# Precision and application range of the quasi-single scattering approximation in the backscattering signal calculation

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The problem of the light reflection from a turbid medium is considered. The method of the single scattering separation into the scattering, leading or not leading to a change in the radiation propagation direction relative to normal to the surface is offered. The reflectance is represented as a series in the scattering multiplicities with single change of the direction. For each multiplicity the precise linear integro-differential equation with homogeneous boundary conditions is derived. The application of the method of discrete ordinates brings to the linear matrix equations. The solution of these equations without application of the small angle approximation in the form of matrix exponential curves is obtained. The application range of the quasi-single scattering approximation depending on the optical parameters of the sensed medium is shown.

At present, the quasi-single scattering approximation is used for interpretation of results of optical remote sensing by means of both lidar and satellite measurements. This paper is devoted to the problem of reflection of solar radiation from the Earth's atmosphere. In the quasi-scattering approximation, the brightness coefficient of the reflected radiation is the first term of the series in the theory of disturbances in terms of backscattering[1]:

$$R_1(\tau,\Omega_0,\Omega)=\int\int_0^{\tau} L_0(t,\Omega_0,\Omega')x_t(\gamma')L_0(\tau-t,\Omega',\Omega)\,\mathrm{d}t\mathrm{d}\Omega', \quad (1)$$

where $\tau$ is the optical thickness of the layer; $\Omega_0 = \{\varphi_0, \theta_0\}$ is the unit vector of the radiation incidence direction; $\Omega = \{\varphi, \theta\}$ is the unit vector of the direction of detecting the radiation; $\varphi$, $\theta$ are the azimuth and polar angles, respectively; $L_0(\tau, \Omega_0, \Omega)$ is the brightness coefficient of the transmitted radiation in the small-angle approximation.[2] The main problem of the use of this approximation is an ambiguity in dividing the scattering phase function $x(\gamma)$ into "sharp" small-angle part $x_o(\gamma)$, describing the forward scattering, and the "obtuse" part $x_t(\gamma)$ describing the backward reflection:

$$x(\gamma) = (1 - a)x_o(\gamma) + a x_t(\gamma), \quad (2)$$

where $a$ is a small parameter; $\gamma$ is the scattering angle.

Solution of the problem of determining $x_t(\gamma)$ can not be formalized and is of a subjective character. But at a successful dividing, a good precision can be reached in interpretation of the reflected signal. The study of beam trajectories in turbid media, carried out with the use of programs of statistical simulation shows that the small-angle approximation does not describe the radiation flux reflected from a real medium even at strongly asymmetry of elastic scattering phase function. So the problem of determination of the application range of the quasi-single scattering approximation becomes urgent.

For reflection of radiation from turbid medium, at least one "strong" scattering is necessary (Fig. 1), which leads to a change of the radiation direction propagation relative to the normal direction to the surface. In the case of reflection from some optically dense media, the portion of "strong" multiple scattering in the reflected signal significantly increases.

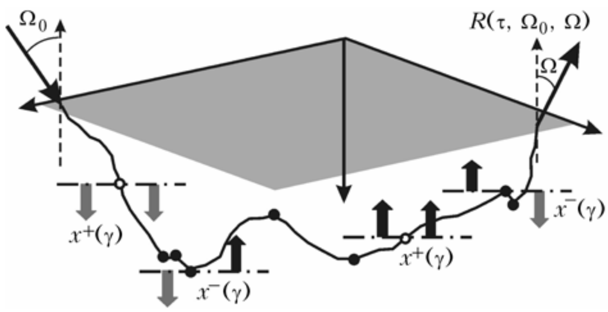


**Fig. 1.** The trajectory of the beam reflected from the turbid medium.

Typical trajectory of a beam reflected from a turbid medium obtained with a program for statistical simulation is shown in Fig. 1. The single scattering albedo $\Lambda = 0.98$ and the Heyney–Greenstein scattering phase function with the parameter $g = 0.95$ were used in the simulation. As is seen against the background of the multiple scattering, which do not lead to a change of the propagation direction relative to the normal to the surface, the "strong" scattering is pronounced, which changes the direction of downward flux to upward, and vise versa. It enables one to divide the scattering phase function into the "positive" part $x^+(\gamma)$

 

describing scattering, which does not lead to the change of the direction of radiation propagation, and the "negative" part $x^{-}(\gamma)$ describing the "strong" scattering. Such division is unambiguous and occurs in each scattering event depending on the direction of incidence of the scattered radiation.

