VARIATIONAL INTERPOLATION METHOD IN RADIATIVE TRANSFER THEORY

V.V. Uchaikin and V.A. Litvinov

Altay State University, Barnaul, 656099
Received July 7, 1988

A new variational interpolation method is reported. The proposed procedure is based on the extension of solutions of certain unperturbed problems to a number of cases where perturbation theory is not valid. The algorithm adopted does not have to be more complicated that that of the first-order perturbation theory. To illustrate the method, examples of solutions of wave and kinetic models for radiative transfer are considered.

The interpretation of atmospheric sounding data is complicated by variability of the state of the medium. It implies that a great number of time-consuming computations have to be performed for different atmospheric conditions. Ways of minimizing the calculations involved have received a great deal of attention. Thus, for example, perturbation theory enables one to reduce the number of computations by applying the solution of the unperturbed problem to the approximation of solutions of perturbed problems.

This paper discusses a new method of interpreting unperturbed solutions to certain problems that has a wider range of aplicability than perturbation theory. The proposed algorithm does not appear to be more complicated than first-order perturbation theory.

Let us calculate the functional

$$J = (D, \phi)$$

of the solution to the linear inhomogeneous equation

$$L\phi = S$$

for a set of operators $L \in L$ and fixed functions D and S. If the domain of interest in the operator space L is sufficiently small the solution of the problem may be given in terms of perturbation theory: the basic operator is chosen to be $L_0 \in L$, the basic solution ϕ_0 and the corresponding conjugate function ϕ_0^+ are found, and the conventional perturbation expansion series is constructed: Refs. 1, 2

$$J = J_0 - (\phi_0^+, V_0\phi_0) + (\phi_0^+, V_0 G_0 V_0 \phi_0) - \ldots$$

Here $V_0 = L - L_0$, $G_0 = L_0^{-1}$ is the Green's function for the unperturbed problem. The concept underlying the algorithm suggested is as follows. Unless the domain L is sufficiently small for perturbation theory to be used, a few "basis" operators $L_1, L_2, \ldots$ from L are to be chosen, and the problem is then solved for each of them (i.e., ϕ_i and ϕ_i^+ are determined, $i = 1, 2, \ldots$). Linear combinations of the basis functions

$$\phi = \sum_i C_i \phi_i, \qquad \phi^+ = \sum_i C_i^+ \phi_i^+$$

are used in the variational problem $\delta J = 0$, where[1]

$$J = (\phi^+, S)\,(D, \phi) / (\phi^+, L\phi)$$

The result is

$$J = \sum_{ij} C_i^+ C_j J_i J_j \Big/ \sum_{kl} C_k^+ C_l L_{kl}$$

with

$$J_i = (\phi_i^+, S) = (D, \phi_i), \quad L_{kl} = (\phi_k^+, L\phi_l) \tag{1}$$

The stationary condition of the form

$$\partial j / \partial C_n^+ = \partial J / \partial C_n = 0$$

results in the equations for the expansion coefficients C_i^+ and C_i:

$$\sum_{ij} C_i^+ C_j \left[J_n L_{ij} - J_j L_{in}\right] = 0,$$

$$\sum_{ij} C_i^+ C_j \left[J_n L_{ij} - J_i L_{nj}\right] = 0.$$

For the case of two basis operators, we have

$$C_1^+ / C_2^+ = -(J_1 L_{22} - J_2 L_{21}) / (J_1 L_{12} - J_2 L_{11})$$

$$C_1 / C_2 = -(J_1 L_{22} - J_2 L_{12}) / (J_1 L_{21} - J_2 L_{11}).$$

Introducing the notation $V_i = L - L_i$ and using the relation

$$(L_1)_{kl} = \begin{cases} J_k, & 1 = l \\ J_l, & 1 = k \end{cases}$$

the coefficient ratios may be written as

$$C_1^+/C_2^+ = -(J_1(V_2)_{22} - J_2(V_2)_{21})/(J_1(V_1)_{12} - J_2(V_1)_1)$$
$$C_1/C_2 = -(J_1(V_2)_{22} - J_2(V_2)_{12})/(J_1(V_1)_{21} - J_2(V_1)_1) \quad (2)$$

It is to be noted that the coefficients C^+ and C coincide when the operators V_1 and V_2 are symmetric ($(V_i)_{kl} = (V_i)_{lk}$).

