AN ANALYTICAL EXPRESSION FOR THE COEFFICIENT OF OPTICAL ATTENUATION BY A PQLYDISPERSE PLATE-CRYSTAL SYSTEM

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An approximate analytic expression for the coefficient of optical attenuation by a polydisperse plate-crystal system is derived within the framework of physical optics. The proposed model implies both neutral behavior of the attenuation coefficient in the visible and its pronounced wavelength dependence in the IR region. A comparison with the integral representation shows that the attenuation coefficient calculated by our algebraic formula is correct to within 3%.

Numerical simulation of optical propagation through various polydisperse media requires cumbersome calculations of attenuation coefficients. The attenuation coefficient is defined by an integral whose integrand contains a factor describing the attenuation cross-section due to a given particle shape. The cross-section, in turn, is found by solution of the problem of wave scattering by a single particle. However, this kind of solution cannot be obtained for all possible particle shapes, while only a small number of those available are expressed by simple relations. Therefore, for the majority of polydisperse models the attenuation coefficient can be found only numerically. Nevertheless, there is a polydisperse model approaching reality that does enable one to carry out integration in analytical form. The present paper discusses this very model.

Consider a collection of circular plates normally oriented to the incident radiation as a model. Each plate is characterized by two linear dimensions, i.e., its radius a and thickness L. As a result, the coefficient of attenuation of optical radiation by this polydisperse medium can be written as

$$\alpha = \iint\limits_{\alpha L} \sigma(a, L, \lambda) N(a, L) da dL. \tag{1}$$

Here $N(a, L)$ is the two-dimensional particle size distribution function; $\sigma(a, L, \lambda)$ is the attenuation cross-section due to a circular plate at the optical wavelength λ. Equation (1) can be simplified by using a priori information on the interrelation between a and L for a single crystal. It has been established experimentally that there exists the following functional relationship between the linear dimensions of the plate crystal[1]:

$$L = f(a) = B(2a)^{\beta}, \tag{2}$$

where $B = 2.020$ and $\beta = 0.449$ are constants, and the values a and L are expressed in μm. Upon substitution of Eq. (2) into Eq. (1) we obtain

$$\alpha = \int\limits_0^{\infty} \sigma(a, L, \lambda)\big|_{L=f(a)} N(a) da \tag{3}$$

The attenuation cross-section σ is generally dependent on the polarization state of the incident radiation[2]. However, for normal incidence on the plate, σ has a simpler form[3]:

$$\sigma=(a, L, \lambda)=2\pi a^2(1-\mathrm{Re}(S)), \tag{4}$$

with

$$S = t \sum_{j=1}^{\infty} \exp(i\delta_j)\, r^{j-1} \tag{5}$$

$$t=4\tilde{n} / (\tilde{n}+1)^2; \qquad r=(n-1)^2/(\tilde{n}+1)^2);$$

$$\delta_j = kL\,[(2j-1)\,\tilde{n}-1],$$

where $k = 2\pi/\lambda$ is the wave number and $\tilde{n} = n + i\text{æ}$ is the complex refractive index. By virtue of Eq. (4), relation (3) can be recast as the sum of two terms, and we have

$$\alpha = \alpha_1 - \alpha_2, \tag{6}$$

where

$$\alpha_1 = \int\limits_0^{\infty} 2\pi a^2 N(a) da \tag{7}$$

and

$$\alpha_2 = \mathrm{Re} \int\limits_0^{\infty} 2\pi a^2 N(a)\; S\big|_{L=f(a)} da. \tag{8}$$

The crystal size distribution function is, as a rule, unimodal and is adequately approximated by the γ-distribution[4]:

$$N(a)=N \frac{\mu^{\mu+1}}{\Gamma(\mu+1)} \frac{1}{a_m} \left(\frac{a}{a_m}\right)^{\mu} e^{-\mu(a/a_m)}. \tag{9}$$

Here N is the particle concentration per unit volume, and a_m and μ are preset distribution parameters.

