Exact and approximate representations for a laser beam in a uniaxial homogeneous medium

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We have considered the problem of monochromatic radiation propagation through a uniaxial homogeneous medium. Two versions are shown for investigating the above-mentioned problem. The scalar approximation (no variations of the polarization state of radiation at propagation are taken into account) and the rigorous approach based on the exact solution of the vector wave equation. The basis for the first version is the suggested general determination of ordinary and extraordinary radiation, while in solving the wave equation we used the Kirchhoff–Helmholtz method generalized to vector problems. In both of these versions, simplification of solutions was considered based on the parabolic approximation. It is shown that in this case the results obtained using the scalar and vector versions for less divergent laser beams will, as assumed, coincide more closely.

Introduction

The above-stated problem consists[1,2] in seeking solutions of the so-called wave equation

$$\mathrm{rot}\,\mathrm{rot}\,\mathbf{E}(\mathbf{r}) - (k^0)^2\tilde{\varepsilon}\mathbf{E}(\mathbf{r}) = 0, \tag{1}$$

where $\tilde{\varepsilon}$ is the second rank tensor of the dielectric constant of the medium (magnetic permeability is assumed unity); $\mathbf{E}(\mathbf{r})$ is the vector of the electric field strength of the monochromatic electromagnetic wave (time dependence – $\exp(-i\omega t)$ – is not indicated); $k^0 = \omega/c$.

The problems of this kind are, apparently, of the greatest practical interest in nonlinear optics, especially in its part dealing with the theory of generation of laser radiation harmonics. Indeed, the overwhelming majority of nonlinear crystals used in practice are uniaxial, and the maximum efficiencies of nonlinear conversion are obtained in the cases when laser radiation turns out to be monochromatic enough. Thus, the solution of Eq. (1) is the first and absolutely necessary stage in solving a nonlinear problem, as it enables one to determine zero (linear) field approximation at the fundamental frequency. In many cases, (it depends on a concrete way of solving the nonlinear problem) the last is used as one of the boundary conditions for the system of nonlinear equations. Note, in this connection, that the main goal of this study exactly consists in seeking solutions to the linear problem, convenient for use in the theory of harmonics' generation.

A typical (at least, for nonlinear optics) method of solution of the equation (1) is based on the so-called slowly varying amplitude approximation.[3] Within the limits of this approximation the solution to Eq. (1) for a beam propagating along the Z axis, is sought in the form[4]:

$$\mathbf{E}(\mathbf{r}) = \left[\mathbf{e}\,A(\mu_1 x, \mu_1 y, \mu_2 z) + \mu_2 \mathbf{U}(\mathbf{r})\right] e^{ikz}, \tag{2}$$

where $\mathbf{e}$ is the constant unit vector of polarization of ordinary ($k = k^0 n_o$), or extraordinary ($k = k^0 n^e$) plane wave directed along the Z axis, and μ_1 and μ_2 are the small parameters connected with the divergence angle α of a laser beam in the following way:

$$\mu_1 \approx \alpha, \ \ \mu_2 \approx \alpha^2 . \tag{3}$$

Substituting expression (2) into Eq. (1) and considering only terms of the μ_2 order, we derive the so-called abridged equations for the complex amplitudes A. In particular, for the extraordinary wave we have[4]:

$$\frac{\partial A}{\partial z} + \rho\frac{\partial A}{\partial x} + \frac{1}{2ik^e}\left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}\right) = 0, \tag{4}$$

where ρ is the anisotropy angle which is equal to zero for ordinary wave.

The pragmatic value of such an approach is of no doubts. The problem, based on the vector equation (1), can be reduced to the scalar equation (4), the solution of which is much easier to derive in all cases. On the other hand, there are a number of moments of the general character, which are to be discussed in detail.

The matter is that the result of the considered method is the function

$$\mathbf{E}(\mathbf{r}) = \mathbf{e}A(\mu_1 x, \mu_1 y, \mu_2 z)\, e^{ikz}, \tag{5}$$

where $\mathbf{e}$ and k are defined in Eq. (2); A is the solution of Eq. (4).

In other words, the function $\mathbf{U}(\mathbf{r})$, entering the Eq. (2) (to take into account the perturbation of polarization state, caused, for example, by diffraction), is omitted from Eq. (5) at all. The most detailed (though, in our opinion, insufficient too) comments

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of this circumstance are given in Ref. 3. The physical meaning of the representation (5) for a weakly diverging laser beam is quite clear by intuition. The less the radiation divergence, the less the vector polarization function of the rigorous solution will differ from a certain constant value determining the polarization of the plane wave. However, the formal aspect of the matter in these circumstances is poor that, in our opinion, is absolutely inadmissible, if we deal with the rigorous theory. The subject of this problem is reduced to the following.

It is easy to see, that any rigorous solution of the equation (1) must satisfy the divergence condition

$$\mathrm{div}(\tilde{\varepsilon}\mathbf{E}) = \mathrm{div}\mathbf{H} = 0, \tag{6a}$$

where $\mathbf{H}$ is the vector of the magnetic field strength.

In the particular case of an isotropic medium we have, instead of Eq. (6a)

$$\mathrm{div}(\mathbf{E}) = \mathrm{div}\mathbf{H} = 0. \tag{6b}$$

Accordingly, and it is quite obvious, that any approximate solution of Eq. (1) should satisfy conditions (6) within the limits of the generality restrictions used. By the direct substitution one can see that Eq. (5) does not meet this requirement at all. One of the results of such a discrepancy is that substituting expression (5) in Eq. (1), we cannot, generally speaking, derive the equation (4). In a particular case, as, for example, in Ref. 3 of a plane non-diffracted wave, no problems arise on the conditions (6).

In this connection the expression (5), apparently, will be more correct as concerning the term certain model representation of the rigorous solution of a linear problem. The term "model representation" in this case should reflect the circumstance that the expression (5) formally is not an approximate solution of Eq. (1); the conditions (6) do not carried hold even approximately (particular cases are not considered).

Nothing forbids the use of other models, if those do not lead to great errors in the result, and the present case is not an exception (though it is impossible at present to check up the correctness of representation (5) for the problem of harmonics generation because of the absence of alternative solutions).

Besides, one more question arises, quite natural in our opinion. What for in that case (*the final result of the form (5) is meant*) we have to form the theory, based on the equation (1)? Most likely, it only complicates the problem as it automatically necessitates proving the form of the $\mathbf{U}$ function in Eq. (2), and then explaining the reasons why this function disappears at substitution of expression (2) into Eq. (1). In our opinion, there exists a more simple version (though, certainly, this is a matter of taste), which we shall present by a solution of Eq. (1) for a field in an isotropic medium, as an example.