Three important features of the proposed algorithm should be pointed out. First, in a sense, it solves the problem of interpolating the solution using two or more given unperturbed solutions in the domain L, whereas conventional perturbation theory is more an extrapolation procedure (i.e., attempts are made to extend the known solution for "the point" L_0 to a domain of L). Second, the solution obtained is not coincident with $J_1(J_2)$ at "points" $L = L_1(L_2)$ but in the neighborhoods of these operators, it also gives the same results as does perturbation theory near $L_1(L_2)$. Actually, for V_1 0 we obtain

$C_2^+ / C_1^+ \equiv \varepsilon^+ \ 0$, $C_2 / C_1 = \varepsilon \ 0$, and

$$J \approx \frac{J_1^2 + (\varepsilon^+ + \varepsilon)\, J_1\, J_2}{J_1 + (V_1)_{11} + (\varepsilon^+ + \varepsilon)\, J_2} \approx J_1 - (V_1)_{11}$$

Third, the calculations of the relevant matrix elements $(V_i)_{kl}$ are no more complicated than those of first-order perturbation theory, and are considerably simpler them the second-order matrix elements.

We now consider the application of this method to kinetic and wave models of radiative transfer. A kinetic model describes the radiation field by the differential photon flux density $\phi(\vec{r}, \vec{\Omega}, \lambda)$, which satisfies the integro-differential equation

$$L\phi = \Omega\nabla\phi + \Sigma_t\phi - \Sigma_s \int_{4\pi} d\Omega'\, g(\Omega\,\Omega')\ \phi(r,\Omega',\lambda) =$$
$$= S(r,\Omega,\lambda),$$

where $\Sigma_t(r, \lambda)$ is the total molecular absorption and scattering cross section per unit length, Σ_s is the scattering coefficient, g is the scattering phase function, $S(r, \Omega, \lambda)$ is the differential photon source density, Ω is the direction of photon travel, λ is the wavelength. The corresponding adjoint equation reads

$$L^+\phi^+ = -\Omega\nabla\phi^+ + \Sigma_t\phi^+ - \Sigma_S \int_{4\pi} d\Omega'\, g(\Omega'\Omega)\ \phi^+(r,\Omega',\lambda) =$$
$$= D(r,\Omega,\lambda).$$

As a numerical example, the simplest problem (call it problem A) of particle-transfer in a homogeneous medium for the case of no scattering will be considered. At the coordinate origin there is a point source emitting particles in the positive z-direction. A detector records the number of particles at the plane $z = t$. It is sufficient to use the reduced phase space $\{z\}$, $S(x) = \delta(z)$, $D = \delta(z - t)$. The operator L depends only on the interaction cross-section Σ:

$$L(\Sigma) = \frac{d}{dz} + \Sigma, \qquad L^+(\Sigma) = -\frac{d}{dz} + \Sigma\ .$$

The basis and conjugate functions are defined as

$$\phi(z) = e^{-\Sigma z}, \qquad \phi^+(z) = e^{-\Sigma(t-z)} \qquad 0 \le z \le t.$$

The solution set is to be constructed in the neighborhood of the basis operator $L_0 = L(\Sigma_0)$, i.e. $J = J(\Sigma)$ is to be found. Perturbation theory yields in this case

$$J(\Sigma) = \left[1 - \Delta\Sigma_0 t + \frac{1}{2}(\Delta\Sigma_0 t)^2 - \ldots\right] J_0, \quad \Delta\Sigma_0 = \Sigma - \Sigma_0.$$

The variational perturbation method requires that at least two basis operators $L_1 = L(\Sigma_1)$ and $L_2 = L(\Sigma_2)$ be chosen, so that $\Sigma_1 < \Sigma_0 < \Sigma_2$ and $(\Sigma_2 - \Sigma_0)\ t = (\Sigma_0 - \Sigma_1)\ t \equiv a$. Let $\Delta\Sigma_0 t = \xi$. Then the matrix elements can be expressed as

$$(V_1)_{11} = (\xi + a)J_1, \quad (V_2)_{22} = (\xi - a)J_2,$$
$$(V_1)_{21} = \frac{\xi + a}{2a}(J_1 - J_2),$$
$$(V_2)_{12} = \frac{(\xi - a)}{2a}(J_1 - J_2),\ J_1 = e^a J_0,\ J_2 = e^{-a} J_0,\ J_0 = e^{-\Sigma_0 t}.$$

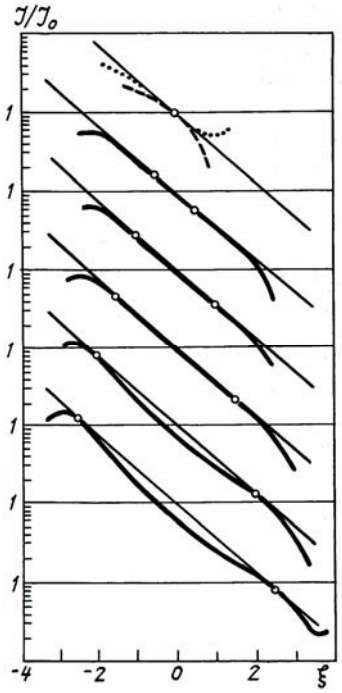

Fig. 1. Problem A: —— – exact solution; · · ·, ----- – first and second-order perturbation theory solutions, respectively; —— variational interpolation method for a = 0.5, 1, 1.5, 2, 2.5; ○ – basic points.