Using Eq. (9), integral (7) becomes

$$\alpha_1 = 2\pi N \frac{\mu+2}{\mu} \frac{\mu+1}{\mu} a_m^2 \equiv D \qquad (10)$$

For α_2 to be expressed analytically it is first recast in the form

$$\alpha_2 = \mathrm{Re}\int_0^\infty S\ 2\pi\{a^2 N(a)\}_{a=F(L)} F'(L)dL. \qquad (11)$$

Here $a = F(L) = 0.5;\ (L/B)^{1/\beta}$ is the inverse function of $L = f(a)$; $F'(L) = dF/dL = \frac{1}{2}\beta B\ (L/B)^{1/\beta-1}$, and S depends solely on the plate thickness L. We introduce the notation $2\pi\{a^2N(a)\}_{a=F(L)}\ F'(L) = \tilde{N}(L)$. Then,

$$\alpha_2 = \mathrm{Re}\int_0^\infty S\ \tilde{N}(L)dL. \qquad (12)$$

The function $\tilde{N}(L)$ differs from the γ-distribution, even though it also has one maximum. To carry out the Integration in Eq. (10) in analytic form, $\tilde{N}(L)$ is replaced by the function $N(L)$. This new function $N(L)$ should be the best approximation of $\tilde{N}(L)$ over the entire integration interval. In other words, the value

$$\rho = \frac{\sqrt{\sum_{1=1}^{n} (\tilde{N}(L_1) - N(L_1))^2}}{n} \qquad (13)$$

should be minimal. In addition, the following normalization condition for $N(L)$ must be satisfied:

$$\int_0^\infty \tilde{N}(L)dL = \int_0^\infty \tilde{N}(L)dL = \int_0^\infty 2\pi a^2 N(a)da \equiv D. \qquad (14)$$

Finally, the form of $N(L)$ should permit the analytic integration of Eq. (12). Function $N(L)$ is sought in the form:

$$N(L) = D\frac{1}{\Gamma(x_1+1)}\frac{1}{x_2}\left[\frac{L}{x_2}\right]^{x_1}\exp\left[-\frac{L}{x_2}\right], \qquad (15)$$

where x_1 and x_2 are constants determined by minimization of ρ. Note that x_1 is a dimensionless quantity, and x_2 is in microns.

Integration of Eq. (12), using Eq. (4), gives

$$\alpha_2 \approx D\cdot \mathrm{Re}(t\cdot P), \qquad (16)$$

where

$$P = \sum_{j=1}^{\infty} \frac{r^{j-1}}{(1+(2j-1)\kappa\cdot kx_2 - i[(2j-1)n-1]k\cdot x_2)^{x_1+1}}. \qquad (17)$$

Substituting Eqs. (16) and (10) into Eq. (6), we finally obtain for the attenuation coefficient

$$a \approx D(1-\mathrm{Re}(t\times P)). \qquad (18)$$

The parameters x_1 and x_2 determined by minimization of ρ by the pattern search method[5] are listed in Table 1. These values are in one-to-one correspondence with the assigned parameters μ and a_m. Substitution of different pairs of values of x_1 and x_2 from Table 1 into Eq. (17) with subsequent analysis readily shows that all the terms for P in Eq. (17) vanish in the visible part of the optical region. Therefore, in this frequency interval the attenuation coefficient $\alpha = D$ exhibits neutral behavior with respect to λ. It follows from Eq. (17) that with the wavelength increase the value of P grows ever so slightly, remaining fairly small. However, it cannot be regarded as approaching zero at IR frequencies and is there determined by the first term only, the other terms being negligible. This means that in this frequency interval reflections within a crystal can be ignored. As a result, the attenuation coefficient in the IR range is related to the parameters of the proposed polydisperse model through the following simple expression

TABLE 1.

