ANALYSIS OF THE PERFORMANCE OF ADAPTIVE OPTICAL SYSTEMS WITH NOISY CONTROL CHANNELS

D.A. Besuglov

Rostov Higher Military Command-Engineering School for the Rocket Forces
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A method for evaluating the effect of noise in the control channels on the Strehl factor is constructed based on an expansion of the quality functional of an adaptive optical system in the Taylor series around the extremum point. The potential statistical characteristics of the Strehl factor are calculated for the Gaussian and Poisson noise.

The real experimentally measured characteristics of controllable flexible mirrors are usually taken into account when analyzing the performance of adaptive optical systems (AOSs) along atmospheric paths. A method for doing this Is proposed In Ref. 1. A method for evaluating the quality of the approximation of the wavefront by the correcting elements, combining different bases of functions and using an arbitrary number of degrees of freedom, is proposed in Ref. 2. However the fact that the response functions of different correcting elements are not ideal is not the only source of errors inherent in AOSs. It is obvious that in the case of "frozen" turbulence the value of the quality functional of an AOS in neighborhood of the extremum point when noise is present in the control channels will differ from the maximum possible value. Expressions enabling estimation of the "signal-to-noise" ratio in the control channels of an AOS in the presence of modulation signals were derived in Ref. 3. This approach, however, does not permit evaluating the potentially possible characteristics of AOSs when there is no modulation.

This paper is devoted to evaluating the effect of noise in the control channels on the performance of adaptive optical systems for aperture sounding in the absence of modulation.

We shall study the problem as follows. An aperture-sounding AOS with multichannel phase modulation focuses coherent radiation which has passed through a turbulent layer of the atmosphere onto a point photodetector. Such a system essentially maximizes the Strehl factor. In spite of the fact that the Strehl factor is effective only for systems which are close to ideal its use is justified in many cases which are important in practice. For example, it is shown in Ref. 4 that maximizing the Strehl factor is equivalent to maximizing the resolution of an optical system. The complex amplitude of the field on a point photodetector of the AOS can be represented in the form

$$\vec{E} = A\sum_{1=1}^{N}\exp(i\Psi_1), \tag{1}$$

where A is the amplitude of the light oscillations generated on the photodetector by each of N elements of the corrector; Ψ_i is the corresponding phase; and, N is the number of control channels. The phase Ψ_i has the form

$$\Psi_1 = \alpha_1 + u_1, \tag{2}$$

where α_i is the perturbation of the phase profile in the i-th channel and u_i is the control which, is introduced for the purpose of correction.

The control signals u_i in the aperture-sounding AOS with multichannel phase modulation are usually segregated with the help of synchronous detectors. In the process, statistically independent noise, caused by different factors, is present in the control channels. We shall study the effect of the noise on the performance of the AOS with multichannel phase modulation assuming that the interaction of the control signal and the noise is additive.

To this end we expand the quality functional J of the AOS in the Taylor series around the extremum point:

$$J = J^0 + \vec{n}^T \text{grad } J(\vec{\Psi}^*) + \frac{1}{2}\vec{n}^T\vec{G}(\vec{\Psi}^*)\vec{n}, \tag{3}$$

where J^o is the value of the quality functional at the extremal point; $\vec{n} = \{n_1, n_2, \ldots, n_N\}$ is the vector of noise in the control channels with statistically independent components, $n_i \ll 1$ and $i = 1, \ldots, N$; $\vec{\Psi}^*$ are the coordinates of the extremum point J^0; and, $\vec{G}(\vec{\Psi}^*)^{2\to} = \dfrac{\partial^* J(\Psi)}{\partial\Psi_i\partial\Psi_j}$ is the Hess matrix of the quality functional J^0 at the extremum point.

Since at the extremum point $\text{grad}J(\vec{\Psi}^*) = 0$ and the noise processes in the control channel are statistically independent the mathematical expectation of the negative increment to the quality functional owing to noise in the control channels $\langle J^0 - J\rangle = \langle\Delta J\rangle$ can be written in the form:

$$\langle \Delta J\rangle = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} g^{*}_{ij}\langle n_i n_j\rangle, \tag{4}$$

where g^{*}_{ij} are the elements of the matrix $\vec{G}(\vec{\Psi}^{*})$ and $\langle n_i n_j\rangle$ are the moments of a random quantity $\vec{n}$.

The variance of the negative increment to the quality functional of the AOS can be found in the form

$$\sigma^2_{\Delta} = \langle \Delta J^2\rangle - \langle \Delta J\rangle^2. \tag{5}$$

Substituting Eq. (4) the expression (5) can be written as

$$\sigma^2_{\Delta} = \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N} g^{*}_{ij}\, g_{lk}\langle n_i n_j n_k n_l\rangle - -\frac{1}{4}\left[\sum_{i=1}^{N}\sum_{j=1}^{N} g^{*}_{ij}\langle n_i n_j\rangle\right]^2. \tag{6}$$

We shall calculate the value of $\langle \Delta J\rangle$ and σ^2_{Δ} for the case of the Gaussian noise with zero mathematical expectation. Following the results of Ref. 3 and using the expression (1) the intensity on a point photodetector in the absence of modulation can be written in the form:

$$J = A^2\left[N + \sum_{\substack{i=1\\ i\neq j}}^{N}\sum_{j=1}^{N}\cos\left(\Psi_i - \Psi_j\right)\right]. \tag{7}$$

The elements of the Hess matrix for such an AOS can be written in the form

$$g_{ij} = \begin{cases} -2A^2\sum\limits_{p=1}^{N}\cos\left(\Psi_i - \Psi_p\right) & \text{at } i=j;\\ 2A^2\cos\left(\Psi_i - \Psi_j\right) & \text{at } i\neq j, \end{cases} \tag{8}$$

and at the extremum point, when all phases are equal $\alpha_i = u_i$,

$$g^{*}_{ij} = \begin{cases} -2A^2 N & \text{at } i=j;\\ 2A^2 & \text{at } i\neq j. \end{cases} \tag{9}$$

Then using Eqs. (4) and (9) the value of $\langle \Delta J\rangle$ can be written as

$$\langle \Delta J\rangle = -\frac{1}{2}\sigma^2\sum_{i=1}^{N} g^{*}_{ij}, \tag{10}$$

where σ^2 is the variance of the Gaussian noise.

