ON THE RESOLUTION OF OPTICAL SYSTEMS

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A correct (Foucalt) definition of the resolution is given in this paper. This concept of the resolving power of an optical system is then used for optimal correction of the images of objects observed with an optical system through a turbulent atmosphere, i.e., under conditions of random refraction along the viewing path.

INTRODUCTION

Assessment of the quality of optical systems working in a turbulent atmosphere with random refraction is one of the most important problems in atmospheric optics[6]. The Foucalt resolution is the most widely used criterion for assessing the quality of imaging systems, including optical ones. It is defined[1–5] by the solution to the equation

$$G(\nu) = K(\nu), \qquad (1)$$

where G is the resultant modulation transfer function (MTF) of the imaging system, K is the threshold contrast detectable by an image analyzer, and ν is the spatial frequency.

It is pertinent to mention here some specific definitions of the resolution: objective resolution, which according to Ref. 5 corresponds to the case when

$$K = M_{thr}\delta;$$

limiting resolution[1], where

$$K = 0;$$

visual resolution[1], where

$$K = K_{vis}.$$

Here M_{thr} is the threshold value of the signal-to-noise ratio (SNR), δ is the relative value of the r. m. s. noise level at the output of the imaging system, and K_{vis} is the threshold contrast discernible by a visual analyzer (according to Selvin[4], $K_{vis} = 0.02$).

In the general case[11] one has

$$K = \sqrt{K_N^2 + K_{vis}^2},$$

where $K_N = M_{thr}\,\delta$ is the component of the threshold contrast due to noise at the imaging system output. If one takes into account the contrast factor of the imaging system, one obtains $K_N = \gamma M_{thr}\delta$, where γ is the contrast coefficient of the latter[11].

It is quite obvious that the definition of the resolution is incorrect, since Eq. (1) can have several solutions and hence the resolution thus defined may be ambiguous. Such a situation can actually occur in practice. An example of this is described in Ref. 2, and concerns a photographic system which uses an objective with a centrally obstructed aperture. In addition, the same situation occurs in aerial mapping cameras due to target displacement during an exposure[2,4]. In these situations, following Fivenskii[4], the lowest-frequency root of Eq. (1) is taken to be the resolution, with the rest of the frequency range being considered to give "false resolution"; that is,

$$R = \min\left\{\nu \ge 0 \mid G(\nu) = K(\nu)\right\}. \qquad (2)$$

However this definition of the resolution is also faulty. Thus, for example, it follows from Eq. (2) that if $G = K$ in the frequency range $[0, \bar{\nu}]$, where $\bar{\nu} > 0$, the resolution will be zero. But this is unacceptable, because the equality $G(\nu) = K(\nu)$ at some frequency ν means that this frequency is in fact still resolvable[2,16]. In that case, the resolution is no lower than $\bar{\nu}$, and hence differs from zero. Moreover, Eq. (1) may have no roots at all, and as a consequence the resolution simply not be defined, which contradicts the requirements for a figure of merit, since such a criterion must be always expressible in terms of a number or a function[9].

THE CORRECT FORMULATION OF THE FOUCALT RESOLUTION CRITERION

As shown in Refs. 2 and 16 the relationship

$$k_0G(\nu) \ge K(\nu)$$

is valid in general, for a resolved frequency ν, where k_0G is the contrast of a sinusoidal resolution target at the imaging system output, and k_0 is the inherent contrast of the target. The latter is included for the sake of generality.

Let us also introduce the continuously resolvable frequency along with simply resolvable one. This variable we define as the resolved frequency ν below which all frequencies, except for negative ones, are resolvable, i.e., if $k_0G(\nu) \ge K(\nu)$, then $k_0G(\nu') \ge K(\nu')$ for any $0 \le \nu' \le \nu$.

It is natural therefore to define the resolution as the maximum resolvable frequency, which is completely consistent with the meaning of resolution, and with the experimental technique described in Ref. 17. The foregoing can be formally written as follows[7]:

$$R = \sup\left\{\bar{\nu} \geq 0 \mid k_0 G(\nu) \geq K(\nu),\ 0 \leq \nu \leq \bar{\nu}\right\}. \quad (3)$$

But even this formulation lacks generality. Firstly, as follows from expression (3), the limiting resolution, corresponding to $K = 0$ (Ref. 1), will be infinite for any imaging system, although it is limited by the first root of the equation $G(\nu) = 0$, as can be shown using Eq. (2). Secondly, the set on the right-hand side of Eq. (3) may be empty because of low inherent contrast or high threshold contrast. As a result, the resolution may turn out to be indeterminate, since the concept of an "exact upper bound" is inapplicable to an empty set[8].