Constants x_1 (upper row) and x_2 (lower row) determined from Eq. (15) by minimization of expression (13) by the pattern search method for $\tilde{n} = 1.31 + i\times 10^{-4}$.

a_m, μm	μ = 1	2	3	4	5	6
100	17.17 2.20	22.12 1.41	27.08 1.05	32.04 0.84	37.02 0.70	41.98 0.60
200	17.17 3.01	22.12 1.92	27.08 1.43	32.05 1.15	37.02 0.96	41.99 0.82
300	17.17 3.61	22.13 2.30	27.08 1.72	32.05 1.38	37.02 1.15	41.99 0.99
400	17.17 4.10	22.13 2.62	27.08 1.95	32.05 1.57	37.02 1.31	41.99 1.13
500	17.17 4.54	22.13 2.89	27.08 2.16	32.05 1.73	37.02 1.45	41.99 1.25
600	17.17 4.92	22.13 3.14	27.08 2.35	32.05 1.88	37.02 1.57	41.98 1.35
700	17.17 5.28	22.13 3.37	27.08 2.51	32.05 2.01	37.02 1.68	41.98 1.44

$$\alpha \approx D\left\{1-\mathrm{Re}\left[\frac{t}{(1+\kappa\cdot kx_2 - i(n-1)kx_2)^{x_1+1}}\right]\right\} \equiv \tilde{\alpha}. \qquad (19)$$

It can be readily seen that Eq. (19) also implies the limiting case, where $n = 1$, $æ = 0$ and $\alpha = \tilde{\alpha} \equiv 0$.

TABLE 2.

Comparative analysis of the attenuation coefficients obtained by exact calculations and our analytic approximation for $\tilde{n} = 1.31 + i \times 10^{-4}$.

$N=1$, $\mu=4$, $a_m=400$			
λ	α	$\tilde{\alpha}$	$(\alpha-\tilde{\alpha})/\alpha \times 100\%$
9.0	$1.901 \cdot 10^6$	$1.944 \cdot 10^6$	-2.24
9.1	1.937	1.980	-2.20
9.2	1.974	2.017	-2.12
9.3	2.013	2.054	-2.00
9.4	2.052	2.090	-1.86
9.5	2.091	2.126	-1.70
9.6	2.129	2.161	-1.53
9.7	2.165	2.194	-1.35
9.8	2.200	2.226	-1.17
9.9	2.233	2.255	-0.99
10.0	2.263	2.281	-0.82
10.1	2.290	2.305	-0.64
10.2	2.314	2.325	-0.47
10.3	2.335	2.342	-0.31
10.4	2.353	2.356	-0.15
10.5	2.367	2.367	0.00
10.6	2.377	2.374	0.14
10.7	2.383	2.377	0.27
10.8	2.386	2.377	0.39
10.9	2.385	2.373	0.50
11.0	2.380	2.365	0.60
11.1	2.371	2.354	0.68
11.2	2.358	2.340	0.76
11.3	2.342	2.323	0.82
11.4	2.323	2.303	0.87
11.5	2.300	2.279	0.92

The attenuation coefficients α and $\tilde{\alpha}$ calculated by Eqs. (3) and (19), respectively, are summarized in Table 2 for different IR wavelengths. The integral in coefficient calculated by Eq. (19) does not exceed 2.5%. Note that numerical integration of rapidly oscillating functions over wide limits is a rather complicated computational problem. Therefore the equation for the attenuation coefficient obtained in this paper (Eq. (19)) is simple not only from the point of view of analysis but it also facilitates computer calculation. It should be pointed out in conclusion that the attenuation cross-section due to plate is only weakly dependent on the incident polarization state[3]. In addition, the value of the attenuation cross-section changes but Eq. (3) was computed numerically. It follows from an analysis of Table 2 that the error in the attenuation slightly as the angle of incidence varies from 0° (normal incidence) up to 7–10°. Hence, Eq. (19) allows for generalization to the real case of the preferred orientation of crystal plates in a polydisperse medium where every plate oscillates near some equilibrium position in the air flow.

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