Retrieving modes of the wave front from an image

G.L. Degtyarev, A.V. Makhan'ko, S.M. Chernyavskii, and A.S. Chernyavskii

A.N. Tupolev State Technical University, Kazan

Received August 26, 2004

A method for retrieving modes of the wave front reproduced by an adaptive optical system from variations of an image is proposed. In projection methods of solution of the phase problem, the vector representation of the problem on determining the generic point of m sets, allowing application of the known methods of its solution at $m = 2$, is introduced.

1. Retrieving modes of the wave front from variations of a point source image

1.1. Let the distortions of amplitude and phase of a wave front at the exit pupil Ω of an optical system be described by the pupil function

$$G(\bar\rho) = A(\bar\rho)\exp[ik\Phi(\bar\rho)],$$

where $\bar\rho = (\xi,\eta)$; $k = 2\pi/\lambda$ is the wave number. The optical system is assumed adaptive, and its adaptive element, considered as a linear system, can change the wave front; therefore, real wave front distortions can be represented in the form $\Phi = \varphi + \Delta\Phi$. Here φ determines the component of wave front distortion, which can be compensated for by an adaptive element, while $\Delta\Phi$ is the residual distortion, not compensated by the adaptive element. It is assumed that the function φ is an element of a finite-dimension space, whose basis is comprised by the actuator responses $\{\Psi_k\}$, $k = \overline{1,N}$. Trial variations of the wave front by the adaptive element lead to variations of the intensity in the image. The task is to determine and compensate for the component φ using these variations. This approach to control of the adaptive element is called multidithering technique and can be implemented in two ways. In the first case, the image sharpness function is specified, and the adaptive system operates by the "hill climbing" principle.[1] The second method involves the determination of the coefficients of expansion of the unknown function $\varphi = \sum_{k=1}^{N} c_k\Psi_k$. This approach was realized in Ref. 2 by making use of variations in the second-order intensity moments in the image and assuming the amplitude at the pupil to be known and the basis functions to be linearly independent in the space with the scalar product

$$(\Psi_i,\Psi_j) = \iint A^2\Psi'_i\,\Psi'_j\,\mathrm{d}\bar\rho, \qquad (1)$$

where Ψ' is the vector of the partial derivatives.

In this section, we discuss the possibility of realizing this approach based on the analysis of variation of the Fourier cosine transform for the cases of both known and unknown amplitude at the pupil.

1.2. The Fourier cosine transform of the intensity distribution I over the focal plane at the spatial frequency $\bar\rho/\lambda f$ (f is the focal length) has the following form (accurate to an insignificant factor)[3]:

$$F_c(I,\bar\rho/\lambda f,\Phi) = \int\!\!\int_{-\infty}^{\infty}(A+\Delta A)A\cos(k\Delta\Phi)\,\mathrm{d}\bar\rho',$$

where

$$A = A(\bar\rho');\ A+\Delta A = A(\bar\rho+\bar\rho');\ \Delta\Phi = \Phi(\bar\rho+\bar\rho') - \Phi(\bar\rho').$$

Then we shall consider small spatial frequencies in the form $\bar\rho/\lambda f = \bar e v/\lambda f$, where $\bar e = \cos\theta\,\bar i + \sin\theta\,\bar j$ is the unit vector of frequency, and $v > 0$ is a small value. The factor $1/\lambda f$ in the expression for the frequency will be omitted.