If the medium is isotropic, the condition (6b) holds and Eq. (1) reduces to the so-called Helmholtz vector equation in a usual way:

$$\nabla^2\mathbf{E}(\mathbf{r}) + k^2\mathbf{E}(\mathbf{r}) = 0. \tag{7}$$

Now if substituting the expression (5) into the Eq. (7), and taking the scalar product of both parts with a constant vector $\mathbf{e}$, we shall obtain, after restricting ourselves to terms of the μ_2 order (see the expression (3)), the short form (4) of this equation. Moreover, the expression (5) appears to be not that model one in relation to Eq. (7) and the approximate solution will coincide with the exact solution of Eq. (7) much better in the case of a less divergent laser beam. Of course, the crux of the matter does not change because of that at all. To say it in simpler words, now we agree that we shall ignore the variation of the wave polarization, regardless of the cause of these variations and start solving the problem with the equation (7). Thus all the further actions like choosing a solution in the form (5), transition to the scalar Helmholtz equation and its subsequent substitution with the abridged equation (4), do not contain already any contradictions. In this sense, such an approach (for definiteness we shall call it the scalar approximation) looks even more preferable.

Rough, tentative estimations show, that the polarization vector in the exact solution (1) (both for isotropic and for uniaxial medium), in its most general case, can be written in the following form

$$\mathbf{e} = \left\{ e_x\,\mathrm{e}^{i\varphi_x}, e_y\,\mathrm{e}^{i\varphi_y}, e_z\,\mathrm{e}^{i\varphi_z} \right\}, \tag{8}$$

where all e_i and φ_i are the functions of coordinates. In this connection, the question quite naturally, in our opinion, arises on whether or not the efficiency of nonlinear transformation depend on that and how strong the vector (8) differs from the polarization vector of the plane wave, used in Eq. (5). As follows from the above-mentioned reasoning that within the limits of the theory of wave harmonics generation developed to the present time it is impossible to answer it because this theory is based on the scalar approximation. Since the formulated question does not seem trivial to the authors and they are going to be concerned with this problem, seeking suitable representations for a vector field in a linear uniaxial medium becomes the absolutely necessary condition.

The work is constructed in following way. At the beginning, in part 1, the transition to the scalar approximation for a field in the uniaxial medium is considered, in our opinion rigorously enough. In part 2 the method, allowing deriving the exact solution of the equation (1) is shown. Unfortunately, it is impossible to use this exact result in achieving our final goal as the solutions of nonlinear equations, in the general case, become too complicated (the results obtained in Refs. 5 and 6 are also hardly suitable for the same reason). It is for this reason that the basic attention in part 2 is paid to the problem of seeking an approximate solution of the equation (1) for the problems on propagation of weakly diverging laser beams. Stated in a different way, it becomes a subject for the parabolic approximation approach, but already in the essentially vector form.

1. Scalar approximation for the field in a uniaxial medium

Let us direct the X-axis of the coordinate system along the optical axis of the medium. The position of the Y and Z axes can be optional. In such a coordinate system, which we shall call the basic one (or ε-coordinates), we have[7] for nonzero $\tilde{\varepsilon}$ tensor component:

$$\varepsilon_{11} = n_e^2 \neq \varepsilon_{22} = \varepsilon_{33} = n_o^2, \qquad (9)$$

where n_o and n_e are the main refractive indices of the uniaxial medium.

Within the limits of this paper, we shall deal with laser beams, the longitudinal axis of which (propagation direction) makes an optional θ angle with the optical axis of the medium in the XZ plane of the basic system of coordinates. The last remark does not limit generality of the considered problem, unless and until the tensor of the dielectric constant is invariant with respect to turns of the coordinate system about the optical axis (axis X). By way of using the parabolic approximation such fields are more convenient if considered in the (x', y', z') coordinate system, in which the Z' axis coincides with a longitudinal beam axis, and the Y' with Y axis of the ε-coordinate system. Therefore, the $X'Y'$ plane turns out to be the beam cross section. We shall term these new coordinates the E-coordinates, thereby emphasizing, that they are connected with a laser beam, instead of the crystal symmetry. The transition to new system is provided by turning counter-clockwise the former one by $\pi/2 - \theta$ angle around the common Y-axis. Some coordinates are expressed through others in the following way:

$$x = \sin\theta\, x' + \cos\theta\, z',\ y = y',\ z = -\cos\theta\, x' + \sin\theta\, z'. \qquad (10)$$

In the ε-coordinates the vector equation (1) is equivalent to the system of three scalar equations:

$$\frac{\partial}{\partial y}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) - k_e^2 E_x = 0, \qquad (11a)$$

$$\frac{\partial}{\partial z}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) - k_o^2 E_y = 0, \qquad (11b)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) - k_o^2 E_z = 0, \qquad (11c)$$

where $k_o = k^0 n_o$, $k_e = k^0 n_e$.

For the divergence condition (6a) in the same system we have

$$\mathrm{div}(\tilde{\varepsilon}\mathbf{E}) = n_e^2 \frac{\partial E_x}{\partial x} + n_o^2\left(\frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right). \qquad (12)$$

Now it is necessary for us to give definitions of the ordinary (o) and extraordinary (e) electromagnetic waves.

Usually, speaking about the o- or e-waves, the definition formulated within the limits of the plane wave approximation, is meant (see, for example, Refs. 1 and 7). Let us remind it. Let the plane wave propagate through a uniaxial medium along the **s** direction. The **s** vector and the principal optical axis of the medium form a plane which is termed the principal optical plane (POP). A wave, the polarization vector of which is perpendicular to POP, is termed the ordinary plane wave. An extraordinary plane wave is that, the polarization vector of which lies in POP.

Let us make a useful for us generalization, considering all the facts. A wave, the electric field strength vector of which has no projection onto the principal optical axis of medium we shall term the o-wave. If the solution is valid in the ε-coordinates, this means, that the **E** field whose E_x component is absent will be the o-wave, i.e.,

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_o(\mathbf{r}) = \{0, E_{oy}(\mathbf{r}), E_{oz}(\mathbf{r})\}. \qquad (13)$$

Let us also formulate the general definition of the extraordinary wave as follows. We shall term a wave, the magnetic field strength vector of which has no projection onto the principal optical axis the e-wave. If the solution is valid in the basic coordinate system, this means, that

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_e(\mathbf{r}) = \{E_{ex}(\mathbf{r}), E_{ey}(\mathbf{r}), E_{ez}(\mathbf{r})\}, \qquad (14a)$$

where

$$\frac{1}{ik^0}\left(\frac{\partial E_{ez}}{\partial y} - \frac{\partial E_{ey}}{\partial z}\right) = H_{ex} = 0. \qquad (14b)$$

Substitute Eq. (13) into formula (12) and after differentiation two relations are derived:

$$-\frac{\partial^2 E_{oy}}{\partial y^2} = \frac{\partial^2 E_{oz}}{\partial y \partial z},\quad -\frac{\partial^2 E_{oz}}{\partial z^2} = \frac{\partial^2 E_{oy}}{\partial y \partial z}. \qquad (15)$$

Now substitute expression (13) into Eqs. (11), use Eqs. (15), and make sure that both the field (13) and any of its components (E_{oy} or E_{oz}) do satisfy the Helmholtz equation (vector or scalar one)

$$\hat{L}_o \mathbf{E}_o = \hat{L}_o E_{oi} = 0, \qquad (16)$$

where $\hat{L}_o = \nabla^2 + k_o^2$, $i = y, z$.

