Unwrapping of the optical field phase from its gradient in the presence of branch points

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In this study, we have realized and tested an algorithm of phase unwrapping from its gradient for the case of wave front dislocations present. The algorithm proposed is based on the complex exponential phase estimator proposed by D. Fried and the least squares method (LSM). In unwrapping the hidden phase, the algorithm does not require the localization and pair matching of the branch points and corrects for the errors of LSM smoothing. High accuracy of the algorithm has been demonstrated by simulating wave front conjugation under conditions of turbulent atmosphere, as an example.

Introduction

To unwrap the phase of an optical wave

$$U(\mathbf{r}) = A(\mathbf{r})\exp[i\phi(\mathbf{r})], \tag{1}$$

where $U(\mathbf{r})$, $A(\mathbf{r})$, and $\phi(\mathbf{r})$ are the complex amplitude, wave amplitude and phase, and $\mathbf{r} = \mathbf{r}(x, y)$ is the two-dimensional vector, from the wave tilts measured with Hartmann sensors or shear interferometers,[1] the least squares method (LSM)[2,3] is widely used by analogy with the radar interferometry.[2] The LSM produces smoothing[4,5] and filtering since it deals with the potential part of the phase gradient vector and does not unwrap the so-called hidden phase[6,7] determined by the solenoidal component of the phase gradient.[2,3,6–8] In the domain where the solenoidal part of the phase gradient is non-zero, $\pm 2\pi n$ phase jumps occur often due to noises.[9] Filtering of these noises is carried out with the LSM.[9,10] Nevertheless, if random, as well as regular optical fields propagate through random media, strong random spatial modulation of the intensity of propagated waves can occur. In such cases, the wave front dislocations arise[11] at the points where the intensity is close to zero, with the nonzero solenoidal component of the phase gradient vector being an indicator of such a situation. In this case, the use of LSM results in loss of valuable information on the phase. The phase of an optical wave can be written as a sum of two terms[7]:

$$\phi(\mathbf{r}) = \phi_{\mathrm{lmse}}(\mathbf{r}) + \phi_{\mathrm{hid}}(\mathbf{r}), \tag{2}$$

where $\phi_{\mathrm{lmse}}(\mathbf{r})$ is the phase unwrapped by LSM, $\phi_{\mathrm{hid}}(\mathbf{r})$ is the hidden phase determined by the solenoidal component of the phase gradient.

The phase gradient, in the discrete representation, can be written in the following form[5,7]:

$$\mathbf{g}(i,k) = \frac{\arg[U(i+1,k)U^*(i,k)]}{d}\mathbf{e}_x + \frac{\arg[U(i,k+1)U^*(i,k)]}{d}\mathbf{e}_y, \tag{3}$$

where i and k determine the pixel position in a two-dimensional array of discrete phase values, d is the distance between neighbor grid nodes; $\mathbf{e}_x$ and $\mathbf{e}_y$ are the unit vectors. The indicator of the dislocations at a point is the change by $\pm 2\pi$ of the principal phase gradient value when circling about the point along a closed trajectory[7]:

$$\mathbf{g}(i,k)\cdot\mathbf{e}_x d + \mathbf{g}(i+1,k)\mathbf{e}_y d - \mathbf{g}(i,k+1)\mathbf{e}_x d - \mathbf{g}(i,k)\mathbf{e}_y d = \begin{cases} \pm 2\pi \text{ if the branch point is inside the circle,} \\ 0 \text{ if there is no a branch point inside the circle.} \end{cases} \tag{4}$$

Positive and negative branch points of the phase appear by pair and are coupled via phase surface discontinuities. Having known the coordinates of the branch points, the hidden phase $\phi_{\mathrm{hid}}(\mathbf{r})$[7] can be determined and thus the full phase can be estimated. Nevertheless, false phase disruptions can appear in calculating the hidden phase at the branch points[5] due to the errors in determination of the paired points thus this lowering the efficiency of the algorithm.[5,12,13]

D. Fried has proposed the phase unwrapping algorithm based on the so-called exponential phase estimator he had introduced to estimate the phase surface from its local tilts, or gradient. A modification of this method allows, like the LSM, one to minimize the noise effect.[10] The algorithm enables one to unwrap full phase $\phi(\mathbf{r})$ in its principal value, i.e.,

$$\phi_{\mathrm{CEE}}(\mathbf{r}) = P[\phi(\mathbf{r})], \tag{5}$$

where $P[...]$ means reduction of the parameter in parenthesis to the interval $(-\pi, \pi]$ of the principal phase value. Then the principal phase value $P[\phi(\mathbf{r})]$ is unwrapped into the phase surface $\phi(\mathbf{r})$ using the information on the positions of branch points.

