Statistical characteristics of the optical transfer function of the "turbulent atmosphere–telescope" system

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In this paper I have theoretically investigated the statistical characteristics of fluctuations of the optical transfer function of the "turbulent atmosphere–telescope" optical system when recording an averaged image. Such statistical characteristics of the optical transfer function fluctuations as the mean value, variance and correlation functions are calculated. The results obtained enable one to assess the image deterioration due to the effect of the atmospheric turbulence and, also, to introduce the quantitative estimation for the concepts of "very long" and "very short" exposures.

The joint action of the atmosphere and the optical system in forming an image of an incoherent source is considered traditionally as a random linear filtration, and the "atmosphere–optical instrument" system is characterized by the optical transfer function.[1] It is known[2–9] that the optical transfer function of the turbulent atmosphere depends essentially on the time of averaging (exposure time). The limiting cases of "very long"[2–5] and "very short"[3,5] exposures were studied extensively. The investigations of intensity fluctuations of optical radiation in the focal plane of a telescopic system depending on the time of averaging (exposure time) were carried out in Ref. 9.

In this paper the statistical characteristics have been calculated of the optical transfer function of the turbulent atmosphere and the telescopic optical system at an arbitrary exposure time.

An instantaneous value of the optical transfer function of the atmospheric turbulence and telescopic optical system can be written in the following form[2–5]:

$$M(\mathbf{p},\ t) =$$

$$= \int\int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{\rho}\, U(\boldsymbol{\rho},t)\, U^*(\boldsymbol{\rho}+\mathbf{p},\ t)\ K(\boldsymbol{\rho})\, K^*(\boldsymbol{\rho}+\mathbf{p}), \quad (1)$$

where $U(\boldsymbol{\rho},t)$ is the complex amplitude of the field at a point $\boldsymbol{\rho}$ at the receiving aperture at the moment t in time created by a point incoherent source located in the subject space; $K(\boldsymbol{\rho})$ is the pupil function of the receiving aperture; $\mathbf{p}$ is the spatial scale.

Since the field $U(\boldsymbol{\rho},t)$ is a random value, because of fluctuations of the air dielectric constant, the optical transfer function $M(\mathbf{p},\ t)$ turned out to be a random value varying in time. Therefore, in recording an object image using a detector with the finite response time (for example, photographing or visual observation) the averaging of the optical transfer function will occur. Thus the optical transfer function can be written in the form:

$$\bar{M}(\mathbf{p},\Delta t) = \frac{1}{\Delta t}\int_0^{\Delta t} \mathrm{d}t M(\mathbf{p},t), \quad (2)$$

where Δt is the time of averaging (exposure time). In view of the fact that the optical transfer function $\bar{M}(\mathbf{p},\Delta t)$ in the general case is also a random value, then to describe it, the method of moments is used. In particular, we consider the following statistical characteristics: mean value, variance, and correlation functions of the optical transfer function fluctuations.

Mean value of the optical transfer function is obtained by averaging Eq. (2) over the ensemble of realizations of dielectric constant of the atmospheric air:

$$\left\langle \bar{M}(\mathbf{p},\Delta t)\right\rangle = \frac{1}{\Delta t}\int_0^{\Delta t} \mathrm{d}t \left\langle M(\mathbf{p},t)\right\rangle =$$

$$= \frac{1}{\Delta t}\int_0^{\Delta t} \mathrm{d}t \int\int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{\rho}\ \Gamma_2(\boldsymbol{\rho},\ \boldsymbol{\rho}+\mathbf{p};\ t,\ t)\, K(\boldsymbol{\rho})\, K^*(\ \boldsymbol{\rho}+\mathbf{p}), \quad (3)$$

where

$$\Gamma_2(\boldsymbol{\rho},\ \boldsymbol{\rho}+\mathbf{p};\ t,\ t) = \langle U(\boldsymbol{\rho},\ t) U^*(\boldsymbol{\rho}+\mathbf{p},\ t)\rangle$$

is the function of mutual coherence of the optical wave field of the second order.[10]

Let the optical wave be plane (for example, the star radiation), i.e.,

$$U(\boldsymbol{\rho},t) = U_0 \exp\left[\psi(\boldsymbol{\rho},t)\right],$$

where U_0 is the optical wave amplitude;

$$\psi(\boldsymbol{\rho},t) = \chi(\boldsymbol{\rho},t) + iS(\boldsymbol{\rho},t)$$

are the fluctuations of the complex phase of the optical wave; $\chi(\boldsymbol{\rho},t)$ describes the fluctuations of a logarithm of the optical wave amplitude; $S(\boldsymbol{\rho},t)$ denotes the phase fluctuations of the optical wave. As known,[10] $\chi(\boldsymbol{\rho},t)$ and $S(\boldsymbol{\rho},t)$ in the applicability domain of the