Hence

$$C_1^+ = C_1 = (\xi - a)\ [2a - 1 + e^{-2a}]$$
$$C_2^+ = C_2 = (\xi + a)\ [2a + 1 - e^{2a}].$$

The calculated results are shown in Fig. 1. The variational method is seen to have a wider range of validity than perturbation theory; e.g. , for $a = 1.5$ a change of an order of magnitude greater or smaller in the functional value is well reproduced, whereas perturbation theory yields reasonable agreement only for variations within 30–40% of the functional value. At larger values of a, the solution tends to give smaller values around $\xi = 0$.

In a wave model, the radiation is described by a function v that obeys the wave equation. The discussion be confined to the parabolic equation for the wave amplitude[3]

$$2ik\frac{\partial v}{\partial z}+\Delta_{\perp}v(z,\rho)+k^2(n(z,\rho)-n_c+i\alpha(z,\rho)/k)v=0 \qquad (3)$$

Here $\rho = (x^2 + y^2)^{1/2}$, $n(z, \rho)$ is the index of refraction, $\alpha(z, \rho)$ is the absorption coefficient, k is the wave number. Of practical interest is the situation where neither diffraction n or absorption processes can be neglected. For the case where $2i\beta \equiv k(n(z, \rho) - n_0 + i\alpha(z, \rho)/k)$ is independent of p, equation (3), subject to the boundary condition $v(0, \rho) = [2\pi\sigma^2]^{-1} \exp(-\rho^2/2\sigma^2)$, is solvable analytically, and we have

$$v(z,\rho) = [2\pi(\sigma^2+iz/k)]^{-1}\exp\left\{-\rho^2/2(\sigma^2+iz/k)\right\}\times \exp\left\{\int_0^z \beta(z')\,dz'\right\} \qquad (4)$$

Let us now find the radiation field amplitude in the plane $z = t$. Then $D(z) = \delta(z - t)\delta(\rho - \rho_0)$ and $J = v(t, \rho_0)$. The coefficient β is taken to be $\beta(\rho, z) = (5–4 \exp(-\rho^2/2\sigma^2)) k^{-1}$. The above problem will be called problem B. For $\sigma|\partial\beta/\partial\rho|z \ll 1$, perturbation theory appears to be valid. The solution of equation (3) for $\beta = \beta_1 = \text{const}$ is chosen as the basis solution. The conjugate function has the form

$$v^+(\vec{z},\vec{\rho})=[2\pi i(t-z)]^{-1}\exp\left\{-(\rho-\rho_0)^2/2i(t-z)\right\}\times \exp\left\{-\int_z^t \beta(z')\,dz'\right\} \qquad (5)$$

Perturbation theory fails for $\Delta\beta t \le 1$. We now apply the variational interpolation method to problem B. Choose two basis solutions correspond to $\beta_2 = 1/k$, $\beta_1 = 5/k$. In that case $\beta_2 \le \beta \le \beta_1$. To calculate the wave amplitude v for a given β we utilize relations (1) and (2) and basis solutions (4) and (5). A numerical integration over z is carried out to find the matrix elements $(V_i)_{kl}$. The calculated results are shown in Fig. 2. The finite difference method is used to obtain the exact solution of Eq. (3). Note that for the situations considered, the basis solutions differ by more than an order of magnitude. Perturbation theory fails to describe this variation of the function $v(z, \rho)$ as expected, whereas the variational interpolation method results in values that are practically the same as those provided by the exact solution.

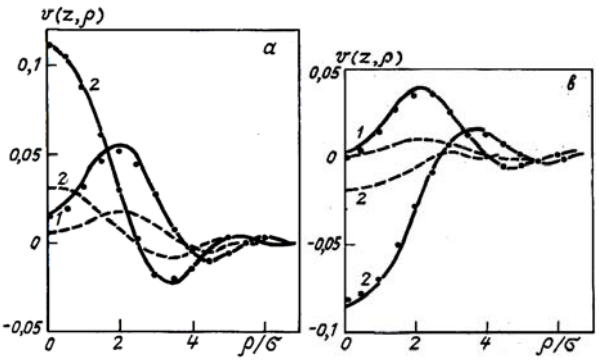

Fig. 2. Problem B: —— - exact solution; ----- – first-order perturbation theory solution; · · · · – variational interpolation method. 1 – Re V; 2 – Im V; a) $\Delta\beta t = 2.4$; b) $\Delta\beta t = 2.8$.

REFERENCES

1. G.I. Marchuk and B.B. Orlov in: Neutron Physics (Gasatomizdat, Moscow, 1961).
2. V.V. Uchaikin Izv. Vyssh. Uchebn. Zaved. Fizika, No. 10, 88 (1968).
3. S.M. Rytov, Ju.A. Kravtsov and V.I. Tatarskii *Introduction to Statistical Radiophysics*, Part 2 (Nauka, Moscow, 1978).