Using the adaptive element, change the wave front by $\alpha\Psi$ and measure the first and second variations of the Fourier cosine transform:

$$\delta F_c(I,\bar\rho,\Phi,\Psi) = \left.\frac{\mathrm{d}F_c(I,\bar\rho,\Phi+\alpha\Psi)}{\mathrm{d}\alpha}\right|_{\alpha=0} =$$

$$= -k\int\!\!\int_{-\infty}^{\infty}(A+\Delta A)A\sin(k\Delta\Phi)\Delta\Psi\,\mathrm{d}\bar\rho';$$

$$\delta^2 F_c(I,\bar\rho,\Phi,\Psi) = \left.\frac{\mathrm{d}^2F_c(I,\bar\rho,\Phi+\alpha\Psi)}{\mathrm{d}\alpha^2}\right|_{\alpha=0} =$$

$$= -k^2\int\!\!\int_{-\infty}^{\infty}(A+\Delta A)A\cos(k\Delta\Phi)(\Delta\Psi)^2\,\mathrm{d}\bar\rho'.$$

Consider the limit

$$\gamma_1(\Psi) = \lim_{v\to 0}\int_0^{2\pi}\delta F_c(I,\bar e v,\Phi,\Psi)\,\mathrm{d}\theta \Big/ \int_0^{2\pi}\delta^2F_c(I,\bar e v,\Phi,\Psi)\,\mathrm{d}\theta =$$

$$= \int\!\!\int_{-\infty}^{\infty}A^2\int_0^{2\pi}\frac{\partial\Phi}{\partial e}\frac{\partial\Psi}{\partial e}\mathrm{d}\theta\,\mathrm{d}\bar\rho' \Big/ \int\!\!\int_{-\infty}^{\infty}A^2\int_0^{2\pi}\left(\frac{\partial\Psi}{\partial e}\right)^2\mathrm{d}\theta\,\mathrm{d}\bar\rho' =$$

$$= \int\!\!\int_{-\infty}^{\infty}A^2\Phi'\Psi'\,\mathrm{d}\bar\rho' \Big/ \int\!\!\int_{-\infty}^{\infty}A^2(\Psi')^2\,\mathrm{d}\bar\rho' = \frac{(\Phi,\Psi)}{\|\Psi\|^2},$$

where $\partial/\partial e$ is the derivative with respect to the direction of the unit vector $\bar{e}$.

If the amplitude at the pupil is known, then the actuator responses can be orthonormalized in accordance with the scalar product (1), and then the limit $\gamma_1(\Psi_k) = (\Phi,\Psi_k)$ is the Fourier coefficient of the function Φ in this basis and

$$\Phi = \sum \gamma_1(\Psi_k)\Psi_k + \Delta\Phi .$$

If the amplitude at the pupil is unknown, then additional measurements and more complex processing are needed. Take the second mixed variation

$$\delta^2 F_c(I,\bar{\rho},\Phi,\Psi,\chi) = \left.\frac{\partial^2 F_c(I,\bar{\rho},\Phi+\alpha\Psi+\beta\chi)}{\partial\alpha\partial\beta}\right|_{\alpha=\beta=0} =$$

$$= -k^2 \int\int_{-\infty}^{\infty} (A+\Delta A)A\cos(k\Delta\Phi)(\Delta\Psi)(\Delta\chi)\,\mathrm{d}\bar{\rho}'$$

as an additional measurement and consider two limits:

$$\gamma_2(\Psi,\chi) = \lim_{v\to 0}\int_0^{2\pi}\delta^2 F_c(I,\bar{e}v,\Phi,\Psi)\,\mathrm{d}\theta \Big/ \int_0^{2\pi}\delta^2 F_c(I,\bar{e}v,\Phi,\chi)\,\mathrm{d}\theta =$$

$$= \int\int_{-\infty}^{\infty} A^2 \int_0^{2\pi}\left(\frac{\partial\Psi}{\partial e}\right)^2 \mathrm{d}\theta\mathrm{d}\bar{\rho}' \Big/ \int\int_{-\infty}^{\infty} A^2 \int_0^{2\pi}\left(\frac{\partial\chi}{\partial e}\right)^2 \mathrm{d}\theta\mathrm{d}\bar{\rho}' =$$