In our of view, the following expression generalizes the definition (3), and at the same time is free of these drawbacks:

$$R = \mu\left\{\bar{\nu} \geq 0 \mid k_0 G(\nu) \geq K(\nu)],\ G(\nu) > 0,\ 0 \leq \nu \leq \bar{\nu}\right\}, \quad (4)$$

where μ is the Lebesgue measure.

This is a correct definition of resolution, in that the resolution calculated using Eq. (4) exists and is unique for any imaging system. This follows from the fact that the set A on the right hand side of Eq. (4) is convex, according to the definition of the continuously resolvable frequency, and is thus only one of the two sets[18]

$$A = [0, a] \text{ or } A = [0, a) \quad a \geq 0, \quad (5)$$

which are Lebesgue-measurable[8]. If the set $A \neq 0$ (is not empty), then by virtue of (5) one can replace the symbol μ in equation (4) with "sup". Moreover, if $k_0 G + K > 0$ everywhere on the positive spatial frequency axis, then Eqs. (3) and (4) become equivalent.

It should be noted here that since the imaging (MTF and threshold contrast) are in general anisotropic, Eq. (4) should be considered to be the system resolution in the direction θ (directional resolution),

$$R_\theta = \mu\left\{\bar{\nu} \geq 0 \mid k_0 G_\theta(\nu) \geq k_\theta(\nu),\ G_\theta(\nu) > 0, 0 \leq \nu \leq \bar{\nu}\right\}, \quad (6)$$

while the total resolution, according to Ref. 10, can be written as

$$\bar{R} = \mu\left\{\bar{\nu} \geq 0 \mid k_0 G_\theta(\nu) \geq K_\theta(\nu),\ G_\theta(\nu) > 0, 0 \leq \nu \leq \bar{\nu},\ 0 \leq \theta \leq 2\pi\right\}, \quad (7)$$

where $G_\theta(\nu) = T(\nu\cos\theta,\ \nu\sin\theta)$; $T(\nu_x,\ \nu_y)$ is the resultant two-dimensional MTF of the imaging system, and θ is the orientation angle of the sinusoidal resolution target[1].

If the imaging capabilities of a system sire isotropic, Eqs. (6) and (7) are then identical to one another and to Eq. (4).

SOLUTION OF OPTIMIZATION PROBLEMS

Any imaging system will always have some probability that its full resolution will actually be realized[2]. The value of this probability, which is a function of threshold contrast[3], will simply be called the resolution probability RP. Of two different imaging systems, the one with the higher value of RP must naturally be considered the better. It is therefore reasonable to solve the problem of optical image correction by maximizing the RP for any given resolution value. In particular, because of the monotonic behavior of RP as a function of threshold SNR[12], one can reformulate the problem of optimal image correction as maximizing the threshold SNR for a given value of imaging system resolution[12].

In our view, this problem is of particular interest, and we shall therefore analyze it for fairly general imaging systems, including optical systems operating in a turbulent atmosphere, i.e., under conditions of random refraction.

ONE-DIMENSIONAL CASE

Let the resolution of a system be

$$R_0 = \mu\left\{\bar{\nu} \geq 0 \mid k_0 G(\nu) \geq M_{thr}\delta,\ G(\nu) > 0, 0 \leq \nu \leq \bar{\nu}\right\},$$

In the most practical case, with $M_{thr}\delta > 0$, one has

$$R_0 = \mu\left\{\bar{\nu} \geq 0 \mid k_0 G(\nu) \geq M_{thr}\delta,\ 0 \leq \nu \leq \bar{\nu}\right\}. \quad (8)$$

For a one-dimensional model of imaging system operation, with the noise being a spatially stationary zero-mean random process, one has

$$G = |\tilde{h}\,\tilde{\varphi}|;\ \delta = \sqrt{\int_{-\infty}^{\infty} S(\nu)\,|\tilde{\varphi}(\nu)|^2 d\nu}\ / B, \quad (9)$$

where $\tilde{h}$ and $\tilde{\varphi}$ are the optical transfer functions (OTF) of the distorting and correcting filters, respectively, S is the spectral density of the noise (a nonnegative, even function[15]), and B is the background in the resolution target image.