The algorithm of unwrapping the phase $\phi(\mathbf{r})$ from its gradient $\mathbf{g}(\mathbf{r})$ using the complex exponential estimator proposed by D. Fried, along with LSM, is discussed in this paper. The peculiarity of the

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algorithm, in contrast to the complex exponential estimator, is that it does not require determination of the positions of phase branch points.

Phase unwrapping algorithm

The phase $\phi(\mathbf{r})$ unwrapping from its gradient $\mathbf{g}(\mathbf{r})$ is carried out in the following way. The D. Fried's complex exponential estimator of the phase is applied to the preset array of the phase gradient $\mathbf{g}(\mathbf{r})$, which yields the array of complex numbers

$$\chi(\mathbf{r}) = A(\mathbf{r}) + iB(\mathbf{r}) = \mathrm{CEE}[\mathbf{g}(\mathbf{r})] = \exp[i\phi(\mathbf{r})], \quad (6)$$

where A and B are the real and imagine parts of a complex number; CEE[...] is the complex exponential estimator algorithm.[10] As follows from Eq. (6), the complex numbers $\chi(\mathbf{r})$ are related to the wave phase by the equation

$$\phi_{\mathrm{CEE}}(\mathbf{r}) = \arg[\chi(\mathbf{r})] = P[\phi(\mathbf{r})]. \quad (7)$$

Equation (7) shows that the full phase $\phi(\mathbf{r})$ is determined in its principal value on the interval $(-\pi, \pi]$. Unwrapping of the phase, within the interval $(-\pi, \pi]$, is carried out by the LSM. As a result, the smooth $\phi_{\mathrm{lmse}}(\mathbf{r})$ phase is obtained. To estimate the hidden phase component, the following calculations are to be performed by Eqs. (6) and (2):

$$\chi(\mathbf{r})/\exp\left[i\tilde{\phi}_{\mathrm{lmse}}(\mathbf{r})\right] = \exp\left[i\tilde{\phi}_{\mathrm{hid}}(\mathbf{r})\right], \quad (8)$$

whence

$$\tilde{\phi}_{\mathrm{hid}}(\mathbf{r}) = \arg\left\{\exp\left[i\tilde{\phi}_{\mathrm{hid}}(\mathbf{r})\right]\right\}. \quad (9)$$

As follows from Eqs. (8) and (9), the hidden phase component is determined for the principal phase value on the interval $(-\pi, \pi]$ and contains the error of smoothing Δ_{sm} produced by the LSM[4]:

$$\tilde{\phi}_{\mathrm{hid}}(\mathbf{r}) = \phi_{\mathrm{hid}} + \Delta_{\mathrm{sm}}.$$

The full phase is obtained by summing $\phi_{\mathrm{lmse}}(\mathbf{r})$ and $\tilde{\phi}_{\mathrm{hid}}$ components:

$$\phi(\mathbf{r}) = \phi_{\mathrm{lmse}}(\mathbf{r}) + \tilde{\phi}_{\mathrm{hid}}(\mathbf{r}). \quad (10)$$

Thus, the phase $\phi(\mathbf{r})$ can be unwrapped from its gradient $\mathbf{g}(\mathbf{r})$. In determining the hidden phase, the information on locations of the branch points and phase surface discontinuities is not required.

Numerical experiment

We have checked the efficiency of unwrapping the optical wave phase by the method proposed in numerical experiments on propagation of a collimated Gaussian beam $U(0,\mathbf{r}) = U_0\exp\{-r^2/(2a^2)\}$ (see Fig. 1) in a turbulent atmosphere with the use of simulation codes.[5,13]

The conditions of wave propagation through a turbulent atmosphere are conveniently characterized by the index of a plane wave scintillation[14]:

$$\beta_0^2 = 1.23C_n^2k^{7/6}L^{11/6}, \quad (11)$$

where C_n^2 is the structure characteristic of the air refractive index fluctuations, $k = 2\pi/\lambda$ is the wave number; λ is the wavelength; L is the path length. According to Refs. 15 and 16, at $\beta_0^2 > 1$ the regime of strong intensity fluctuations occurs and the intensity distribution across the beam breaks into speckles, i.e., strong spatial amplitude modulation of the optical field occurs. Figure 2 shows the intensity distribution for the field $U(\mathbf{r}, L)$ propagated through a turbulent atmosphere ($L = 1.5$ km, $a = 4$ cm, $\lambda = 1.06$ μm, $C_n^2 = 7.7 \cdot 10^{-14}$ m$^{-2/3}$, $\beta_0^2 = 5$).