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method of smooth perturbations have normal laws of the probability distribution, then

$$\Gamma_2(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;t_1,t_2) = U_0^2 \exp\left[-\frac{1}{2}D(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2,t_1-t_2)\right], \quad (4)$$

where $D(\boldsymbol{\rho},t)$ is the space-time structure function of complex phase fluctuations of the plane optical wave.[10]

Having substituted Eq. (4) into the Eq. (3) we obtain:

$$\left\langle \bar{M}(\mathbf{p},\Delta t)\right\rangle =$$

$$= U_0^2 \exp\left[-\frac{1}{2}D(p)\right]\int\int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{\rho}\, K(\boldsymbol{\rho})\, K^*(\boldsymbol{\rho}+\mathbf{p}), \quad (5)$$

where $D(p) = D(\mathbf{p},0)$ is the spatial structure function of the complex phase fluctuations of a plane optical wave at the spacing of observation points being equal to p.

Hence it follows that $\left\langle \bar{M}(\mathbf{p},\Delta t)\right\rangle$ does not depend on exposure time and is equal to the value of the optical transfer function at "infinitely" delays,[2,3,5] i.e.,

$$\left\langle \bar{M}(\mathbf{p},\Delta t)\right\rangle = \lim_{\Delta t\to\infty}\frac{1}{\Delta t}\int_0^{\Delta t}\mathrm{d}t\left\langle M(\mathbf{p},t)\right\rangle = \left\langle \bar{M}(\mathbf{p})\right\rangle.$$

The variance of fluctuations of the optical transfer function can be written as follows:

$$\sigma_M^2(\mathbf{p},\Delta t) = \left\langle \bar{M}^2(\mathbf{p},\Delta t)\right\rangle - \left\langle \bar{M}(\mathbf{p})\right\rangle^2 = \frac{1}{\Delta t^2}\int_0^{\Delta t}\int_0^{\Delta t}\mathrm{d}t'\,\mathrm{d}t'' \times$$

$$\times \int\int\int\int_{-\infty}^{\infty}\mathrm{d}\boldsymbol{\rho}'\mathrm{d}\boldsymbol{\rho}''\,\Gamma_4(\boldsymbol{\rho}',\ \boldsymbol{\rho}'+\mathbf{p};\ \boldsymbol{\rho}''+\mathbf{p},\ \boldsymbol{\rho}'';\ t',t'') \times$$

$$\times\ K(\boldsymbol{\rho}')K^*(\boldsymbol{\rho}'+\mathbf{p})K^*(\boldsymbol{\rho}'')K(\boldsymbol{\rho}''+\mathbf{p}) - \left\langle \bar{M}(\mathbf{p})\right\rangle^2, \quad (6)$$

where

$$\Gamma_4(\boldsymbol{\rho}',\ \boldsymbol{\rho}'+\mathbf{p};\ \boldsymbol{\rho}''+\mathbf{p},\ \boldsymbol{\rho}'';\ t',t'') =$$

$$= \langle U(\boldsymbol{\rho}',t')U^*(\boldsymbol{\rho}'+\mathbf{p},t')U(\boldsymbol{\rho}''+\mathbf{p},t'')U^*(\boldsymbol{\rho}'',t'')\rangle$$

is the fourth order coherence function of the optical wave.[10]

Accepting the same assumptions as previously in deriving the function of the second order mutual coherence of the optical wave field and considering that $\left\langle \chi^2(\boldsymbol{\rho},t)\right\rangle \ll 1$ (this condition is well fulfilled for the plane wave and the paths penetrating the entire thickness of the Earth's atmosphere at zenith angles $\le 80°$) we derive

$$\Gamma_4(\boldsymbol{\rho}',\ \boldsymbol{\rho}'+\mathbf{p};\ \boldsymbol{\rho}''+\mathbf{p},\ \boldsymbol{\rho}'';\ t',t'') \cong$$

$$\cong U_0^4 \exp\left[-D(\mathbf{p},0)+\frac{1}{2}D(\boldsymbol{\rho}'-\boldsymbol{\rho}''-\mathbf{p},\ t'-t'') + \right.$$

$$\left. +\frac{1}{2}D(\boldsymbol{\rho}'-\boldsymbol{\rho}''+\mathbf{p},t'-t'') - D(\boldsymbol{\rho}'-\boldsymbol{\rho}'',t'-t'')\right]. \quad (7)$$

