SYNTHESIS OF AN ALGORITHM FOR ESTIMATING THE WAVEFRONT STATE FOR AN ADAPTIVE OPTICAL SYSTEM

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The problem of compensating for wavefront distortions is considered. An algorithm for estimating of the state of the wavefront from a remote point source is synthesized for an adaptive optical system with separate correction for aberrations of different orders. The algorithm allows one to minimize the rms error of estimation. The Hartmann sensor is used as a measure of the correlation errors.

An algorithm for the functioning of an optimal observer of the state of the wavefront from a remote point source is obtained for a system with separate correction of the common wavefront insertion defocusing, and residual local distortions. The Hartmann sensor is used as a measure of the correction errors.

According to the current thinking, the spectrum of the phase fluctuations of the wavefront is bounded at the high spatial-frequency end. For this reason the instantaneous realization of the wavefront may be represented by the spatial-temporal correlation field. Based on the considered correlation scheme, we represent the function of the phase distortions as follows:

$$\Phi(\rho, t) = b_1(t)\, R_1(\rho) + b_2(t)\, R_2(\rho) + b_3(t)\, R_3(\rho) + \Phi_L(\rho, t) \qquad (1)$$

where ρ is the vector of the normalized coordinates, positioning the given point with respect to the receiving aperture center using either polar coordinates $\rho = \{\rho, \varphi\}$ or Cartesian coordinates $\rho = \{u, v\}$ with $u = x/R_0$; $v = y/R_0$; $\rho = (u^2 + v^2)^{1/2}$ is the metric distance along the appropriate axes; R_0 is the aperture radius;

$$R_1(\rho) = \frac{Z_1(\rho)}{} = \rho\cos\varphi = \frac{x}{R_0},$$

$$R_2(\rho) = \frac{Z_2(\rho)}{} = \rho\sin\varphi = \frac{y}{R_0}$$

are functions of the wavefront inclinations with respect to the x and y axes in the plane of the receiving aperture; $R_3(\rho) = \frac{Z_3(\rho)}{\sqrt{3}} = 2\rho^2 - 1$ is the defocusing function; $Z_1(\rho)$, $Z_2(\rho)$, $Z_3(\rho)$ — are Zernike polymonials; $b_1(t)$ — are the mode coefficients which are random functions of time; $\Phi_L(\rho, t)$ — are the residual phase distortions of the wavefront corresponding to aberrations of astigmatism and aberrations of higher orders.

As input actions on the correction system we shall take the inclinations of the wavefront, which are partial derivatives of the phase distortion function with respect to two mutually orthogonal directions.

$$\psi(\rho, t) = \frac{\partial\Phi(\rho, t)}{\partial u} = b_1(t) + 4ub_3(t) + \psi_L(\rho, t),$$

$$\theta(\rho, t) = \frac{\partial\Phi(\rho, t)}{\partial v} = b_2(t) + 4ub_3(t) + \theta_L(\rho, t), \qquad (2)$$

where $\psi_L(\rho, t) = \frac{\partial\Phi_L(\rho, t)}{\partial u}$, $\theta_L(\rho, t) = \frac{\partial\Phi_L(\rho, t)}{\partial v}$, are the residual local inclinations of the wavefront.

Let us represent the receiving aperture in the form of a square array consisting of N^2 elementary cells. Suppose that within each cell ($p \in \Omega_1$) the wavefront inclinations are characterized by some averaged values $\Phi_1(t)$, $\theta_1(t)$, $i = 1, \ldots, N^2$. Then taking into account Eq. (2), one may represent the wavefront inclination within the i-th cell as follows:

$$\psi_1(t) = b_1(t) + 4ub_3(t) + \psi_{L1}(t).$$

$$\theta_1(t) = b_2(t) + 4vb_3(t) + \theta_{L1}(t). \qquad (3)$$

We shall characteristize the set of wavefront inclinations with respect to the axes x and y by the column vectors $\psi = \{\psi_i\}$ and $\theta = \{\theta_i\}$, $i = 1, \ldots, N^2$, respectively.