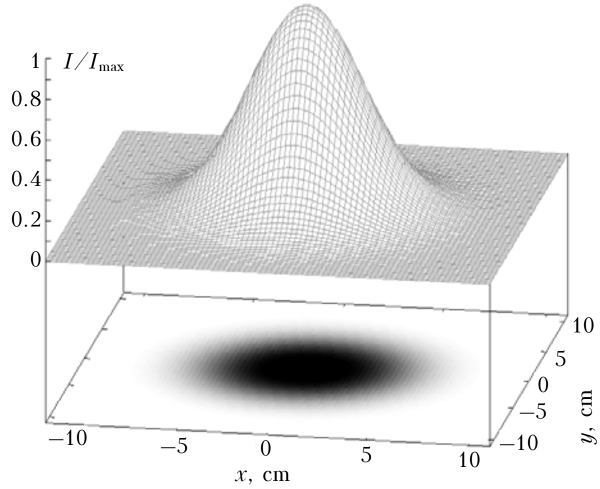

Fig. 1. Normalized intensity distribution for the initial Gaussian beam.

Let us make use of the wave front reciprocity.[17] If a forward propagated wave travels backward after its phase changed sign at the end of the path, $[\phi(\mathbf{r}) \to -\phi(\mathbf{r})]$, then, in the beginning of the path, the back wave field takes the initial distribution. The change of phase from $\phi(\mathbf{r})$ to $-\phi(\mathbf{r})$ is equivalent to the $U(\mathbf{r}, L)$ field complex conjugation operation: $U^*(\mathbf{r}, L) = A(\mathbf{r})\exp[-i\phi(\mathbf{r})]$. Hence, if the phase determined from the gradient measurements is incorrect at the wave front conjugation, then no initial field distribution will be obtained.

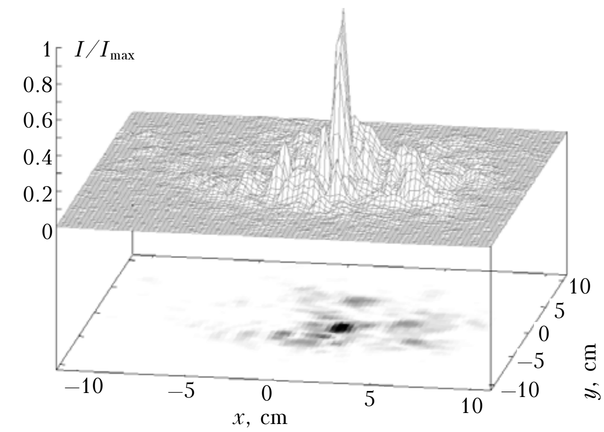

Fig. 2. Intensity distribution of a Gaussian beam propagated through the turbulent atmosphere, $\beta_0^2 = 5$.

Using the $U(\mathbf{r}, L)$ field modeled for the case of a turbulent atmosphere we have calculated, by Eq. (3), the phase gradient. The optical wave phase shown in Fig. 3 was obtained from the gradient with the use of the complex exponential phase estimator.

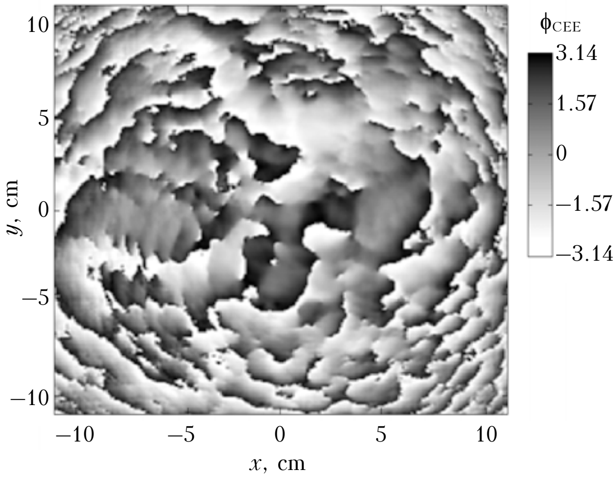

Fig. 3. Distribution of the phase of a Gaussian beam field unwrapped using the Fried's complex exponential phase estimator.[9]

As is seen from Fig. 3, phase values are in the $(-\pi, \pi]$ interval, i.e., they bounded by the principal value. Analysis of the phase distribution shows that it contains wave front dislocations.