It turns out that $\sigma_M^2(\mathbf{p},\Delta t)$ depends not only on the time of averaging, but also on the shape and dimensions of the entrance pupil of the telescopic system. Therefore, for the subsequent considerations it is required to set a specific type of the function of entrance pupil of the receiving aperture. Assume that a fluctuating wave is incident on the circular objective over an area $S_R = \pi R^2$, where R is the radius of the receiving aperture and the function of the pupil of the receiving aperture can be chosen in the form of the quadratic exponential function[5,6]:

$$K(\boldsymbol{\rho}) = K_0 \exp\left(-\frac{1}{2R^2}\rho^2\right), \quad (8)$$

where K_0 is the amplitude transmission of the telescope on the optical axis of the system. In this case, $\left\langle \bar{M}(\mathbf{p})\right\rangle\Big|_{\mathbf{p}=0} = \pi U_0^2 R^2$ is the value of the mean optical transfer function at zero spatial frequency.

When calculating the integral using one spatial variable (6) we derive a simple expression for the variance of the optical transfer function:

$$\sigma_M^2(\mathbf{p},\Delta t) \cong \frac{\pi U_0^4 K_0^4 R^2}{2\Delta t^2}\exp\left(-\frac{1}{2R^2}p^2\right)\times$$

$$\times\int_0^{\Delta t}\int_0^{\Delta t}\mathrm{d}t'\mathrm{d}t''\int\int_{-\infty}^{\infty}\mathrm{d}\boldsymbol{\rho}\exp\left(-\frac{1}{2R^2}\rho^2\right)\times$$

$$\times\exp\left[-D(\mathbf{p},0)+\frac{1}{2}D(\boldsymbol{\rho}-\mathbf{p},t'-t'') + \right.$$

$$\left. +\frac{1}{2}D(\boldsymbol{\rho}+\mathbf{p},t'-t'') - D(\boldsymbol{\rho},t'-t'')\right] - \left\langle \bar{M}(\mathbf{p})\right\rangle^2. \quad (9)$$

For telescopic systems of small dimensions ($R < L_0$, where L_0 is the outer scale of the atmospheric turbulence) using the Taylor hypothesis of "freezing," the structural function of complex phase can be written in the form:

$$D(\boldsymbol{\rho},t) = 0.73 C_\varepsilon^2 k^2 h |\boldsymbol{\rho} - \mathbf{V}t|^{5/3},$$

where $k = 2\pi/\lambda$, λ is the radiation wavelength in the free space; C_ε^2 is the ground value of the structure parameter of fluctuations of the dielectric constant of the turbulent atmosphere; $h = C_\varepsilon^{-2}\int_0^{\infty}\mathrm{d}x C_\varepsilon^2(x,\theta)$ is the effective optical thickness of the active layer of the atmospheric turbulence; $C_\varepsilon^2(x,\theta)$ is the altitude profile of the structure parameter of the fluctuations of the dielectric constant of the turbulent atmosphere depending on the zenith angle θ; $\mathbf{V}$ is the wind velocity in the direction perpendicular to the direction of light propagation.

Asymptotic analysis of Eq. (9) has shown that in the region of small spatial scales: $p \ll D^{-3/5}(R)R$ and $p \ll R$ the variance of fluctuations of the optical transfer function is of the form:

$$\sigma_M^2(\mathbf{p},\Delta t) \cong \pi^2 U_0^4 K_0^4 R^4 f\left(\frac{V\Delta t}{R}\right) D(R) \left(\frac{p}{R}\right)^2, \quad (10)$$

where

$$f(x) \cong \begin{cases} 0.69\left(1-\frac{1}{48}x^2\right), & x \le 1, \\ 0.99x^{-1/3}, & x > 1. \end{cases}$$

In the intermediate region of spatial scales: $D^{-3/5}(R)R \ll p \ll R$ (which takes place only in the case when the structure function of the fluctuations of the complex phase over the aperture size is large as compared with unity), the variance of fluctuations of the optical transfer function equals:

$$\sigma_M^2(\mathbf{p},\ \Delta t) \cong \pi^2 U_0^4 K_0^4 R^4 \varphi\left(\frac{V\Delta t}{R}\right) \exp\left[-\frac{1}{2}\left(\frac{p}{R}\right)^2\right], \quad (11)$$

where

$$\varphi(x) \cong \begin{cases} \frac{1}{2} D^{-6/5}(R)\left(1-\frac{1}{12}x^2\right), & x \le 1, \\ \frac{1}{\sqrt{2}} x^{-1}, & x > 1. \end{cases}$$

And, finally, for large spatial scales ($p > R$) $\sigma_M^2(\mathbf{p},\Delta t) \cong 0$, the variance of the optical transfer function is small because of the presence of the following factor: $\exp\left[-\frac{1}{2}\left(\frac{p}{R}\right)^2\right]$.