$$= \int\int_{-\infty}^{\infty} A^2(\Psi')^2\,\mathrm{d}\bar{\rho}' \Big/ \int\int_{-\infty}^{\infty} A^2(\chi')^2\,\mathrm{d}\bar{\rho}' = \frac{\|\Psi\|^2}{\|\chi\|^2}$$

and

$$\gamma_3(\Psi,\chi) = \lim_{v\to 0}\int_0^{2\pi}\delta^2 F_c(I,\bar{e}v,\Phi,\Psi,\chi)\,\mathrm{d}\theta \Big/$$

$$\Big/ \left(\int_0^{2\pi}\delta^2 F_c(I,\bar{e}v,\Phi,\Psi)\,\mathrm{d}\theta\right)^{1/2}\left(\int_0^{2\pi}\delta^2 F_c(I,\bar{e}v,\Phi,\chi)\,\mathrm{d}\theta\right)^{1/2} =$$

$$= \int\int_{-\infty}^{\infty} A^2 \int_0^{2\pi}\frac{\partial\Psi}{\partial e}\frac{\partial\chi}{\partial e}\,\mathrm{d}\theta\mathrm{d}\bar{\rho}' \Big/$$

$$\Big/ \left(\int\int_{-\infty}^{\infty} A^2 \int_0^{2\pi}\left(\frac{\partial\Psi}{\partial e}\right)^2 \mathrm{d}\theta\mathrm{d}\bar{\rho}'\right)^{1/2}\left(\int\int_{-\infty}^{\infty} A^2 \int_0^{2\pi}\left(\frac{\partial\Psi}{\partial e}\right)^2 \mathrm{d}\theta\mathrm{d}\bar{\rho}'\right)^{1/2} =$$

$$= \int\int_{-\infty}^{\infty} A^2\Psi'\chi'\mathrm{d}\bar{\rho}' \Big/ \left(\int\int_{-\infty}^{\infty} A^2(\Psi')^2\,\mathrm{d}\bar{\rho}'\right)^{1/2} \times$$

$$\times\left(\int\int_{-\infty}^{\infty} A^2(\chi')^2\,\mathrm{d}\bar{\rho}'\right)^{1/2} = \frac{(\Psi,\chi)}{\|\Psi\|\|\chi\|}.$$

The function φ will be sought from the condition of minimum discrepancy

$$\left\|\Phi - \sum \tilde{n}_k\Psi_k\right\|^2 = \int\int_{-\infty}^{\infty} A^2\left(\Phi' - \sum_{k=1}^{N} c_k\Psi_k'\right)^2 \mathrm{d}\bar{\rho}'$$

with respect to the expansion coefficients. The extreme condition leads to the system of linear equations

$$\sum \tilde{n}_k(\Psi_k,\Psi_j) = (\Phi,\Psi_j),\ \ j = \overline{1,N},$$

which can be presented in the form

$$\sum \tilde{n}_k \frac{(\Psi_k,\Psi_j)}{\|\Psi_k\|\|\Psi_j\|}\frac{\|\Psi_k\|}{\|\Psi_j\|} = \frac{(\Phi,\Psi_j)}{\|\Psi_j\|^2},\ \ j = \overline{1,N}$$

or

$$\sum c_k\gamma_3(\Psi_k,\Psi_j)\gamma_2^{1/2}(\Psi_k,\Psi_j) = \gamma_1(\Psi_j). \qquad (2)$$

The system (2) is non-degenerate, because the actuator basis is linearly independent in the space with the scalar product (1) by definition, and therefore this system has a unique solution.