Figure 4 presents the central parts of the intensity and phase distributions of the optical field $U(\mathbf{r}, L)$ with the wave front dislocations marked. The positions of the dislocations were obtained by Eq. (4).

Phase unwrapping was carried out using the LSM. As was noted above, the method does not reconstruct the phase completely. The method does not unwrap the phase component caused by the wave front dislocations as well as the part of phase function caused by the smoothing error.[4] The result of phase unwrapping from its principal value (see Fig. 4*b*) by the LSM is shown in Fig. 5. It is evidently a smooth distribution without phase discontinuities.

The hidden phase component was obtained by Eqs. (8) and (9) by applying the complex exponential phase estimator and the LSM. The full phase was calculated by Eq. (10) (Fig. 6) as the sum of the obtained components and then it was used in forming the reversed wave

$$U^{*}(\mathbf{r},L) = A(\mathbf{r})\exp\left\{-i\left[\phi_{\mathrm{lmse}}(\mathbf{r}) + \tilde{\phi}_{\mathrm{hid}}(\mathbf{r})\right]\right\}. \quad (12)$$

While this wave propagates backward through the medium with the same inhomogeneities as in the case of the forward propagation, the initial intensity distribution of the Gaussian beam is completely reconstructed (see Fig. 1). If one takes into account only the $\phi_{\mathrm{lmse}}(\mathbf{r})$ phase obtained with the LSM, then the optical field is not restored by the wave $U^{*}(\mathbf{r}) = A(\mathbf{r})\exp[-i\phi_{\mathrm{lmse}}(\mathbf{r})]$ propagated backward (Fig. 7).

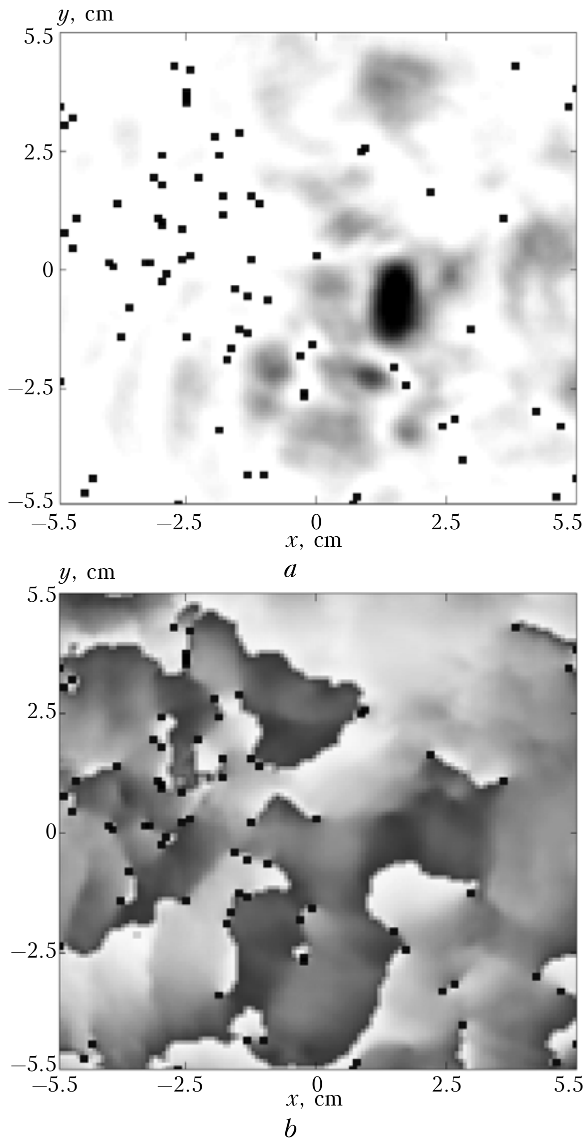

Fig. 4. Intensity (*a*) and phase (*b*) distributions in the central part of the beam cross section with the marked branch points of the phase.