Now it is necessary to do something similar for the e-wave. Note, first that from expression (14b) two equalities follow:

$$\frac{\partial^2 E_{ez}}{\partial y \partial z} = \frac{\partial^2 E_{ey}}{\partial z^2},\quad \frac{\partial^2 E_{ey}}{\partial y \partial z} = \frac{\partial^2 E_{ez}}{\partial y^2}, \qquad (17)$$

and after differentiation of the relation (12) three equalities more are obtained:

$$\beta^2 \frac{\partial^2 E_{ex}}{\partial x^2} = -\frac{\partial^2 E_{ey}}{\partial x \partial y} - \frac{\partial^2 E_{ez}}{\partial x \partial z},$$

$$-\frac{\partial^2 E_{ex}}{\partial x \partial y} = \frac{1}{\beta^2}\left(\frac{\partial^2 E_{ey}}{\partial y^2} + \frac{\partial^2 E_{ez}}{\partial y \partial z}\right),$$

$$-\frac{\partial^2 E_{ex}}{\partial x \partial z} = \frac{1}{\beta^2}\left(\frac{\partial^2 E_{ey}}{\partial y \partial z} + \frac{\partial^2 E_{ez}}{\partial z^2}\right), \qquad (18)$$

where $\beta = n_e/n_o$.

Now, substituting Eqs. (17) and (18) into Eq. (11), we are convinced that both the field (14) and any of its components do satisfy the equation

$$\hat{L}_e \mathbf{E}_e = \hat{L}_e E_{ei} = 0, \qquad (19)$$

where

$$\hat{L}_e = \beta^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_e^2,\ i = x, y, z.$$

We shall term equation (19) the Helmholtz equation for extraordinary wave.

Let us consider two equations for Green's functions:

$$\hat{L}_o g_o(\mathbf{r}, \mathbf{r}_0) = -4\pi\delta(\mathbf{r} - \mathbf{r}_0), \qquad (20a)$$

$$\hat{L}_e g_e(\mathbf{r}, \mathbf{r}_0) = -4\pi\delta(\mathbf{r} - \mathbf{r}_0). \qquad (20b)$$

The solution of Eq. (20a) is well-known[8] and looks like

$$g_o(\mathbf{r}, \mathbf{r}_0) = e^{ik_o R} / R,$$

$$R = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}. \qquad (21)$$

The equation (20b) is solved similarly and the result is

$$g_e(\mathbf{r}, \mathbf{r}_0) = e^{ik_e R'} / R'\beta,$$

$$R' = \sqrt{\frac{(x - x_0)^2}{\beta^2} + (y - y_0)^2 + (z - z_0)^2}. \qquad (22)$$

By means of these Green's functions the solutions of the equations (16) and (19) are as follows[2,9]:

$$\mathbf{E}_{o,e}(\mathbf{r}_0) = \frac{1}{4\pi} \int_S \left(g_{o,e} \frac{\partial \mathbf{E}_{o,e}}{\partial \mathbf{n}} - \mathbf{E}_{o,e} \frac{\partial g_{o,e}}{\partial \mathbf{n}} \right) dS, \qquad (23)$$

where the subscript "o" refers to the ordinary wave (the equation (16)), and the subscript "e" to the extraordinary wave (the equation (19)); S is the optional closed surface; $\mathbf{n}$ is the outer normal to S. Using the Eq. (20), it is easy to see that formulas (23) are really the exact solutions of the Eqs. (16) and (19).

Generally speaking, the transition to the scalar approximation can be considered completed. Indeed, if we agree upon the presentation of the laser beam, according to the problem statement, in the form

$$\mathbf{E}(\mathbf{r}) = \mathbf{e}U(\mathbf{r}) = \mathbf{e}A(\mu_1 x, \mu_1 y, \mu_2 z)\, e^{ik_{o,e} z}, \qquad (24)$$

than, substituting formula (24) into the expression (23), we immediately find the solution of the corresponding equations for the scalar amplitudes $U_{o,e}(\mathbf{r})$ of the vector fields $\mathbf{E}_{o,e}(\mathbf{r})$. However, we still need to find solutions (and corresponding abridged equations) for slowly varying amplitudes (see Eq. (24)) of the o- and e-waves propagating along the direction at an optional θ angle with the optical axis.

For this purpose we choose the $X'Y'$ plane of the E-coordinate system as surface S entering Eq. (23) (thus the beam axis will coincide with the Z' axis) and close it by the hemisphere of infinite radius in the $z' > 0$ half-space. By use of expression (10) we write the solution of Eq. (23) in the E-coordinate system. After simple, but cumbersome calculations (we omit these here) we derive instead of Eq. (23) the following relationship (primes coordinate designations are omitted)

$$U_{o,e}(\mathbf{r}_0) = \frac{1}{4\pi} \iint_{-\infty}^{+\infty} \left(U_{o,e} \frac{\partial g_{o,e}}{\partial z} - g_{o,e} \frac{\partial U_{o,e}}{\partial z} \right)_{z=0} dx\,dy, \qquad (25)$$

where the integral over the hemisphere is considered to be equal to zero.[2,9]

Green's functions $g_o(\mathbf{r}, \mathbf{r}_0)$ and $g_e(\mathbf{r}, \mathbf{r}_0)$ in expressions (25) are of the forms (21) and (22), in the E-coordinate system. Thus the form (21) remains the same, and for Eq. (22) we obtain

$$g_e(\mathbf{r}, \mathbf{r}_0) = \sqrt{a} \exp\left(ik^e R'\right) / \beta R', \qquad (26)$$

where

$$R' = \sqrt{\frac{a^2}{\beta^2}(x - x_0 + \rho z_0 - \rho z)^2 + (y - y_0)^2 + (z - z_0)^2},$$

$$\rho \approx \tan\rho = \sin\theta \cos\theta \left(\beta^2 - 1\right) / a$$

is the anisotropy angle.

Turning the coordinate system does not change the equation (16) and instead of Eq. (19) we obtain

$$\hat{L}_e U_e = 0, \qquad (27)$$

where

$$\hat{L}_e = \frac{\partial^2}{\partial z^2} + 2\rho \frac{\partial^2}{\partial x \partial z} + \frac{b}{a} \frac{\partial^2}{\partial x^2} + \frac{1}{a} \frac{\partial^2}{\partial y^2} + \left(k^e\right)^2,$$

$$b = \cos^2\theta + \beta^2 \sin^2\theta.$$

By direct substitution we see, that Green's function (26) is proved to be the exact solution of the equation

$$\hat{L}_e g_e(\mathbf{r}, \mathbf{r}_0) = -4\pi\delta(\mathbf{r} - \mathbf{r}_0). \qquad (28)$$

From this we conclude, that the integral (25) for the e-wave is the exact solution of the equation (27). It is obvious, that at $\theta = \pi/2$ Eq. (27) coincides with the Eq. (19). If one takes in Eq. (27) that $\rho = 0$, thus assuming the medium isotropic then equation (16) follows.