2. Method of increasing the dimensionality and projection methods for solution of the phase problem

2.1. Many inverse problems of optics can be formulated as a geometric problem on finding a generic point of the given closed sets

$$x \in \bigcap_{s=1}^{m} V_s, \qquad (3)$$

where the sets V_s specify the *a priori* properties of the solution, the results, and the measurement conditions. The papers by Bregman[4] and Gurin with co-authors[5] started the solution of the problem (3) by iteration methods in the infinite-dimensional spaces. These methods are based on the property of the projection $P_s x$ of the point x onto the set V_s. In the optics problems, these methods have gained a wide use starting from the papers by Gerchberg and Saxton, Youla, Levi, Stark, et al.[6] The most complete theoretical results are obtained for the case of convex sets of the complex Hilbert space H and the iteration alternate projection algorithm

$$x_{n+1} = T_1T_2\ldots T_m x_n = Tx_n, \qquad (4)$$

starting from the initial arbitrary point $x_0 \in H$, where

$$T_s x = x + \lambda_s(P_s x - x) = \left[I + \lambda_s(P_s - I)\right]x.$$

The parameters of the method $\lambda_s \in (0,2)$ are referred to as the relaxation parameters. The algorithm (4) is easy to implement and guarantees weak convergence. In the phase problems sets are not convex and therefore, the convergence of the algorithm (4) for these problems is not guaranteed.

The experience of applying the algorithm (4) showed[5,6] that in some cases, the algorithm converges very slowly, and therefore it calls for a modification.

Consider two approaches to the modification of the algorithm (4). In the first one, proposed by Levi and Stark,[6] the relaxation parameters are chosen so that the sequence of the algorithm (4) is also minimizing for some criterion $J(x)$, being a measure of closeness of the point x to the sets V_s. In this case, it becomes possible to optimize the relaxation parameters from the condition of quickest descent to the minimum of the criterion. This approach was successfully realized in Ref. 6 for the case of two sets ($m = 2$), provided that for the criterion

$$J(x) = \| P_1 x - x \| + \| P_2 x - x \| \tag{5}$$

the sequence of the algorithm (4) is minimizing, if the relaxation parameters meet the corresponding inequalities given in Ref. 6. For $m > 2$, it was proposed to reduce the intersection of the sets in Eq. (3) to the intersection of two combined sets, joining several properties of the solution of an inverse problem. This naturally leads to a more complicated determination of the projection onto the combined sets and, consequently, to a more complicated iteration algorithm.

There is one more circumstance connected with this approach, which is worth noting here as well. If the sequence (4) is minimizing, then it is one of the possible minimizing sequences for the criterion (5). Therefore, we can change the form of the minimizing sequence from the condition of the quickest descent.

The second approach to accelerate the convergence of the algorithm (4) was proposed by Gurin with co-authors[5] also for two sets, but convex ones. In this approach, the successive approximations starting from x_0 are determined as follows. At every iteration step, three projections $y_1 = P_1 x_n$, $y_2 = P_2 y_1$, $y_3 = P_1 y_2$ are calculated and used to determine the approximation $x_{n+1} = y_1 + \lambda(y_3 - y_1)$. The points y_1 and y_3 belong to the set V_1 and determine the ray $x = y_1 + \lambda(y_3 - y_1)$. The point y_2 and the vector $y_1 - y_2$ determine a hyperplane, rigorously separating the set V_2 and the point y_3, if $y_3 \notin V_1 \cap V_2$. The parameter λ is found from the condition of ray intersection with the hyperplane:

$$\mathrm{Re}(y_1 + \lambda(y_3 - y_1) - y_2, y_1 - y_2) = 0.$$

The intersection point is taken as x_{n+1}.

It is noteworthy that this approach is a little bit different from the regular algorithm (4) of selecting the next approximation in favor of accelerating the convergence. In the case $m > 2$, the combined algorithm was proposed in Ref. 6: to find successive approximations by the scheme (4), but sometimes to do the “accelerating” steps for different pairs of sets.

