Fig. 1 are tabulated below.

k	p_k	$\mathbf{d}^{(0)}$	$\mathbf{d}^{(1)}$	$\mathbf{d}^{(2)}$
–9	0	0	0	0
–8	0	0	0	0
–7	0	0	$-1.2175786 \cdot 10^{-8}$	$-1.069838 \cdot 10^{-7}$
–6	0	0	–0.0000016	$9.36842 \cdot 10^{-6}$
–5	0	0	0.0001787	–0.00085
–4	0	0	0.0014626	–0.004666
–3	0	0	–0.0315657	0.074327
–2	0.000828	0	0.1883776	–0.349716
–1	0.018714	0	–0.7887922	1.325481
0	–0.045872	1	0	–2.089173
1	–0.073234	0	0.7887922	1.325481
2	0.398704	0	–0.1883776	–0.349716
3	0.816354	0	0.0315657	0.074327
4	0.398704	0	–0.0014626	–0.004666
5	–0.073234	0	–0.0001787	–0.00085
6	–0.045872	0	0.0000016	$9.36842 \cdot 10^{-6}$
7	0.018714	0	$1.2175786 \cdot 10^{-8}$	$-1.069838 \cdot 10^{-7}$
8	0.000828	0	0	0
9		0	0	0
–8	0	0	0	0
–7	0	0	0	0
–6	0	0	0	0.000006
–5	0	0	0.00001	–0.000133
–4	0	0	0.00067	–0.001774
–3	0	0	0.01932	0.063947
–2	0.004968	0	0.15275	–0.420757
–1	0.004968	0	–0.75028	1.638971
0	–0.083230	1	0	–2.560522
1	0.083230	0	0.75028	1.638971
2	0.697206	0	–0.15275	–0.420757
3	0.697206	0	0.01932	0.063947
4	0.083230	0	–0.00067	–0.001774
5	–0.083230	0	–0.00001	–0.000133
6	0.004968	0	0	0.000006
7	0.004968	0	0	0
8	0	0	0	0

Wavelet packages

Introduce the generalized concept of wavelets – wavelet packages. The wavelet packages are constructed by the following scheme[5]: some wavelet with its scaling function is selected. Double-resolution relationships are written for it:

$$\varphi(x) = \sum_{k=0}^{N} p_k \, \varphi(2x - k); \tag{27}$$

$$\psi(x) = \sum_{k=0}^{N} q_k \, \varphi(2x - k),\ q_k = (-1)^k p_{-k+1}. \tag{28}$$