It is possible to use more simple form[2,9] instead of Eq. (25) since integration in Eq. (25) is carried out over the plane $z = 0$:

$$U_{o,e}(\mathbf{r}_0) = \frac{1}{2\pi} \iint_{-\infty}^{+\infty} U_{o,e}(x, y) \left(\frac{\partial g_{o,e}}{\partial z} \right)_{z=0} dx\,dy. \qquad (29)$$

It is proved in Ref. 9, that formula (29) presents the unique exact solution of the equation (16) for the o-wave, which at $z_0 \to 0$ coincides with the boundary condition $U_0(x, y)$, determined (set) in this plane. The theorem on uniqueness of solution (29) for the e-wave can, probably, be proved by analogy, but we did not consider this question.

Substitute the scalar amplitude U from expression (24) into Eqs. (16) and (27), collect the terms, proportional to μ_2, and pass to the equations for complex slowly varying amplitudes of the ordinary and extraordinary waves:

$$\frac{\partial A_o}{\partial z}+\frac{1}{2ik_o}\left(\frac{\partial^2 A_o}{\partial x^2}+\frac{\partial^2 A_o}{\partial y^2}\right)=0, \tag{30}$$

$$\frac{\partial A_e}{\partial z}+\rho\frac{\partial A_e}{\partial x}+\frac{1}{2ik^e}\left(\frac{b}{a}\frac{\partial^2 A_e}{\partial x^2}+\frac{1}{a}\frac{\partial^2 A_e}{\partial y^2}\right)=0. \tag{31}$$

The equation (30) represents the required abridged equation for an ordinary wave in a uniaxial medium. But Eq. (31) requires one more modification.

For the second term of Eq. (31) to be of the second order of smallness ($\sim \mu_2$), it is necessary to require fulfillment of the condition

$$|\rho| \sim \left|\beta^2-1\right| \sim \mu_1. \tag{32}$$

This means, that the considered uniaxial medium is a weakly anisotropic. This restriction is not essential according to the harmonic generation theory, since the majority of nonlinear crystals used in practice are just of this type. However, if Eq. (32) takes place, this means, that now terms of the third order of smallness ($\sim \mu_1\mu_2$) will appear in Eq. (31). However, eliminating those from this equation, i.e., restricting ourselves to solving the problem in the second approximation, we derive the required abridged equation (4) instead of Eq. (31).

The expressions (29), in which it is necessary to conserve the terms proportional to $\sim \mu_2$, i.e., to write down the solutions with the same accuracy, as the equations itself will obviously be the solutions of the equations (30) (for the *o*-wave) and (4) (for the *e*-wave). For this purpose we do the following (we shall refer all relations for the *e*-wave since taking the $\rho = 0$ transforms them into similar expressions for the *o*-wave).

Note, that for weakly diverging beams the following condition holds

$$\frac{|x-x_0+\rho z_0-\rho z|}{z_0-z} \sim \frac{|x_0|}{z_0} \sim \frac{|y-y_0|}{z_0-z} \sim \mu_1. \tag{33}$$

The equation (33) allows one to present the expression for R' from expression (26) in the form of a power series

$$R' = z_0 - z + \frac{a^2}{b^2}\frac{(x-x_0+\rho z_0-\rho z)^2}{2(z_0-z)} +$$
$$+a\frac{(y-y_0)^2}{2(z_0-z)} + O\left(\mu_1^4, \mu_1^6 \ldots\right) =$$
$$= z_0 - z + \frac{(x-x_0+\rho z_0-\rho z)^2}{2(z_0-z)} +$$
$$+\frac{(y-y_0)^2}{2(z_0-z)} + O\left(\mu_1^3,\, \mu_1^4 \ldots\right), \tag{34}$$

where $O\left(\mu_1^3, \mu_1^4 \ldots\right)$ are the values of the μ_1^3, $\mu_1^4 \ldots$, orders of magnitude and we have accounted the expression (32).

Now we assume, that the following inequality is valid

$$k^e z_0 \gg 1 \text{ or } \left(ik^e - \frac{1}{R'}\right) \approx ik^e, \tag{35}$$

and calculate the derivative in expression (29), conserving the terms $\sim \mu_2$ for R' in the exponent and restrict ourselves to the zero approximation for R', standing outside of exponential function. As a result, we have instead of Eq. (29)

$$U_e(\mathbf{r}_0) = A_e(\mathbf{r}_0)\, e^{ik^e z_0}, \tag{36a}$$

where

$$A_e(\mathbf{r}_0) = -\frac{ik^e}{2\pi z_0} \times$$
$$\times \iint_{-\infty}^{+\infty} U_e(x,y,0) \exp\left[ik^e \frac{(x-x_0+\rho z_0)^2+(y-y_0)^2}{2z_0}\right] \mathrm{d}x\mathrm{d}y. \tag{36b}$$

By direct substitution we see, that expression (36b) is the exact solution of Eq. (4), and, at $\rho = 0$, the exact solution of Eq. (30).

Let us note the relation, which is important for us

$$\left(\frac{\partial g_{o,e}}{\partial x}\right) \sim \left(\frac{\partial g_{o,e}}{\partial y}\right) \sim \mu_1 \left(\frac{\partial g_{o,e}}{\partial z}\right). \tag{37}$$

The validity of relation (37), obviously, follows directly from Eq. (33). Nevertheless, we attract especial attention to this moment since we shall often refer to it in what follows.

Thus, at this stage we consider the question on the scalar approximation for the electromagnetic field in the uniaxial and homogeneous medium closed and, in conclusion, we would like to note once more the basic, in our opinion, result among all the above-mentioned ones. By this we mean the expressions (13) and (14) that define the ordinary and extraordinary waves in a uniaxial medium. Using just these definitions, it is possible to make an absolutely rigorous transition from the wave equation (1) to the Helmholtz equations (16) and (19) for scalar amplitudes of the *o*- and *e*-waves. The subsequent reasoning and calculations contain no any essentially new moments, and we give those here, mainly because we shall have a need for those in the below discussion.

2. Account of vector properties of laser beams in uniaxial media

Let the Green's tensor $\tilde{G}(\mathbf{r},\mathbf{r}_0)$ (of the second rank) is the solution of the equation

$$\operatorname{rot}\operatorname{rot}\tilde{G} - (k^0)^2\tilde{\varepsilon}\tilde{G} = 4\pi\left[\tilde{\chi}\operatorname{rot}\delta(\mathbf{r}-\mathbf{r}_0)\right], \tag{38}$$

where $\tilde{\chi}$ is the unit second rank tensor.