Both of the considered approaches for acceleration of the convergence of the algorithm (4) are justified only for $m = 2$. At $m > 2$ their application complicates the algorithm. The restriction of this kind can be overcome by applying the method of increasing the dimensionality.[7]

2.2. Method of increasing the dimensionality. Consider the direct product

$$\bar{H} = \prod_{s=1}^{m-1} H_s, \;\; H_s = H$$

and two sets in it:

$$\bar{V}_1 = \{\bar{x} = (x_1, \dots, x_{m-1}) : x_1 = \dots = x_{m-1} \in V_1\},$$

$$\bar{V}_2 = \{\bar{x} = (x_1, \dots, x_{m-1}) : x_s \in V_{s+1}\}.$$

Then the problem (3) is equivalent to the problem of finding the point $\bar{x} \in \bar{H}$ from the condition

$$\bar{x} \in \bar{V}_1 \cap \bar{V}_2. \tag{6}$$

At $\bar{H}$, define the scalar product

$$(\bar{x}, \bar{y})_{\bar{H}} = \sum_{s=1}^{m-1} \alpha_s (x_s, y_s)_H, \;\; \alpha_s > 0, \;\; \alpha_1 + \dots + \alpha_{m-1} = 1.$$

The operator of projection onto the set $\bar{V}_1$ is determined by the condition

$$\left\| \bar{x} - P_{\bar{V}_1} \bar{x} \right\| = \min_{\bar{y} \in \bar{V}_1} \| \bar{x} - \bar{y} \| = \min_{y \in V_1} \left(\sum_{s=1}^{m-1} \alpha_s \| x_s - y \|^2 \right)^{1/2}.$$

Because

$$\sum_{s=1}^{m-1} \alpha_s \| x_s - y \|^2 = \sum_{s=1}^{m-1} \alpha_s \| x_s \|^2 - \left\| \sum_{s=1}^{m-1} \alpha_s x_s \right\|^2 + \left\| \sum_{s=1}^{m-1} \alpha_s x_s - y \right\|^2,$$

$$P_{\bar{V}_1} \bar{x} = (y, \dots, y),$$

where

$$y = P_1 \sum_{s=1}^{m-1} \alpha_s x_s.$$

The operator of projection onto the set $\bar{V}_2$ is determined analogously:

$$\left\| \bar{x} - P_{\bar{V}_2} \bar{x} \right\| = \min_{\bar{y} \in \bar{V}_2} \| \bar{x} - \bar{y} \| = \min_{y_s \in V_{s+1}} \left(\sum_{s=1}^{m-1} \alpha_s \| x_s - y_s \|^2 \right)^{1/2} =$$

$$= \left(\sum_{s=1}^{m-1} \alpha_s \left\| x_s - P_{V_{s+1}} x_s \right\|^2 \right)^{1/2},$$

therefore

$$P_{\bar{V}_2} \bar{x} = (P_2 x_1, \dots, P_m x_{m-1}).$$

The transition from the Hilbert space H to the Hilbert space $\bar{H}$ on the direct product allowed the initial problem (3) to be transformed into the problem (6) and the two considered approaches to modification of the alternate projection method to be applied to the solution of the problem (6) and ultimately of the problem (3).

The method of alternate projection of the problem (6) with optimization of the relaxation parameters λ_1 and λ_2 by the Levi–Stark method takes the form

$$\bar{x}_{n+1}(\lambda_1,\lambda_2) = T_{\bar{V}_1}(\lambda_1) T_{\bar{V}_2}(\lambda_2)\bar{x}_n,$$

where λ_1 and λ_2 are determined at every step from the condition

$$J(\bar{x}_{n+1}(\lambda_1,\lambda_2)) = \min_{\mu_1,\mu_2 \ge 0}\Big(\left\| P_{\bar{V}_1}\bar{x}_{n+1}(\mu_1,\mu_2) - \bar{x}_{n+1}(\mu_1,\mu_2)\right\| +$$

$$+\left\| P_{\bar{V}_2}\bar{x}_{n+1}(\mu_1,\mu_2) - \bar{x}_{n+1}(\mu_1,\mu_2)\right\|\Big).$$