To find σ^2_{Δ} we shall* study five possible cases:

$$\begin{aligned} &\langle n_i n_i n_j n_j\rangle' = \mu_2^2;\\ &\langle n_i n_j n_j n_j\rangle = \mu_{13};\\ &\langle n_i n_j n_l n_l\rangle = \mu_{112},\\ &\langle n_i n_i n_i n_i\rangle = \mu_4;\\ &\langle n_i n_j n_l n_k\rangle = \mu_{1111}; \end{aligned} \tag{11}$$

where μ are the moments of the random quality $\vec{n}$.

For the Gaussian distribution $\mu_1 = 0$, $\mu_2 = \sigma^2$, and $\mu_4 = 3\sigma^4$. In this case σ^2_{Δ} will be given by the expression

$$\sigma^2_{\Delta} = \frac{1}{4}\left\{\mu_4\sum_{j=1}^{N} g^{*2}_{jj} + \sigma^4\sum_{i=1}^{N}\sum_{j=1}^{N}\left[g^{*}_{ji}g^{*}_{ji\,i\neq j} + + g^{*}_{ji}g^{*}_{ij}\right]\right\}. \tag{12}$$

Then the relations for the mathematical expectation and variance of the Strehl factor will be given by

$$\Gamma_{m_{St}} = 1 - \sigma^2; \tag{13}$$

$$\Gamma_{\sigma^2_{St}} = \frac{\sigma^4(N+1)}{N^2} \simeq \frac{\sigma^4}{N}. \tag{14}$$

The expressions for $\pi_{m_{St}}$ and $\pi_{\sigma^2_{St}}$ corresponding to the Poisson noise can also be found. To this end we shall examine expressions analogous to the expressions (11). Since for the Poisson quantities $\mu_1 = \lambda$; $\mu_2 = \lambda$; $\mu_3 = \lambda$; and, $\mu_4 = 3\lambda^2 + \lambda$, we have

$$\begin{aligned} &\langle n_i n_i n_j n_j\rangle = \mu_2^4 = \lambda^4 + 2\lambda^3 + \lambda^2;\\ &\langle n_i n_j n_j n_j\rangle = \mu_{31} = \lambda^4 + 3\lambda^3 + \lambda^2;\\ &\langle n_i n_j n_l n_l\rangle = \mu_{211} = \lambda^4 + \lambda^3;\\ &\langle n_i n_i n_i n_i\rangle = \mu_4 = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda;\\ &\langle n_i n_j n_l n_k\rangle = \mu_{1111} = \lambda^4;\\ &\langle n_i n_i\rangle = \mu_2 = \lambda_2 + \lambda;\\ &\langle n_i n_j\rangle = \mu_1^2 = \lambda^2. \end{aligned}$$

Then the corresponding expressions for $\pi_{m_{St}}$ and $\pi_{\sigma^2_{St}}$, neglecting second and higher order infinitesimals can be written in the form:

$$\pi_{m_{\mathrm{St}}} = 1 - \lambda \tag{15}$$

$$\pi_{\sigma^2_{\mathrm{St}}} = \frac{\lambda}{N} \tag{16}$$

Analysis of the expressions (13), (14), (15), and (16) shows that the Poisson noise has the greatest effect on the Strehl factor (for $\lambda \ll 1$, $\sigma^2 \ll 1$, and $\lambda > \sigma^4$). Thus when calculating the potential characteristics of the AOS with noisy control channels they must be taken into account first. The effect of noise on the performance of the AOS can be reduced by using standard filtering methods. For example, for the weighted summation algorithm proposed in Ref. 5 the values of the mathematical expectation and the Strehl factor can be written in the form:

$$\pi_{m_{\mathrm{St}_2}} = 1 - \frac{\lambda}{N+1}; \quad \pi_{\sigma^2_{\mathrm{St}_2}} = \frac{2\lambda}{N^2} \tag{17}$$

CONCLUSIONS

The expressions (4) and (6) derived in this work enable analysis of the performance of the aperture-sounding AOS with multichannel phase modulation. The moments of the noise characteristics, for the noise in the control channels, must be known a priori. The approach proposed in this work makes it possible to evaluate the potential performance characteristics of the AOS in the absence of modulation.

Analysis of the operation of the aperture-sounding AOS with multichannel phase modulation showed that the Poisson noise makes the main contribution to the decrease in the Strehl factor. In Ref. 6 is shown that in the mathematical description of the process of photodetection the operation of the photodetector is limited by the Poisson shot noise, if photomultiplication with a high multiplication factor is employed in the detector. It is obvious that photomultipliers should not be used as photodetectors in aperture-sounding AOSs which maximize the intensity at the photodetector. When only thermal Gaussian noise is taken into account in the operation of the photodetector the effect of such noise can be reduced by standard filtering methods.

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