From the results obtained it is concluded that with the increase of exposure time the fluctuations $M(\mathbf{p},\Delta t)$ decrease and at $\Delta t \gg \Delta t_0 = R/V$ they disappear at all. Figure 1 shows the dependence of the ratio of normalized variance of fluctuations of the optical transfer function $\sigma_M^2(\mathbf{p},\ \Delta t)/\sigma_M^2(\mathbf{p},\ 0)$ on the normalized time of averaging $\Delta t/\Delta t_0$.

Curve *1* was constructed for the region of small spatial scales ($p \ll D^{-3/5}(R)R$ and $p \ll R$), i.e., when

$$\sigma_M^2(\mathbf{p},\ \Delta t)/\sigma_M^2(\mathbf{p},\ 0) \cong f(\Delta t/\Delta t_0)\ f^{-1}(0),$$

and curve *2* was constructed for the intermediate region of spatial scales ($D^{-3/5}(R)R \ll p \ll R$):

$$\sigma_M^2(\mathbf{p},\ \Delta t)/\sigma_M^2(\mathbf{p},\ 0) \cong \varphi(\Delta t/\Delta t_0)\varphi^{-1}(0).$$

The behavior of these curves shows that the typical scale of variation of the variance $\sigma_M^2(\mathbf{p},\Delta t)$ depending on the time of averaging Δt is the ratio of the aperture size to the module of the mean wind velocity (Δt_0). Thus, the value of time of averaging Δt_0 can be considered a quantitative criterion of notions "very long" ($\Delta t \gg \Delta t_0$) and "very short" ($\Delta t \ll \Delta t_0$) exposures.

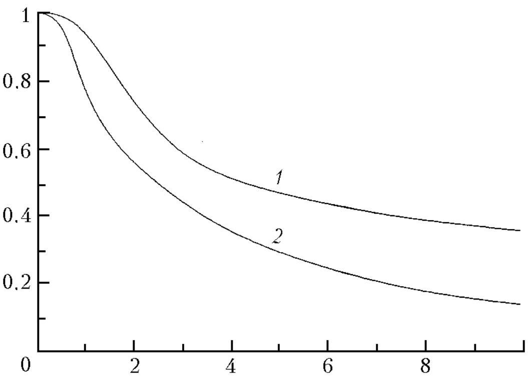

Fig. 1. The behavior of the normalized variance of fluctuations of the optical transfer function depending on the normalized time of averaging under different propagation conditions: at $D(R) < 1$ (*1*); at $D(R) > 1$ (*2*).

To determine the statistical relation of fluctuations of the optical transfer function at different spatial scales we consider the correlation function of fluctuations of the optical transfer function at different spatial scales:

$$B_M(\mathbf{p}_1, \mathbf{p}_2; \Delta t) =$$
$$= \left\langle \bar{M}(\mathbf{p}_1, \Delta t) \bar{M}^*(\mathbf{p}_2, \Delta t) \right\rangle - \left\langle \bar{M}(\mathbf{p}_1) \right\rangle \left\langle \bar{M}(\mathbf{p}_2) \right\rangle.$$

The integral expression for the correlation function has the structure similar to that of Eq. (9), therefore the asymptotic analysis can easily be conducted by the same method as for the variance of the optical transfer function.

In the region of small spatial scales: $p_{1,2} \ll R$ and $p_{1,2} \ll D^{-3/5}(R)R$ almost complete correlation of fluctuations takes place on all scales

$$b_M(\mathbf{p}_1, \mathbf{p}_2; \Delta t) = \frac{B_M(\mathbf{p}_1, \mathbf{p}_2; \Delta t)}{\sigma_M(\mathbf{p}_1, \Delta t)\sigma_M(\mathbf{p}_2, \Delta t)} \cong 1.$$

In the intermediate region ($D^{-3/5}(R)R \ll p_{1,2} \ll$ $\ll R$) the correlation decreases by the exponential function:

$$b_M(\mathbf{p}_1, \mathbf{p}_2; \Delta t) \cong \exp\left[-\left(\frac{|\mathbf{p}_1 - \mathbf{p}_2|}{p_{\text{cor}}}\right)^{5/3}\right],$$

where $p_{\text{cor}} = 2D^{-3/5}(R)R$, i.e., the region of correlation of fluctuations of the $M(\mathbf{p},\Delta t)$ turned out to be limited by the speckle size. It should be noted that the normalized correlation function of fluctuations of the optical transfer function at different spatial frequencies practically does not depend on time of averaging.