Let us postmultiply expression (1) by $\tilde{G}$, and premultiply expression (38) by $\mathbf{E}$ and subtract the second product from the first one. As a result, we have

$$(\mathrm{rot\,rot}\,\mathbf{E})\,\tilde{G}-\mathbf{E}\,\mathrm{rot\,rot}\,\tilde{G}-\left(k^0\right)^2\left(\tilde{\varepsilon}\mathbf{E}\tilde{G}-\mathbf{E}\tilde{\varepsilon}\tilde{G}\right)=$$
$$=-4\pi\mathbf{E}\,\mathrm{rot}\left[\tilde{\chi}\,\delta(\mathbf{r}-\mathbf{r}_0)\right]. \qquad (39)$$

Then use the following definitions[10]:

$$\tilde{\varepsilon}\,\mathbf{E}=\mathbf{U},\;\; U_\alpha=\sum_\beta \varepsilon_{\alpha\beta}\,E_\beta;$$

$$\mathbf{E}\,\tilde{\varepsilon}=\mathbf{U},\;\; U_\alpha=\sum_\beta E_\beta\,\varepsilon_{\beta\alpha};$$

$$\tilde{\varepsilon}\tilde{G}=\tilde{U},\;\; U_{\alpha\beta}=\sum_\gamma \varepsilon_{\alpha\gamma}\,G_{\gamma\beta},$$

where variable subscripts take the values from unity up to three.

Note, that for symmetric tensors $\tilde{\varepsilon}$ the third term in the left part of Eq. (39) goes to zero, so integrating it over an arbitrary volume V limited by the S surface with an external normal $\mathbf{n}$ we obtain

$$\mathrm{rot}\,\mathbf{E}(\mathbf{r}_0)=\frac{1}{4\pi}\int_V\left\{(\mathrm{rot\,rot}\,\mathbf{E})\tilde{G}-\mathbf{E}\,\mathrm{rot\,rot}\,\tilde{G}\right\}\mathrm{d}V, \qquad (40)$$

in which we have taken into account that for all $\mathbf{r}_0$, not lying on S, the following condition holds

$$\int_V \mathbf{E}\,\mathrm{rot}\left[\tilde{\chi}\delta(\mathbf{r}-\mathbf{r}_0)\right]\mathrm{d}V=$$
$$=\int_V\left\{\delta(\mathbf{r}-\mathbf{r}_0)\,\mathrm{rot}\,\mathbf{E}-\mathrm{rot}\left[\mathbf{E}\delta(\mathbf{r}-\mathbf{r}_0)\right]\right\}\mathrm{d}V=$$
$$=\int_V\delta(\mathbf{r}-\mathbf{r}_0)\,\mathrm{rot}\,\mathbf{E}\,\mathrm{d}V-\int_S\delta(\mathbf{r}-\mathbf{r}_0)[\mathbf{nE}]\,\mathrm{d}S=\mathrm{rot}\,\mathbf{E}(\mathbf{r}_0).$$

By definition of the second rank tensor we have

$$\tilde{G}=\sum_\alpha \mathbf{e}_\alpha\,\mathbf{G}_\alpha=\sum_\beta \mathbf{G}_\beta^{(\mathrm{T})}\mathbf{e}_\beta,$$

$$\mathbf{G}_\alpha=\sum_\beta G_{\alpha\beta}\,\mathbf{e}_\beta,\;\; \mathbf{G}_\beta^{(\mathrm{T})}=\sum_\alpha \mathbf{e}_\alpha\,G_{\alpha\beta},$$

where $\mathbf{e}_\alpha$ are the basis vectors.

Using this definition, we find that

$$(\mathrm{rot\,rot}\,\mathbf{E})\,\tilde{G}-\mathbf{E}\mathrm{rot\,rot}\,\tilde{G}=$$
$$=\sum_\alpha\left\{(\mathrm{rot\,rot}\,\mathbf{E})\,\mathbf{G}_\alpha^{(\mathrm{T})}-\mathbf{E}\,\mathrm{rot\,rot}\,\mathbf{G}_\alpha^{(\mathrm{T})}\right\}\mathbf{e}_\alpha=$$
$$=\sum_\alpha \mathrm{div}\left[(\mathrm{rot}\,\mathbf{E})\mathbf{G}_\alpha^{(\mathrm{T})}+\mathbf{E}\,\mathrm{rot}\,\mathbf{G}_\alpha^{(\mathrm{T})}\right]\mathbf{e}_\alpha \qquad (41)$$

for the integrand functions in formula (40) where we used the well-known vector identity

$$\mathrm{div}[\mathbf{AB}]=\mathbf{B}\,\mathrm{rot}\,\mathbf{A}-\mathbf{A}\,\mathrm{rot}\,\mathbf{B}.$$

Substituting expression (41) into Eq. (40), and interchanging the summation and integration order, we obtain by virtue of the Gauss–Ostrogradsky theorem the following expression

$$\mathrm{rot}\,\mathbf{E}(\mathbf{r}_0)=$$
$$=\frac{1}{4\pi}\sum_\alpha\int_S\left\{\mathbf{n}\left[\mathrm{rot}\,\mathbf{E}\,\mathbf{G}_\alpha^{(\mathrm{T})}\right]+\mathbf{n}\left[\mathbf{E}\,\mathrm{rot}\,\mathbf{G}_\alpha^{(\mathrm{T})}\right]\right\}\mathrm{d}S\,\mathbf{e}_\alpha. \qquad (42)$$

Now make use of the permutation rule in a mixed product and apply it to Eq. (42). Then again interchange the summation and integration order we pass from the vectors to tensors and obtain the final result

$$\mathbf{H}(\mathbf{r}_0)=\frac{1}{4\pi\,i\,k^0}\times$$
$$\times\int_S\left\{[\mathbf{En}]\,\mathrm{rot}\,\tilde{G}+[\mathrm{rot}\,\mathbf{En}]\,\tilde{G}\right\}\mathrm{d}S, \qquad (43)$$

where we used the Maxwell's equation

$$\mathrm{rot}\,\mathbf{E}=ik^0\mathbf{H}. \qquad (44)$$

If one writes the tensor equation (38) in the main coordinate system, it is possible to see, that for uniaxial media it splits into three independent systems each consisting of three scalar equations, separate for each of the $\tilde{G}$ tensor columns. Solving these systems of equations, we find that

$$G_{11}=0,\;\; G_{21}=\frac{\partial g_o}{\partial z},\;\; G_{31}=-\frac{\partial g_o}{\partial y},$$

$$G_{12}=-\frac{\partial g_e}{\partial z},\;\; G_{22}=\frac{\partial^3 I}{\partial x\,\partial y\partial z},\;\; G_{32}=\frac{\partial g_o}{\partial z}+\frac{\partial^3 I}{\partial x\,\partial z^2},$$

$$G_{13}=\frac{\partial g_e}{\partial y},\;\; G_{23}=-\frac{\partial g_o}{\partial x}-\frac{\partial^3 I}{\partial x\,\partial y^2},\; G_{33}=-G_{22}, \qquad (45)$$

where $g_o(\mathbf{r},\mathbf{r}_0)$ and $g_e(\mathbf{r},\mathbf{r}_0)$ are the Green's functions (21) and (22). Also we have used the designation

$$I(\mathbf{r},\mathbf{r}_0)=\frac{\Delta_0}{2\pi^2}\iiint_{-\infty}^{+\infty}\frac{\mathrm{e}^{i\alpha(\mathbf{r}-\mathbf{r}_0)}}{\left(k_o^2-\alpha^2\right)\left(k_e^2-\alpha^2+\Delta_0\alpha_x^2\right)}\mathrm{d}\boldsymbol{\alpha}, \qquad (46)$$

$$\Delta_0=\left(n_o^2-n_e^2\right)/\,n_o^2,\quad \alpha^2=\alpha_x^2+\alpha_y^2+\alpha_z^2.$$

If $\Delta_0=0$ the integral (46) also equals to zero, and expressions (43) and (45) present the rigorous solution of the equation (1) for a field in the isotropic medium.