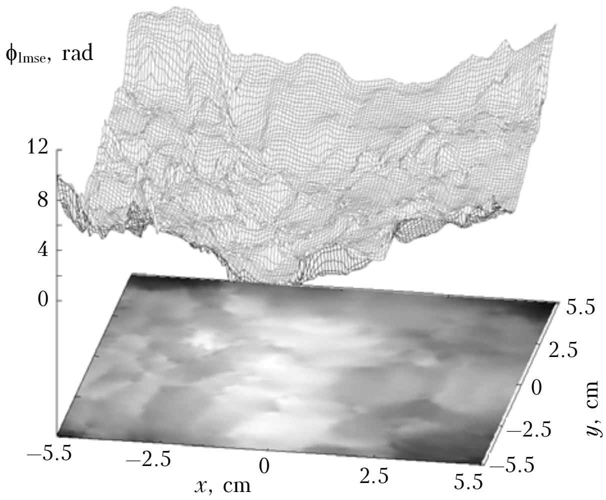

Fig. 5. LSM-unwrapped phase distribution.

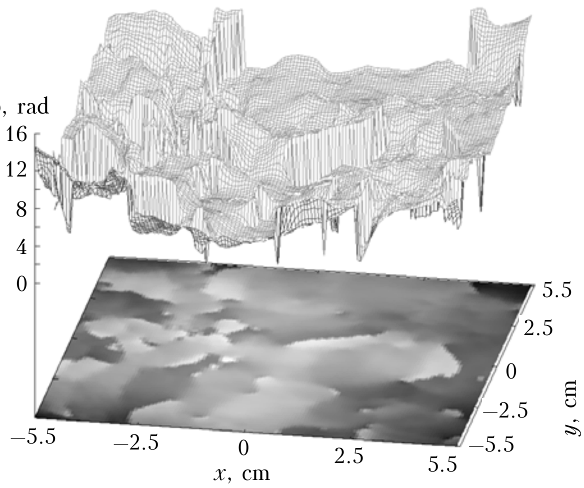

Fig. 6. Phase distribution calculated by Eq. (10).

The fact that the reversed wave (12) formed completely unwraps the initial Gaussian distribution of an optical beam after being propagated backward is an evidence of the exact phase unwrapping by the algorithm proposed.

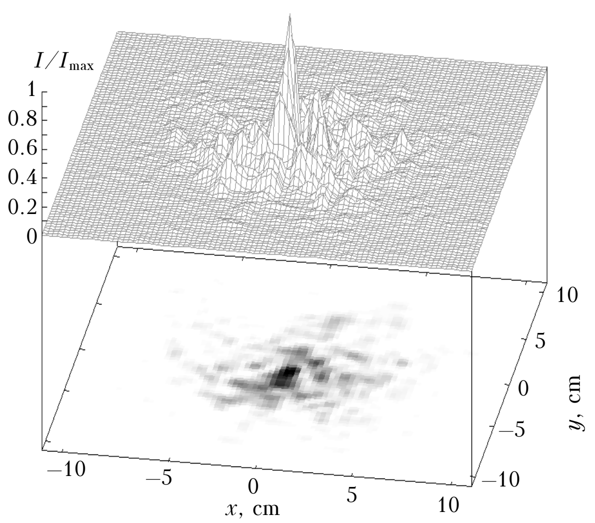

Fig. 7. Intensity distribution of a Gaussian beam after backward propagation if conjugating only the phase unwrapped using the LSM.

Conclusion

The phase unwrapping algorithm has been realized for the case of phase front dislocations. The algorithm is based on the D. Fried complex exponential phase estimator[10] and the least squares method. In contrast to Ref. 10, it does not require localization of branch points and determination of their parity and corrects for the errors introduced in smoothing by the least squares method. The algorithm does not allow the phase to be unwrap in the case of screw dislocations since the topology of such dislocations assumes phase jumps exceeding 2π, because the algorithm proposed calculates the hidden phase $\tilde{\phi}_{\text{hid}}$ only within the limits of the principal phase value (9). However the number of such jumps is negligible for the considered types of problems and does not influence the efficiency of the algorithm.

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