The method of alternate projection of the problem (6) with the Gurin's modification is reduced, at each of the iterations, to the determination of three projections

$$\bar{y}_1 = P_{\bar{V}_1}\bar{x}_n\,,\ \bar{y}_2 = P_{\bar{V}_2}\bar{y}_1\,,\ \bar{y}_3 = P_{\bar{V}_1}\bar{y}_2$$

and the next approximation

$$\bar{x}_{n+1} = \bar{y}_1 + \lambda(\bar{y}_3 - \bar{y}_1),$$

where λ is determined from the condition

$$\mathrm{Re}(\bar{y}_1 + \lambda(\bar{y}_3 - \bar{y}_1) - \bar{y}_2,\ \bar{y}_1 - \bar{y}_2) = 0.$$

2.3. In the Levi–Stark method, first the sequence (4) is specified accurate to the relaxation parameters, and then the measure of closeness of the point to the intersection sets, for which this sequence can be minimizing, is determined. In Ref. 7, the alternative approach was considered, in which the measure of closeness is specified between the points – representatives of the sets, and the minimizing sequence is determined from this measure. This approach lifts the restriction on the number of intersection sets of the problem (3), and the problem is reduced to finding a quickly convergent minimizing sequence.

For the problem (3), the approach functional was considered[7]:

$$J(x, x_1, \ldots, x_m) = \sum_{s=1}^{m} \alpha_s \|x - x_s\|^2,$$

where $x \in H$ and $x_s \in V_s$, $s = \overline{1,m}$. At the solutions of the problem (3) $x = x_1 = \ldots = x_m$, the approach functional achieves the minimum. The method of constructing the minimizing sequence of the approach functions is described in Ref. 7. The representation of this method in the vector form will allow us to establish the relation of this method with already considered algorithms.

Let the space be

$$\bar{H} = \prod_{s=1}^{m} H_s\,,\ H_s = H\,,$$

and two sets in it are:

$$\bar{V}_1 = \{\bar{x} = (x_1,\ldots,x_m) : x_1 = \ldots = x_m \in H\}$$

and

$$\bar{V}_2 = \{\bar{x} = (x_1,\ldots,x_m) : x_s \in V_s\},$$

for which the projection operators are, respectively,

$$P_{\bar{V}_1}\bar{x} = (y_1,\ldots,y_m),$$

where

$$y_s = \sum_{s=1}^{m} \alpha_s x_s,\ s = \overline{1,m}$$

and

$$P_{\bar{V}_2}\bar{x} = (P_1 x_1,\ldots,P_m x_m).$$

In the vector variables, the approach functional takes the form

$$J(\bar{x}_1,\bar{x}_2) = \|\bar{x}_1 - \bar{x}_2\|^2_{\bar{H}},\ \bar{x}_1 \in \bar{V}_1,\ \bar{x}_2 \in \bar{V}_2.$$

The approach functional is equal to the square distance between the two points from different sets. The characteristic

$$J(\bar{x}_1) = \min_{\bar{x}_2 \in \bar{V}_2} J(\bar{x}_1,\bar{x}_2) = J(\bar{x}_1, P_{\bar{V}_2}\bar{x}_1)$$

is a measure of closeness of the point $\bar{x}_1$ to the set $\bar{V}_2$.

The method of coordinate minimization by the variables $\bar{x}_1$ and $\bar{x}_2$ leads to the algorithm of the form (4):

$$\bar{x}_{1n+1} = T_{\bar{V}_1}T_{\bar{V}_2}\bar{x}_{1n}\,,\ \bar{x}_{2n} = T_{\bar{V}_2}\bar{x}_{1n}\,,\ \bar{x}_{10} \in \bar{V}_1\,, \quad (7)$$

and therefore calls for modification. For this purpose, at every iteration the projections are calculated

$$\bar{y}_1 = P_{\bar{V}_2}\bar{x}_{1n}\,,\ \bar{y}_2 = P_{\bar{V}_1}\bar{y}_1\,,\ \bar{y}_3 = P_{\bar{V}_2}\bar{y}_2\,,\ \bar{y}_4 = P_{\bar{V}_1}\bar{y}_3\,.$$