In this case one should expect that the brightness coefficient of the reflected radiation is expanded in terms of multiplicity of the "strong" scattering:

$$R(\tau,\Omega_0,\Omega) = \sum_{k=1}^{N} R_k(\tau,\Omega_0,\Omega). \tag{3}$$

If consider only "strong" scattering, we obtain more precise, than quasi-single scattering, approximation. Analysis of the solution, taking into account the multiple "strong" scattering, enables us to determine the conditions, under which the contribution of multiple "strong" scattering is inessential, and, hence, the upper boundary of applicability of the quasi-single scattering approximation.

The brightness coefficient of the reflected radiation is characterized by the reflection function, which is the solution of the boundary problem of the transfer equation. But the transfer equation itself contains surplus information about the behavior of the radiation flux in the medium depth, therefore, it is better to determine the reflection function from equations containing only this function as unknown. Such equations are known. They are the nonlinear integro-differential Ambartsumyan–Chandrasekar equations.[3] Let us write the Ambartsumyan–Chandrasekar equation for the brightness coefficient of the reflected radiation $R(\tau, \Omega_0, \Omega)$ using the division of the scattering phase function into $x^{+}(\gamma)$ and $x^{-}(\gamma)$:

$$\begin{aligned}
&\frac{d}{d\tau}R(\tau,\Omega_0,\Omega) + \frac{\eta+\xi}{\eta\xi}R(\tau,\Omega_0,\Omega) - \\
&-\frac{\Lambda}{2}\int_{\Omega_+} x^{+}(\Omega_0-\Omega')R(\tau,\Omega',\Omega)\frac{d\Omega'}{\eta'} - \\
&-\frac{\Lambda}{2}\int_{\Omega_-} R(\tau,\Omega_0,\Omega')x^{+}(\Omega'-\Omega)\frac{d\Omega'}{\eta'} = \frac{\Lambda}{4}x^{-}(\Omega_0-\Omega) + \\
&+\Lambda\int_{\Omega_+}\int_{\Omega_-} R(\tau,\Omega_0,\Omega')x^{-}(\Omega'-\Omega'')R(\tau,\Omega',\Omega)\frac{d\Omega'}{\eta'}\frac{d\Omega''}{\eta''},
\end{aligned} \tag{4}$$

where $\xi = \cos\theta_0$, $\eta = \cos\theta$, $\Omega_{\pm}$ are the normal directions to the surface. It is easily seen that the scattering phase function in the Ambartsumyan–Chandrasekar is automatically divided into the "positive" and "negative" parts within the range of integration. Hence, the "positive" and "negative" parts of the scattering phase function are the parts of the real scattering phase function, however, we keep the upper indices for clearness.

Let us write the boundary condition for the integro-differential equation (4) in the case of reflection from a free layer:

$$R(\tau,\Omega_0,\Omega)\Big|_{\tau=0} = 0. \tag{5}$$

The left part of Eq. (4) describes scattering of radiation without a change of the direction of motion relative to the normal direction to the surface. The right part, on the contrary, contains terms with "strong" scattering. If to use the expansion of the brightness coefficient of the reflected radiation in terms of multiplicity of "strong" scattering (3), then the nonlinear integro-differential equation (4) is divided into linear equations for each multiplicity of the "strong" scattering; $k = 2n + 1$, $n = 0, 1, 2, \ldots$ are odd terms corresponding to the reflected radiation:

$$\begin{aligned}
&\frac{d}{d\tau}R_1(\tau,\Omega_0,\Omega) + \frac{\eta+\xi}{\eta\xi}R_1(\tau,\Omega_0,\Omega) - \\
&-\frac{\Lambda}{2}\int_{\Omega_+} x^{+}(\Omega_0-\Omega')R_1(\tau,\Omega',\Omega)\frac{d\Omega'}{\eta'} - \\
&-\frac{\Lambda}{2}\int_{\Omega_-} R_1(\tau,\Omega_0,\Omega')x^{+}(\Omega'-\Omega)\frac{d\Omega'}{\eta'} = \\
&= \frac{\Lambda}{4}x^{-}(\Omega_0-\Omega),\ k = 1,\ (n = 0),
\end{aligned} \tag{6}$$

$$\begin{aligned}
&\frac{d}{d\tau}R_{2n+1}(\tau,\Omega_0,\Omega) + \frac{\eta+\xi}{\eta\xi}R_{2n+1}(\tau,\Omega_0,\Omega) - \\
&-\frac{\Lambda}{2}\int_{\Omega_+} x^{+}(\Omega_0-\Omega')R_{2n+1}(\tau,\Omega',\Omega)\frac{d\Omega'}{\eta'} - \\
&-\frac{\Lambda}{2}\int_{\Omega_-} R_{2n+1}(\tau,\Omega_0,\Omega')x^{+}(\Omega'-\Omega)\frac{d\Omega'}{\eta'} = \\
&= \Lambda\sum_{i=0}^{n-1}\int_{\Omega_+}\int_{\Omega_-} R_{2i+1}(\tau,\Omega_0,\Omega')x^{-}(\Omega'-\Omega'')\times \\
&\times R_{2(n-i)-1}(\tau,\Omega',\Omega)\frac{d\Omega'}{\eta'}\frac{d\Omega''}{\eta''},\ k > 1,\ (n \ge 1).
\end{aligned} \tag{7}$$

Let us solve Eqs. (6) and (7) by the method of discrete ordinates (DO).[3,4] To do it, let us expand $R_k(\tau,\Omega_0,\Omega)$ and the scattering phase function in terms of azimuth harmonics:

$$R_k(\tau,\Omega,\Omega_0) = \sum_{m=0}^{\infty} R_k^m(\tau,\eta,\xi)\cos m(\varphi_0-\varphi); \tag{8}$$

$$x(\gamma) = x(\Omega_0-\Omega) = \sum_{m=0}^{\infty} x^m(\eta,\xi)\cos m(\varphi_0-\varphi). \tag{9}$$

Replace the integrals in Eqs. (6) and (7) with quadrature sums, introducing the nodes $\mu_i$ and weights $a_i$ of the quadrature formula, as well as the grid $\{\eta_i, \xi_j\}$ by variables $\eta$, $\xi$:

$$\eta_i = \mu_i,\ 0 < \eta_i \le 1,\ i = 0,1,\ldots,N;$$

$$\xi_j = \mu_j,\ 0 < \xi_j \le 1,\ j = 0,1,\ldots,N. \qquad (10)$$

For the introduced grid, let us determine the matrices of azimuth reflection harmonics $\overleftrightarrow{R_k}(\tau)$ (missing the index $m$) of "positive" $\overleftrightarrow{x^+}$ and "negative" $\overleftrightarrow{x^-}$ parts of the scattering phase function as follows:

$$\overleftrightarrow{R_k}(\tau) = R_k^m(\tau,\eta_i,\xi_j),\ \overleftrightarrow{x^+} = x^m(\eta_i,\xi_j),$$
$$\overleftrightarrow{x^-} = x^m(-\eta_i,\xi_j). \qquad (11)$$