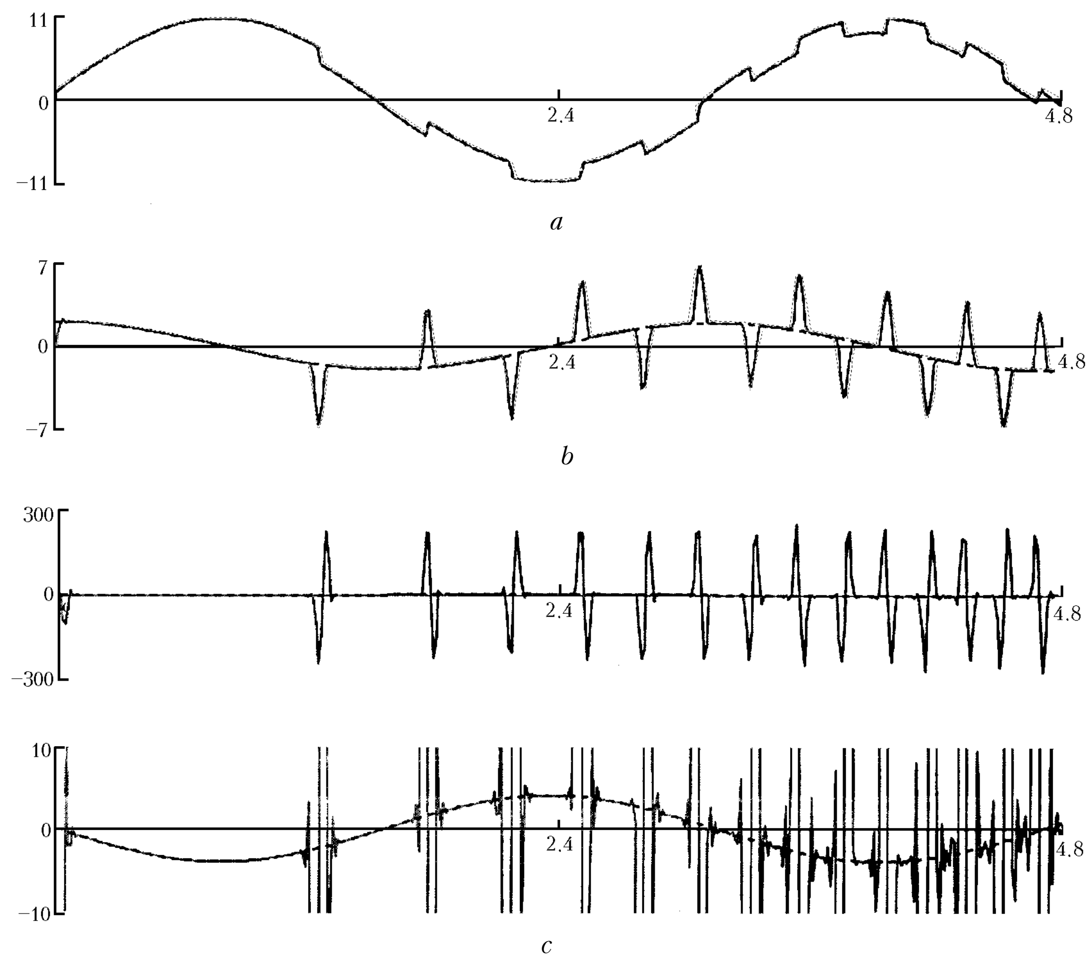

Fig. 3. Model function $f(x)$ and its wavelet restoration (a); $\frac{df(x)}{dx}$ and its wavelet resolution (b); $\frac{d^2f(x)}{dx^2}$ and its wavelet resolution (c).

The wavelet $\psi(x)$ is substituted for the scaling function $\varphi(x)$ in Eqs. (27) and (28), and the following equations are obtained:

$$\mu_3(x) = \sum_{k=0}^{N} p_k \psi(2x - k), \quad (29)$$

$$\mu_4(x) = \sum_{k=0}^{N} q_k \psi(2x - k),\ q_k = (-1)^k p_{-k+1}. \quad (30)$$

This procedure can be continued, and finally we get a family of orthogonal functions called wavelet packages:

$$\mu_{2l}(x) = \sum_{k=0}^{N} p_k \mu_l(2x - k), \quad (31)$$

$$\mu_{2l+1}(x) = \sum_{k=0}^{N} q_k \mu_l(2x - k), \quad (32)$$

where $\mu_0(x) = \varphi(x)$, $\mu_l(x) = \psi(x)$, $l = 0, 1, 2 \ldots$.

The obtained basis of wavelet packages has better localization and is an adaptive basis. When resolving function $f(x)$ into the wavelet packages basis

$$f(x) = -\sum_{k,j,n=0} a_{k,j}\, \mu_n(2^j x - k) \quad (33)$$

along with localization index k and scaling index j there is also frequency index n. When choosing basis, the information-entropy criterion is used

$$E = -\sum_{k=0}^{N} |a_k|^2 \log(|a_k|^2), \quad E \to \min, \quad (34)$$

where a_k are the coefficients of signal resolution in the basis of wavelet packages.

Criterion (33) presents some principle of "similarity" of the approximated and approximating functions. The large value of E-entropy is indicative of the uniform contribution of each component, and the small value of E means that the resolved function is concentrated in a small number of basis directions.