Of practical interest is the possibility of determining the $M(\mathbf{p},\Delta t)$ using one time of averaging by statistical characteristics of fluctuations of the optical transfer function obtained for another time of averaging. For this case we consider the correlation function of the form:

$$B_M(\mathbf{p};\Delta t_1,\Delta t_2)=\left\langle \bar{M}(\mathbf{p},\Delta t_1)\bar{M}^*(\mathbf{p},\Delta t_2)\right\rangle-\left\langle \bar{M}(\mathbf{p})\right\rangle^2,$$

which characterizes the statistical relation of fluctuations of the optical transfer function averaged during the exposure time Δt, with fluctuations of the $M(\mathbf{p},\Delta t)$ with the exposure times Δt_1 and Δt_2.

In the case when $\Delta t_1 \ll \Delta t_0$, and $\Delta t_2 = \Delta t_1 + \Delta t$ it can be shown that the normalized correlation function

$$b_M(\mathbf{p};\Delta t_1,\Delta t_2)=\frac{B_M(\mathbf{p};\Delta t_1,\Delta t_2)}{\sigma_M(\mathbf{p},\Delta t_1)\sigma_M(\mathbf{p},\Delta t_2)}$$

has the following form in the region of small spatial scales p ($p \ll R$ and $p \ll D^{-3/5}(R)R$):

$$b_M(\mathbf{p};\Delta t_1,\Delta t_2)\cong\begin{cases}1-\dfrac{3}{96}\left(\dfrac{\Delta t}{\Delta t_0}\right)^2, & \Delta t\le\Delta t_0,\\[2ex] 0.70\left(\dfrac{\Delta t}{\Delta t_0}\right)^{-1/6}, & \Delta t>\Delta t_0,\end{cases}$$

in the intermediate area of variation of the spatial scale p ($D^{-3/5}(R)R \ll p \ll R$):

$$b_M(\mathbf{p};\Delta t_1,\Delta t_2)\cong\begin{cases}1-\dfrac{1}{8}\left(\dfrac{\Delta t}{\Delta t_0}\right)^2, & \Delta t\le\Delta t_0,\\[2ex] 0.79\left(\dfrac{\Delta t}{\Delta t_0}\right)^{-1/2}, & \Delta t>\Delta t_0.\end{cases}$$

Consequently, the fluctuation correlation of the $M(\mathbf{p},\Delta t)$ with different times of averaging decreases with increasing time Δt, in this case the typical scale of the correlation function is the value Δt_0 (time of transport of mean wind velocity of turbulent inhomogeneities at a distance being equal to the typical scale of the aperture).

One of the standard characteristics of the image quality, obtained with an optical system, is the integral resolution. The integral resolution of the optical system, at a finite duration of the averaging, can be written in the form[3,6–8]

$$\bar{\Re}(\Delta t)=\frac{1}{\Delta t}\int_0^{\Delta t}\mathrm{d}t\,\Re(\Delta t)=\int\!\!\int_{-\infty}^{\infty}\mathrm{d}\mathbf{p}\,\bar{M}(\mathbf{p},\Delta t). \qquad (12)$$

The mean value of the integral resolution, obtained by averaging Eq. (12) over the ensemble of realizations of fluctuations of the atmospheric dielectric constant as well as the mean value of the optical transfer function (5) does not depend on the time of averaging. For the optical system, which pupil function is described by Eq. (8), the mean value of the integral resolution is of the form

$$\left\langle\bar{\Re}\right\rangle=\left\langle\bar{\Re}(\Delta t)\right\rangle=\lim_{\Delta t\to\infty}\frac{1}{\Delta t}\int_0^{\Delta t}\mathrm{d}t\,\Re(t)=$$

$$=2\pi^2U_0^2K_0^2R^4\int_0^{\infty}\mathrm{d}p\,p\exp\left[-\frac{1}{4}p^2-\frac{1}{2}D(R)\,p^{5/3}\right].$$

The dependence of the mean value of the integral resolution $\left\langle\bar{\Re}\right\rangle$ on the atmospheric turbulence parameters is characterized by a single parameter – structure function of the fluctuations of the complex phase $D(R)$, calculated for the aperture size. The increase in the structure function of the complex phase fluctuations $D(R)$ results in a decrease of the resolution of the optical system.

In conclusion, it should be noted that all the results, obtained for the pupil function of the form (8), can also be obtained for the aperture with a sharp edge. In this case the discrepancy of the results will be observed only in the area of large spatial scales $p > R$, being of insignificant practical interest.

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