For opto-atmospheric systems with long enough temporal averaging, one can write[6]

$$\tilde{h} = \tilde{h}_a\,\tilde{h}_o,$$

where $\tilde{h}_a$ is the OTF of the turbulent atmosphere whose influence on the beam bearing the image of the observed object may be modeled with a linear (isoplanatic) low-frequency filter; $\tilde{h}_0$ is the OTF of the receiving aperture of the system, S is the spectral density of fluctuation in the refractive index of the air, which quantitatively models the random refraction.

Based on Eqs. (5), (8) and (9), this optimization problem can be reduced to a search for the maximum of the functional

$$I(\Phi) = \inf_{0\le\nu<R_0} \frac{H(\nu)\Phi(\nu)}{\int_{-\infty}^{\infty} S(\nu)\Phi(\nu)d\nu} \quad (10)$$

assuming that

1) $R_0 > 0$;
2) H, Φ, S are nonnegative even functions everywhere on the frequency axis;
3) $H, \Phi > 0$ on $[0, R_0)$;
4) $0 < \int_{-\infty}^{\infty} S(\nu)\Phi(\nu)d\nu < \infty.$ (11)

Here $h = \left|\tilde{h}\right|^2$; $\Phi = \left|\tilde{\varphi}\right|^2$. The quantity $k_0 B\sqrt{I}$ in this case has the physical meaning of the minimum SNR over the whole range of continuously resolvable frequencies, and is thus the maximum possible threshold SNR at which the resolution R_0 is obtained for a system with fixed characteristics H, Φ.

Making use of a series of simple inequalities

$$\int_{-\infty}^{\infty} S(\nu)\Phi(\nu)d\nu \ge \int_{0}^{R_0} S(\nu)\Phi(\nu)d\nu \ge \inf_{0\le\nu<R_0} H(\nu)\Phi(\nu) \times \int_0^{R_0} \frac{S(\nu)}{H(\nu)}d\nu$$

one finds that the maximum of the functional (10), taking into account the constraints (11), is equal to

$$I_{\max} = \left[2\int_0^{R_0} \frac{S(\nu)}{H(\nu)}\, d\nu \right]^{-1},$$

and is reached when

$$\Phi_{\text{opt}}(\nu) = \begin{cases} 1/H(\nu) & |\nu| < R_0 \\ 0 & |\nu| \ge R_0 \end{cases} \quad (12)$$

Thus, the maximum threshold SNR, at which some given value of the objective resolution can be achieved is $k_0 B\sqrt{I_{\max}}$. One can see from (12) that the OTF of the correction filter is the reciprocal of the OTF of the distorting filter in the frequency range $(-R_0, R_0)$, and is zero outside it.

TWO-DIMENSIONAL CASE

In this case the optimization problem is reduced to the .problem of finding a maximum of the functional

$$I(\Phi) = \inf_{\substack{0\le\nu<\bar{R}_0 \\ 0\le\theta\le2\pi}} \frac{H(\nu\cos\theta,\nu\sin\theta)\,\Phi(\nu\cos\theta,\nu\sin\theta)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\nu_x,\nu_y)\,\Phi(\nu_x,\nu_y)\,d\nu_x d\nu_y} \quad (13)$$

with

1) $\bar{R}_0 > 0$;

2) H, Φ, S nonnegative even functions everywhere on the frequency plane;

3) $H(\nu_x, \nu_y)\Phi(\nu_x, \nu_y) > 0$, $\nu_x^2 + \nu_y^2 < \bar{R}_0^2$; (14)

4) $0 < \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\nu_x, \nu_y)\Phi(\nu_x, \nu_y)d\nu_x d\nu_y < \infty.$

The quantities $\bar{R}_0$, H, Φ, S, $k_0 B\sqrt{J}$ have the same meanings as in the one-dimensional case.