Investigations of the correlation properties of the mode coefficients $b_l(t)$, $l = 1, 2, 3$ of the first and second order aberrations (inclinations and defocusing) have shown that they can be sufficiently well approximated by the exponential correlation function[1]. Then for the discrete time $t = kT$ (T is the period of measurement) one can describe mathematical models of the total wavefront inclination and the inclination due to defocusing by stochastic differential equations:

$$b_l(kt) = \beta_l b_l(kT-T) + \eta_l(kT-T), \quad l = 1,2,3 \qquad (4)$$

where $\beta_l = \exp(-T/\tau_0)$; τ_0 is the time constant of correlation for coefficient b_1; η_l – is a discrete white time-series with zero mean and variance $d_{\eta l} = M\{b_l^2(kt)\}(1 - \beta_l^2) = d_{bl}(1 - \beta_l^2)$ and $M\{\cdot\}$ is the mathematical expectation operator.

Based on Eq. (2) we assume that the spatial-temporal correlation functions of the residual local inclinations to the first approximation have the following form:

$$\Gamma_i(\rho,\tau) = d_{Li}\exp\left[-\left[\frac{\rho^2}{\rho_0^2} + \frac{\tau^2}{\tau_0^2}\right]^{1/2}\right] \qquad (5)$$

where d_{Li} is the variance of the local distortions; ρ_0 is the radius of the spatial correlation; τ_0 is the time constant of the correlation of the local distortions. In this case the mathematical model of the residual local inclinations is described in discrete time by the vector-matrix stochastic differential equation:

$$a_L(kT) = A_\alpha \alpha_L(kT-T) + \eta_{L\alpha}(kT). \qquad (6)$$

$$(\alpha = \psi, \theta)$$

where $\alpha_L = \{\alpha_{Li}\}$ is a column vector of the local wavefront inclinations, consisting of N^2 components $A_\alpha = \{\beta_{\alpha ik}\}$ is a matrix with the elements

$$\beta_{\alpha ik} = \exp\left[-\left[\frac{|\rho_i - \rho_k|^2}{\rho_0^2} + \frac{T^2}{\tau_0^2}\right]^{1/2}\right];$$

$i, k = 1...N^2$; $\eta_{\alpha L} = \{\eta_{Li}\}$ is a column vector of the discrete white time-series with the covariance matrix $D_{\eta\alpha L} = \text{diag}\{d_{\eta L\alpha}\}$, where $d_{\eta Li} = d_{Li}(1 - \beta_{\alpha li}^2)$.

As one can see, the state of the wavefront as a whole is determined by the vector $z^T = [b_1\psi_L^T b_2\theta_L^T b_3]$, which consists of $(3 + 2N^2)$ components.

Combining expressions (4) and (6), we obtain a generalized recursion equation for the wavefront state:

$$\mathbf{z}(kT) = A\mathbf{z}(kT-T) + \eta(kT-T), \qquad (7)$$

where A is the block matrix of the state transitions; $\eta(kT) = [\eta_1, \eta_{\psi L}^T, \eta_2, \eta_{\theta L}^T, \eta_3]^T$ is the column vector of the discrete white time-series with the covariance matrix D_η.

If the images of the pupils of the Hartmann diaphragm in the photodetector array interference of the fields which create these images), then the orthogonal components of the shifts of image's "center of gravity" on each photodetector are proportional to the wavefront inclinations at the output of the corrector within the area confined by the coordinates of the center of the corresponding pupil of the diaphragm. Therefore the column vector $\mathbf{u}_s(kT)$ of the output signals of the two-dimensional sensor may be represented as

$$\mathbf{u}_s(kT) = k_s[\gamma(kT) - \gamma_0(kT)] + \mathbf{u}_f(kT) \qquad (8)$$

where $\gamma(kT) = \begin{bmatrix}\psi(kT)\\ \theta(kT)\end{bmatrix}$ is the combined column vector of the inclinations of the wavefront incident upon the corrector; $\gamma_0(kT) = \begin{bmatrix}\psi_0(kT)\\ \theta_0(kT)\end{bmatrix}$ is the combined vector of the shifts in the field of the wavefront inclinations introduced by the corrector; k_S is the slope of the device characteristic with respect to the wavefront inclination; $\mathbf{u}_f(kT)$ is the column vector of the measurement errors which are mutually uncorrelated white time-series with variances $d_{fi}(kT)$.