From Eq. (38) directly follows the condition

$$\mathrm{div}\left(\tilde{\varepsilon}\tilde{G}\right)=0. \qquad (47)$$

Substituting formula (45) into Eq. (47), we obtain, that the unknown function (46) is to be the exact solution of the equation

$$\frac{\partial^2 I}{\partial y^2}+\frac{\partial^2 I}{\partial z^2}=\beta^2 g_e-g_o. \qquad (48)$$

Let us introduce the designations

$$\mathbf{B}=[\mathbf{En}]=\{B_1,B_2,B_3\},$$
$$\mathbf{D}=[\mathrm{rot}\,\mathbf{En}]=\{D_1,D_2,D_3\}. \qquad (49)$$

Using Eq. (45) and after the simple calculations for expression (43) we find

$$H_x(\mathbf{r}_0)=\frac{1}{4\pi i k^0}\int_S \mathrm{d}S\left\{D_2\frac{\partial g_o}{\partial z}-D_3\frac{\partial g_o}{\partial y}-B_1\left(\frac{\partial^2 g_o}{\partial y^2}+\frac{\partial^2 g_o}{\partial z^2}\right)+B_2\left(\frac{\partial^2 g_o}{\partial x\partial y}\right)+B_3\left(\frac{\partial^2 g_o}{\partial x\partial z}\right)\right\}, \tag{50a}$$

$$H_y(\mathbf{r}_0)=\frac{1}{4\pi i k^0}\int_S \mathrm{d}S\left\{-D_1\frac{\partial g_e}{\partial z}+D_2\frac{\partial^3 I}{\partial x\partial y\partial z}+D_3\left(\frac{\partial g_o}{\partial x}+\frac{\partial^3 I}{\partial x\partial z^2}\right)+B_1\frac{\partial g_o}{\partial x\partial y}-\right.$$
$$\left.-B_2\left(\frac{\partial^2 g_e}{\partial z^2}+\frac{\partial^2 g_o}{\partial x^2}+\frac{\partial^4 I}{\partial x^2\partial z^2}\right)+B_3\left(\frac{\partial^2 g_e}{\partial y\partial z}+\frac{\partial^4 I}{\partial x^2\partial y\partial z}\right)\right\}, \tag{50b}$$

$$H_z(\mathbf{r}_0)=\frac{1}{4\pi i k^0}\int_S \mathrm{d}S\left\{D_1\frac{\partial g_e}{\partial y}-D_2\left(\frac{\partial g_o}{\partial x}+\frac{\partial^3 I}{\partial x\partial y^2}\right)-D_3\frac{\partial^3 I}{\partial x\partial y\partial z}+B_1\frac{\partial g_o}{\partial x\partial z}+\right.$$
$$\left.+B_2\left(\frac{\partial^2 g_e}{\partial y\partial z}+\frac{\partial^4 I}{\partial x^2\partial y\partial z}\right)-B_3\left(\frac{\partial^2 g_o}{\partial x^2}+\frac{\partial^2 g_e}{\partial y^2}+\frac{\partial^4 I}{\partial x^2\partial y^2}\right)\right\}. \tag{50c}$$

We shall consider the expression (50) the exact solution of a vector problem for a field in the uniaxial medium. It follows, from formulas (21), (22), and (49) that for all $x_i = x, y, z$ and $x_{0i} = x_0, y_0, z_0$

$$\frac{\partial g_{o,e}}{\partial x_i}=-\frac{\partial g_{o,e}}{\partial x_{0i}},\quad \frac{\partial I}{\partial x_i}=-\frac{\partial I}{\partial x_{0i}}. \tag{51}$$

Using the relations (51), it is easy to see the first general property of the solution sought. For any boundary conditions (at any values of the vectors **B** and **D**) the field divergence (50) is equal to zero.

Let us construct two functions:

$$I_o(\mathbf{r},\mathbf{r}_0)=\frac{R^2}{k_o^2p^2}g_o(\mathbf{r},\mathbf{r}_0), \tag{52a}$$

$$I_e(\mathbf{r},\mathbf{r}_0)=-\frac{R'^2}{k_o^2p^2}g_e(\mathbf{r},\mathbf{r}_0), \tag{52b}$$

where $p^2=(y-y_0)^2+(z-z_0)^2$.

Now calculate the exact value of the expression

$$\frac{\partial^2 I_o}{\partial y^2}+\frac{\partial^2 I_o}{\partial z^2}=$$
$$=-g_o\left\{1+\frac{1}{ik_o}\left(\frac{1}{R}-\frac{2R'}{p^2}\right)+\frac{1}{(ik_o)^2}\left(\frac{4R^2}{p^4}-\frac{2}{p^2}-\frac{1}{R^2}\right)\right\}. \tag{53}$$

It is easy to see, that if the condition

$$ik_o\frac{p^2}{R}\gg 1 \tag{54}$$

holds, the Eq. (53) transforms into the following equality:

$$\frac{\partial^2 I_o}{\partial y^2}+\frac{\partial^2 I_o}{\partial z^2}=-g_o. \tag{55a}$$

The basic negative moment of the derived solution is its dependence on the integral (46), whose exact value we did not manage to determine. In this connection, there exists a need in seeking ways for calculating its approximate value. Below we shall show one of such possibilities.

Similarly it is possible to show, that under condition (54) the following equality is valid

$$\frac{\partial^2 I_e}{\partial y^2}+\frac{\partial^2 I_e}{\partial z^2}=\beta^2 g_e. \tag{55b}$$

As a result we arrive at a conclusion that, if the condition (54) holds, the integral (46) can be presented in the form of a sum of two functions (52), and this sum approximately satisfies the condition (48). Actually, by investigating the condition (54), it is possible to see, that for large enough z_0 values (see expression (35)) it does not hold in the only case when the beam propagates almost along the optical axis. However, this situation is of no interest, at least for the harmonic generation theory. For this reason it is possible to believe, that the considered approximation does not impose more restrictions, than the condition (35), which is necessary for the parabolic approximation in principle.