In fact, the iterations (7) with the unit relaxation parameters are carried out twice. The obtained projections specify the ray $\bar{x}(\lambda) = \bar{y}_2 + \lambda(\bar{y}_4 - \bar{y}_2)$, contained in $\bar{V}_1$, because $\bar{V}_1$ is a linear set, and the hyperplane in $\bar{H}$ is

$$\mathrm{Re}(\bar{x} - \bar{y}_3,\ \bar{y}_3 - \bar{y}_2) = 0.$$

If $\bar{y}_3 \notin \bar{V}_1 \cap \bar{V}_2$, then, according to the property of projections,

$$\mathrm{Re}(\bar{x} - \bar{y}_3,\ \bar{y}_3 - \bar{y}_2) \ge 0,\ \forall \bar{x} \in \bar{V}_2\,,$$

and

$$\mathrm{Re}(\bar{y}_2 - \bar{y}_3,\ \bar{y}_3 - \bar{y}_2) = -\|\bar{y}_2 - \bar{y}_3\|^2 < 0,$$

therefore the hyperplane separates the set $\bar{V}_2$ and the point $\bar{y}_2$. Substitute a ray point into the equation of hyperplane

$$\mathrm{Re}(\bar{y}_2 + \lambda(\bar{y}_4 - \bar{y}_2) - \bar{y}_3,\ \bar{y}_3 - \bar{y}_2) = 0.$$

According to the projection property, the points meet the following inequality

$$\mathrm{Re}(\bar{y}_3 - \bar{y}_2,\ \bar{y}_4 - \bar{y}_2) \ge \|\bar{y}_2 - \bar{y}_4\|^2 > 0,$$

and therefore there is a number

$$\Lambda = \|\bar{y}_3 - \bar{y}_2\|^2 / \mathrm{Re}(\bar{y}_3 - \bar{y}_2, \bar{y}_4 - \bar{y}_2) \geq$$
$$\geq \|\bar{y}_3 - \bar{y}_2\| / \|\bar{y}_4 - \bar{y}_2\| \geq 1,$$

at which the ray $\bar{x}(\lambda)$ at the point $\bar{x}(\Lambda)$ intersects the hyperplane.

The approximation $\bar{x}_{1n+1}$ will be sought in the form $\bar{x}_{1n+1} = = \bar{x}(\lambda_*)$, where λ_* will be found from the condition

$$J\left(\bar{x}(\lambda_*), P_{\bar{V}_2}\bar{x}(\lambda_*)\right) = \min_{\lambda \in [1,\Lambda]} J\left(\bar{x}(\lambda), P_{\bar{V}_2}\bar{x}(\lambda)\right).$$

Thus, the method from Ref. 7 for solution of the problem (3) includes the ideas from Refs. 5 and 6 and can be applied to solving the problem for intersection of more than two convex sets.

References

1. V.G. Taranenko and O.I. Shanin, *Adaptive Optics* (Radio i Svyaz', Moscow, 1990), 110 pp.
2. S.M. Chernyavskii, Proc. SPIE 4341, 116–125 (2000).
3. M. Born and E. Wolf, *Principles of Optics* (Pergamon, New York, 1959).
4. L.M. Bregman, Dokl. Akad. Nauk SSSR 162, No. 3, 487–490 (1965).
5. L.G. Gurin, B.T. Polyak, and E.V. Raik, Zh. Vychisl. Mat. Mat. Fiz. 7, No. 6, 1213–1228 (1967).
6. H. Stark, ed., *Image Recovery: Theory and Applications* (Academic Press, New York, 1987).
7. S.M. Chernyavskii, Proc. SPIE 5026, 112–116 (2002).