Using expression (14) one obtains

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\nu_x,\nu_y)\,\Phi(\nu_x,\nu_y)\,d\nu_x d\nu_y =$$

$$= \int_0^{\infty}\int_0^{2\pi} S(\nu\cos\theta,\nu\sin\theta)\,\Phi(\nu\cos\theta,\nu\sin\theta)\nu\,d\nu\,d\theta \ge$$

$$\ge \int_0^{\bar{R}_0}\int_0^{2\pi} S(\nu\cos\theta,\nu\sin\theta)\,\Phi(\nu\cos\theta,\nu\sin\theta)\nu\,d\nu\,d\theta \ge$$

$$\ge \inf_{\substack{0\le\nu<\bar{R}_0 \\ 0\le\theta\le2\pi}} H(\nu\cos\theta,\nu\sin\theta)\,\Phi(\nu\cos\theta,\nu\sin\theta) \times$$

$$\times \int_0^{\bar{R}_0}\int_0^{2\pi} \frac{S(\nu\cos\theta,\ \nu\sin\theta)}{H(\nu\cos\theta,\ \nu\sin\theta)}\,\nu d\nu\,d\theta =$$

$$= \inf_{\substack{0\le\nu<R_0 \\ 0\le\theta\le2\pi}} H(\nu\cos\theta,\nu\sin\theta)\,\Phi(\nu\cos\theta,\nu\sin\theta) \times$$

$$\times \iint_{\nu_x^2+\nu_y^2<\bar{R}_0^2} \frac{S(\nu_x,\nu_y)}{H(\nu_x,\nu_y)}\,d\nu_x d\nu_y$$

Then it follows that the maximum of the functional (13) under the conditions (14) is equal to

$$I_{\max} = \left[\iint_{\nu_x^2+\nu_y^2<\bar{R}_0^2} \frac{S(\nu_x,\nu_y)}{H(\nu_x,\nu_y)}\,d\nu_x d\nu_y \right]^{-1}$$

and is reached when

$$\Phi_{\text{opt}}(\nu_x,\nu_y) = \begin{cases} 1/H(\nu_x,\nu_y) & \nu_x^2+\nu_y^2 < \bar{R}_0^2 \\ 0 & \nu_x^2+\nu_y^2 \ge \bar{R}_0^2 \end{cases} \quad (15)$$

Thus the maximum threshold SNR at which the value $\bar{R}_0$ can be attained is $k_0 B\sqrt{J_{\max}}$. One can see from (15) that the OTF of the optimal correction filter is the reciprocal of that of the distorting filter within a region of radius $\bar{R}_0$ whose center coincides with the coordinate origin, and is zero outside this circle. An example of a nonnegative band-limited, finite function, whose Fourier transform has no nulls. It is sometimes useful to image different parts of an object at different resolutions[13]. To retain the possibility of optimal filtering, the OTF of the distorting filter must then never vanish in the corresponding frequency range (see Eqs. (12) and (15)). However, it is quite obvious that this condition can be violated if the objective resolution of an arbitrary distorting filter (receiving aperture of an optical system) varies over a wide enough range. Since the scattering functions of distorting filters of many types of imaging systems, for example those that use penetrating radiation[14], are nonnegative, finite and band-limited, it is practical to search for a function that has the aforementioned properties and whose Fourier transform is nowhere zero.

An example of such a function is

$$f(x) = \begin{cases} 2x/a^2, & 0 \le x \le a \\ 0, & \text{elsewhere,} \end{cases}$$

where a is a parameter.

Another example is the function $\varphi(x) = f(x) \cdot (f(-x)$, where the asterisk denotes convolution. In contrast to the previous function, this one is even. The Fourier transforms of these functions are

$$\tilde{f}(z) = 2\,\frac{(1+iz)\exp(-iz)-1}{z^2}$$

and

$$\tilde{\varphi}(z) = 4/z^4[z^2 + 2\,(1-\cos z) - 2z\sin z],$$

where $z = 2\pi\nu a$.

In the two-dimensional case one can use the product of these one-dimensional functions. The foregoing functions can be used for apodized (partially obstructed) receiving apertures of opto-atmospheric systems. The apodization, in turn, can provide for optimal correction of images of objects observed varying resolution.

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