One can easily convince oneself that the vector $\gamma(kT)$ is linearly connected with the wavefront state vector

$$\gamma(kT) = C\mathbf{z}(kT) \qquad (9)$$

For the models of the wavefront state (7), the output (9), and measurements (8), the algorithm of the functioning of the observer which is optimal according to the criterion of minimum variance of the estimation error is defined by the following recursion relations[3]:

$$\hat{\mathbf{z}}(kT|kT-T) = A\hat{\mathbf{z}}(kT-T);$$

$$D_z(kT|kT-T) = AD_z(kT-T)A^T + D_\eta;$$

$$D_z(kT) = D_z(kT|kT-T)\left[I - C^T\left(CD_z(kT|kT-T)C^T + \frac{D_f(kT)}{k_s^2}\right)^{-1}C\right]D_z(kT|kT-T); \qquad (10)$$

$$K(kT) = D_z(kT)C^T k_s D_f^{-1};$$

$$\hat{\mathbf{z}}(kT) = \hat{\mathbf{z}}(kT-T) + K(kT)[\mathbf{u}_s(kT) - k_s C\hat{\mathbf{z}}(kT|kT-T) + k_s\gamma_0(kT)];$$

where $\hat{z}(kT)$ is the estimate of the wavefront state; $\hat{z}(kT/kT - T)$ is the wavefront state extrapolated to the next current moment of measurement; $D_z(kT) = M\{[z(kT) - \hat{z}(kT)][z(kT) - \hat{z}(kT]^T\}$ is the matrix of second central moments (covariances) of the errors of the state estimation;

$$D_z(kT|kT-T) = M\{[\mathbf{z}(kT) - \hat{\mathbf{z}}(kT|kT-T)] \times [\mathbf{z}(kT) - \hat{\mathbf{z}}(kT|kT-T)]^T\}$$

is the covariance matrix of the state extrapolation errors; I is the identity matrix; $K(kT)$ is the gain coefficient matrix; $D_f(kT) = \text{diag}\{d_{fi}(kT)\}$, $i = 1, \ldots, 2N^2$.

In order for the given recursion algorithm to work, one has to have information about the initial state $Z(k_0T)$ and its covariance matrix $D_z(kT)$. In this case because of the asymptotic stability of an optimal observer[3], the requirements on the quality of information about the initial state are not very strict.

The main difficulty in the realization this algorithm is connected with necessity of knowing information about the vector $\gamma_0(kT)$ of shifts introduced by the corrector. If the operation of the corrector satisfies the condition

$$\gamma_0(kT) = C\hat{z}(kT/kT - T)\mathring{\gamma}_0(kT) \tag{11}$$

where $\mathring{\gamma}_0$ is the centralized random vector of errors introduced by the corrector and characterized by the covariance matrix $D_{\Delta\bar{\gamma}}(kT)$, one can transform relation (10) to the form:

$$\hat{z}(kT) = \hat{z}(kT)/kT - T) + K(kT)\,u_g(kT)$$

i.e. one can eliminate $\gamma_0(kT)$ from the state ˆ estimation algorithm.

The additional error due to the uncertainty of the state of the corrector should be taken into account by adding the matrix $k_g^2 D_{\Delta\bar{\gamma}}(kT)$ to the matrix $D_f(kT)$.

The method put forward here enables one to synthesize an algorithm for the optimal action of the observer when statistical data on the external influence are known. The algorithm gives the minimal rms error of estimation of the wavefront distortions.

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