If now the integral (46), written in form of a sum of two functions (52), is substituted into the solution (50) it quite naturally breaks into two solutions. The terms, depending on the $g_o(\mathbf{r},\mathbf{r}_0)$ function, will, obviously, describe behavior of the *o*-wave, and all the rest of the *e*-wave. Because $g_{o,e}$ functions are exact solutions of the equations (20), components of the **H** vector from Eq. (50) will satisfy the Helmholtz scalar equations (16) or (19). Using the Maxwell's equation

$$\mathrm{rot}\,\mathbf{H}=-ik^0(\hat{\varepsilon}\mathbf{E}), \tag{56}$$

it is also possible to easily determine the $\mathbf{E}_{o,e}$ vector components. These will satisfy both of the equations (16) and (19) and the condition (6a) imposed on the divergence. One can easily see from expression (50), that the $\mathbf{H}_e$ vector has no projection onto the optical axis ($H_{ex} = 0$), and by means of equality (55) it is possible to show, that the solution for the $\mathbf{E}_o$ vector will possess the same property.

Thus, all the tests, which could be done without specifying the boundary conditions, show that expressions (50) really determine the solution of the vector problem sought, the accuracy of which is regulated by meeting the condition (54). Nevertheless, one very important item is not clear: whether or not the solution (50) will conform to the requirement of uniqueness. Some doubts can certainly arise concerning this aspect based on the following quite appreciable circumstance. If an arbitrary function of the only argument x is added to the sum of functions (52), the new sum will satisfy the condition (48) as well. We, personally, think that it is impossible to decide on which of these approximate versions does coincide with the exact value of the integral (46) better. The uniqueness of the solution must either be proved rigorously, or one has to try to establish the clear relation of the solution (50) to solutions derived in the scalar approximation, for which the uniqueness theorems have already been proved. The second version looks, in our opinion, preferable and below we shall try to realize it, while hoping at the same time to simplify the expressions (50) which seem to us still too cumbersome.

We shall do this in the following way. First, we write down the expressions (50) in the E-coordinate system, thereby, obtaining solution of the problem on beam propagation along a direction at an arbitrary angle θ to the optical axis. Then we assume the beam a weakly diverging one and agree to seek the problem solution in the first approximation. The last note means, that by virtue of expression (37) all derivatives (the second and higher ones) of Green's functions with respect to transverse coordinates (x and y) are equal to zero. We consider, that functions (52) in the E-coordinates can be presented in the form of [everywhere below it is necessary to use formula (26) for g_e]

$$I_o = \frac{1}{\sin^2\theta\, k_o^2} g_o + O(\mu_1^2),\quad I_e = \frac{1}{\sin^2\theta\, k_o^2} g_e + O(\mu_1^2), \tag{57}$$

and we obtain for $\mathbf{E}_o$ and $\mathbf{H}_e$ vectors the following

$$\mathbf{E}_o(\mathbf{r}_0) = -\frac{1}{4\pi k_o^2}\times$$

$$\times\int\limits_{-\infty}^{+\infty}\!\!\int \Bigg\{ -\mathbf{i}C\left[D_2\frac{1}{S}\frac{\partial^2 g_o}{\partial y\partial z} + B_1 k_o^2\frac{\partial g_o}{\partial y} + B_3\frac{C}{S}\frac{\partial^3 g_o}{\partial y\partial z^2}\right] +$$

$$+\mathbf{j}\left[-D_2 k_o^2 g_o - D_3\frac{1}{S}\frac{\partial^2 g_o}{\partial y\partial z} - B_1 k_o^2\left(C\frac{\partial g_o}{\partial x} - S\frac{\partial g_o}{\partial z}\right) +\right.$$

$$\left.+B_2\frac{C}{S}\frac{\partial^3 g_o}{\partial y\partial z^2} - B_3 k_o^2\left(S\frac{\partial g_o}{\partial x} + C\frac{\partial g_o}{\partial z}\right)\right] -$$

$$-\mathbf{k}S\left[D_2\frac{1}{S}\frac{\partial^2 g_o}{\partial y\partial z} + B_1 k_o^2\frac{\partial g_o}{\partial y} + B_3\frac{C}{S}\frac{\partial^3 g_o}{\partial y\partial z^2}\right]\Bigg\}\mathrm{d}x\mathrm{d}y; \tag{58}$$

$$\mathbf{H}_e(\mathbf{r}_0) = \frac{1}{4\pi i\, k^0}\times$$

$$\times\int\limits_{-\infty}^{+\infty}\!\!\int \Bigg\{ -\mathbf{i}C\left[D_1\frac{\partial g_e}{\partial y} + D_3\frac{C}{S}\frac{1}{k_o^2}\frac{\partial^3 g_e}{\partial y\partial z^2} +\right.$$

$$\left.+B_2\left(S\frac{\partial^2 g_e}{\partial y\partial z} - \frac{C^2}{S}\frac{1}{k_o^2}\frac{\partial^4 g_e}{\partial y\partial z^3}\right)\right] +$$

$$+\mathbf{j}\left[D_1\left(C\frac{\partial g_e}{\partial x} - S\frac{\partial g_e}{\partial z}\right) + D_3\beta^2\left(S\frac{\partial g_e}{\partial x} + C\frac{\partial g_e}{\partial z}\right) + B_2 k_e^2 g_e\right] +$$

$$+\mathbf{k}S\left[D_1\frac{\partial g_e}{\partial y} + D_3\frac{C}{S}\frac{1}{k_o^2}\frac{\partial^3 g_e}{\partial y\partial z^2} +\right.$$

$$\left.+B_2\left(S\frac{\partial^2 g_e}{\partial y\partial z} - \frac{C^2}{S}\frac{1}{k_o^2}\frac{\partial^4 g_e}{\partial y\partial z^3}\right)\right]\Bigg\}\mathrm{d}x\mathrm{d}y, \tag{59}$$

where we used the $C = \cos\theta$, $S = \sin\theta$ designations, to simplify the writing.

Now it is necessary to concretize the boundary conditions. Turning to Eqs. (58) and (59), we notice, that for all $z_0 > 0$ the vectors of the electric field strength of the o- and e-waves can be presented in the form

$$\mathbf{E}_o = \{\mu_1 E_{ox}, E_{oy}, \mu_1 E_{oz}\},\quad \mathbf{E}_e = \{E_{ex}, \mu_1 E_{ey}, \mu_1 E_{ez}\}. \tag{60}$$

Besides, all the vector components (60) have a slowly varying amplitudes, i.e.,

$$(\mathbf{E}_{o,e})_i = (A_{o,e})_i(\mu_1 x, \mu_1 y, \mu_2 z)\,\mathrm{e}^{ik_o^e z}. \tag{61}$$