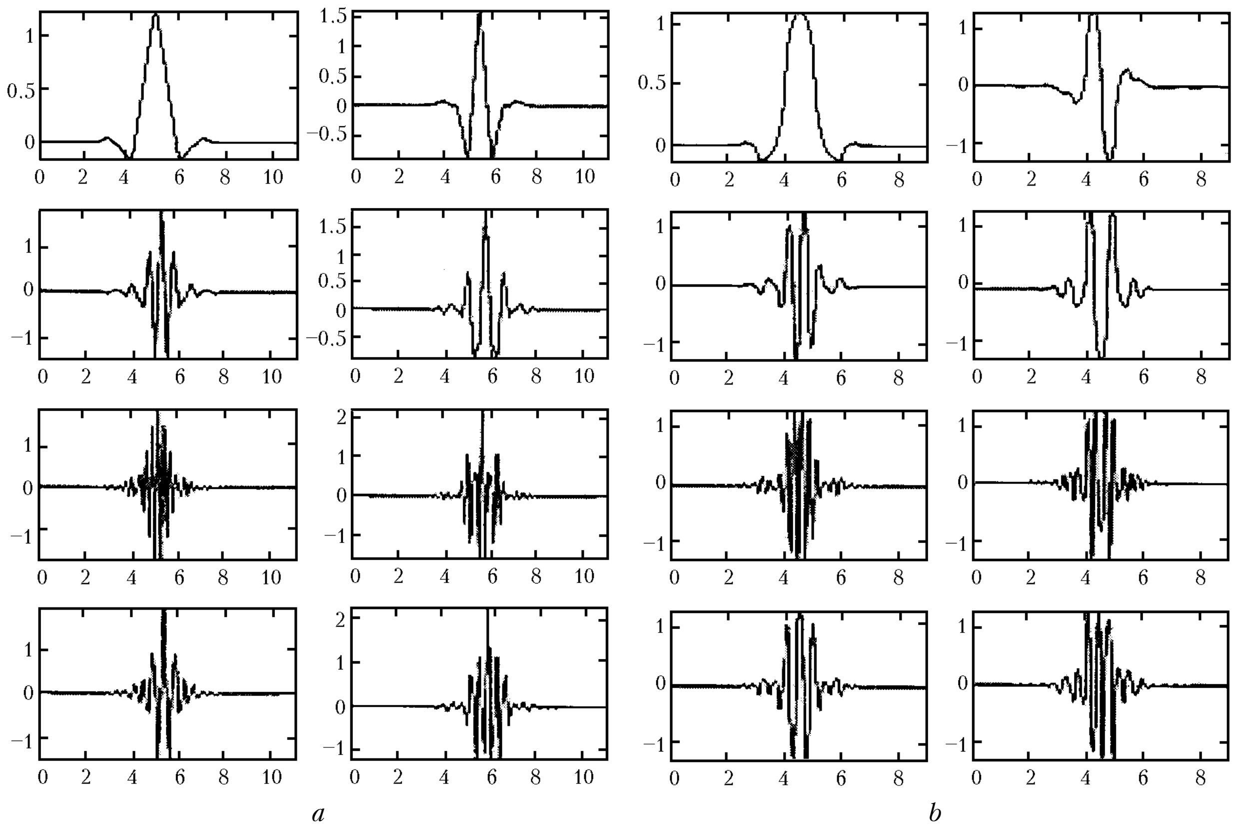

Fig. 5. Wavelet packages.

This means that when resolving different signals into the same basis, different basis functions from the same package will contribute into signal formation.

I have obtained wavelet packages based on the wavelets derived in Refs. 3 and 4. Figures 5*a* and *b* show the plots for the wavelet packages obtained from the wavelets (see Fig. 1), whose coefficients are given in the Table.

Conclusion

It has been shown that when separating singularities it is necessary to choose wavelets with zero high-order moments. Such wavelets increase the relative contribution of signal terms of high order of smallness. It has been explained how the pattern of evolution of the signal and its derivative can be visualized through selection of the proper wavelet form. Wavelets have no analytical representation therefore the described algorithms of formation of matrices representing differential and inverse operators in wavelet bases are important. Wavelet packages presented in this paper allow the signal to be resolved in the optimal way. Due to selective choice of basis functions, the adaptive packages constructed best fit to the analyzed functions based on the information-entropy criterion.

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