Since we solve the problem in the first approximation, the boundary conditions (i.e., components of **B** and **D** vectors) which are to be substituted into expressions (58) and (59), should be written in the first approximation too. As a result, the vector components (49) of the ordinary wave are written in the first approximation as follows

$$B_1 = -SE_y \sim \mu_1^0,\ B_2 = E_x \sim \mu_1,\ B_3 = CE_y \sim \mu_1^0; \tag{62a}$$

$$D_1 = -ik_o SE_x \sim \mu_1,\ D_2 = -ik_o E_y \sim \mu^0,$$

$$D_3 = ik_o CE_x \sim \mu_1. \tag{62b}$$

Boundary conditions for the extraordinary wave take the form

$$B_1 = -SE_y \sim \mu_1,\ B_2 = E_x \sim \mu_1^0,\ B_3 = CE_y \sim \mu_1, \tag{63a}$$

$$D_1 = -ik^e SE_x \sim \mu_1^0,\ D_2 = -ik^e E_y \sim \mu_1,$$

$$D_3 = ik^e CE_x \sim \mu_1^0. \tag{63b}$$

By substituting formulas (62) into Eq. (58), and canceling the new formed terms of the second order of smallness we obtain the final expression for the vector of electric field strength of the o-wave

$$\mathbf{E}_o(\mathbf{r}_0) = \mathbf{i}\left(\frac{1}{ik_o}\frac{C}{S}\frac{\partial T_o}{\partial y_0}\right) + \mathbf{j}T_o - \mathbf{k}\left(\frac{1}{ik_o}\frac{\partial T_o}{\partial y_0}\right), \tag{64a}$$

where

$$T_o(\mathbf{r},\mathbf{r}_0)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\int E_y(x,y)\left[\frac{\partial g_o}{\partial z}\right]_{z=0}\mathrm{d}x\mathrm{d}y. \quad (64b)$$

Substitute the boundary conditions (63) into the solution (59) and make use of the Maxwell's equation (56). Consider also that in the E-coordinates the nonzero components of the tensor of the dielectric constant are determined in the following way:

$$\varepsilon_{11}=n_o^2 b,\ \ \varepsilon_{22}=n_o^2,\ \ \varepsilon_{33}=n_o^2 a,\ \ \varepsilon_{13}=\varepsilon_{31}=n_o^2\rho\, a. \quad (65)$$

After that, keeping terms of zero- and the first-order of smallness, we obtain, for the electric field strength vector of the extraordinary wave its final form

$$\mathbf{E}_e(\mathbf{r}_0)=\mathbf{i}(aT_e)-\mathbf{j}\left(\frac{C}{S}\frac{1}{ik^e}\frac{\partial T_e}{\partial y_0}\right)-\mathbf{k}\left(\frac{1}{ik^e}\frac{\partial T_e}{\partial x_0}+\rho T_e\right), \quad (66a)$$

where

$$T_e(\mathbf{r},\mathbf{r}_0)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\int E_x(x,y)\left[\frac{\partial g_e}{\partial z}\right]_{z=0}\mathrm{d}x\mathrm{d}y. \quad (66b)$$

Let us make some comments. The position of an optical axis in the E-coordinate system is obviously determined by the vector

$$\mathbf{O}=\{\sin\theta, 0, \cos\theta\}. \quad (67)$$

Then it is easy to see, that the scalar products of the vectors (67) and (66), and also of the vectors (67) and (59) identically go to zero. Hence, by definitions (13) and (14), the field (64) appears to be the o-wave, and (66) – the e-wave. All the vector components (64) and (66) precisely satisfy all the Helmholtz equations (16) and (27), and the vectors themselves in their first approximation form satisfy the divergence condition (6a). Finally, we have to note most important thing that in zero approximation the solutions (64) and (66) coincide with the corresponding exact solutions obtained in the scalar approximation. The small vector components (64) and (66) are determined by simple differentiation of this scalar solution. Thus, each of the solutions (64) and (66) determines the unique vector which coincides at $z_0 = 0$ with the corresponding boundary condition set in this plane.

It follows from the comments adduced, that the problem on propagation of the inherently vector field through a uniaxial and homogeneous medium may be considered in the first approximation completely solved. All the requirements, which such a solution must satisfy, have appeared to be met. Since the solutions (64) and (66) have been obtained as a result of formal simplifications of the general expression (50) there are serious grounds to suppose, that the representation (50) itself is the exact solution of the wave equation (1).

Conclusion

Let us note one thing of the general character, which we did not discuss in this paper at all. The main idea of it is in the answer to the question on whether or not the wave of one type can be the source of wave of another type in the uniaxial medium. In other words, whether or not it is possible to set such boundary conditions at the entrance to the uniaxial medium, which would guarantee the presence in the medium of the wave of only one type. From the above-mentioned solutions (64) and (66) it follows that, generally speaking, the answer to the first question should be positive, and to the second – negative. For example, if the o-wave has the E_x component (or "acquires" it in the process of propagation) this component according to expression (66) must become the source of the e-wave (the quantitative aspect of the question is not discussed, and we are interested only in a basic possibility). Similarly, we draw a conclusion that in the uniaxial medium the o-wave, which was not present at the entrance to the uniaxial medium, can appear.

From that the next question naturally arises on whether such a mutual influence (or, perhaps, interaction) between the o- and e-waves is, generally speaking, usual property of the process of field propagation through a uniaxial medium or it follows from the series of approximations, we have used for simplification of the rigorous solution (50). It is most "suspicious," in this sense, the possibility of rigorously presenting the integral (46) in the form of a sum of two functions of the form (52) since this is the basis for splitting the general solution (50) into two separate ones, one for the o- and other for e-wave. Maybe, there is a rigorous proof of the possibility or impossibility of splitting the function (46) into two terms, but now we do not know this. Certainly, the answer would be found if the exact value of this integral could be calculated. However, at present we failed to find it. It's quite another matter, that the indirect information on this problem can be obtained in considering the special case of an isotropic medium.

Let the beam be propagating along the Z-axis of the E-coordinate system in an isotropic medium. Let us choose an arbitrary direction (67) and try to find the solution of equation (1), which would have no projection onto this direction. If we shall manage to find such a solution, it would be a full analog of the o-wave, but for the case of isotropic medium. In a similar way, using Eq. (14), one can arrange an "isotropic" e-wave. It is clear, that such "quasi" – o- and e-wave will have absolutely identical properties except only for different orientations of the polarization vectors.

As a result, such a statement of the problem makes it possible to turn to the question on the mutual influence again, but now already between the "quasi-waves." In so doing we assume that from the standpoint of the property we are interested in it is of no importance for which cause (natural or artificial one) the chosen direction appears in the medium. In this way, the probability of achieving the final goal of the calculations essentially increases as the passage to an isotropic medium automatically removes all the problems with calculating the integral (46).

We are not ready to discuss in detail such a scheme of solution now and therefore restrict ourselves to the following remark. Actually, we have already derived the rigorous solution of a standard problem on the field in an isotropic medium as it is only needed to turn to formula (50) and suppose that $n_o = n_e = n$. Note however, that in this case the integral (46) also goes to zero. This also seems to be quite curious. If we think about the propagation of the "quasi-waves," then in order to obtain corresponding solutions the integral (46) (in a more simple, "isotropic" form) should be entered into expression (50) again, but already in some artificial way.

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