Then the integro-differential equations (6) and (7) are reduced to the matrix differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\overleftrightarrow{R_1}(\tau)-\left(\frac{\Lambda}{2}\overleftrightarrow{x^+}\overleftrightarrow{S}-\overleftrightarrow{E}\right)\overleftrightarrow{\psi}\overleftrightarrow{R_1}(\tau)-\overleftrightarrow{R_1}(\tau)\overleftrightarrow{\psi}\left(\frac{\Lambda}{2}\overleftrightarrow{S}\overleftrightarrow{x^+}-\overleftrightarrow{E}\right)=\frac{\Lambda}{4}\overleftrightarrow{x^-}, \qquad (12)$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\overleftrightarrow{R_{2n+1}}(\tau)-\left(\frac{\Lambda}{2}\overleftrightarrow{x^+}\overleftrightarrow{S}-\overleftrightarrow{E}\right)\overleftrightarrow{\psi}\overleftrightarrow{R_{2n+1}}(\tau)-$$
$$-\overleftrightarrow{R_{2n+1}}(\tau)\overleftrightarrow{\psi}\left(\frac{\Lambda}{2}\overleftrightarrow{S}\overleftrightarrow{x^+}-\overleftrightarrow{E}\right)=\overleftrightarrow{F_n}(\tau), \qquad (13)$$

$$\overleftrightarrow{F_n}(\tau)=\Lambda\sum_{i=0}^{n-1}\overleftrightarrow{R_{2i+1}}(\tau)\overleftrightarrow{\psi}\overleftrightarrow{S}\overleftrightarrow{x_m^-}\overleftrightarrow{S}\overleftrightarrow{\psi}\overleftrightarrow{R_{2(n-i)-1}}(\tau),$$

where $\overleftrightarrow{E}$ is the unit diagonal matrix; $\overleftrightarrow{S} = \mathrm{diag}(a_i)$; $\overleftrightarrow{\psi} = \mathrm{diag}(1/\mu_i)$.

Consider the reflection from a semi-infinite medium. In this case, the dependence of the brightness coefficient of radiation reflected from the optical thickness disappears, and the differential term in Eqs. (12) and (13) is

$$\overleftrightarrow{A}\,\overleftrightarrow{R_{2n+1}} + \overleftrightarrow{R_{2n+1}}\,\overleftrightarrow{A}' = \overleftrightarrow{F_n}, \qquad (14)$$

where

$$\overleftrightarrow{A} = \left(\overleftrightarrow{E} - \frac{\Lambda}{2}\overleftrightarrow{x^+}\overleftrightarrow{S}\right)\overleftrightarrow{\psi};\ \ \overleftrightarrow{F_0} = \frac{\Lambda}{4}\overleftrightarrow{x^-},$$

$$\overleftrightarrow{F_n} = \Lambda\sum_{i=0}^{n-1}\overleftrightarrow{R_{2i+1}}\,\overleftrightarrow{\psi}\overleftrightarrow{S}\overleftrightarrow{x_m^-}\overleftrightarrow{S}\overleftrightarrow{\psi}\overleftrightarrow{R_{2(n-i)-1}},\ \ n \ge 1;$$

the accent means the transposed matrix.

Solution of the linear matrix equation (14) is studied in detail in the matrix theory,[15] and one can write it in the form of convolution:

$$\overleftrightarrow{R_1} = \int_0^\infty \exp\left(-\overleftrightarrow{A}t\right)\frac{\Lambda}{4}\overleftrightarrow{x^-}\exp\left(-\overleftrightarrow{A}'t\right)\mathrm{d}t, \qquad (15)$$

$$\overleftrightarrow{R_{2n+1}} = \int_0^\infty \exp(-\overleftrightarrow{A}t)\overleftrightarrow{F_n}\exp\left(-\overleftrightarrow{A}'t\right)\mathrm{d}t, \qquad (16)$$

where exp(.) is the matrix exponent. Then calculate the brightness coefficient of reflected radiation using Eqs. (3) and (8):

$$\overleftrightarrow{R} = \sum_{m=0}^{M}\sum_{n=0}^{N}\overleftrightarrow{R_{2n+1}}\cos m(\varphi_0-\varphi). \qquad (17)$$

The matrix structure of the solution allowed us to realize calculations by Eqs. (15)–(17) using the mathematical software package MatLab. At the modern level of development of computers, calculation with very high precision needs several seconds for any degree of asymmetry of the scattering phase function.

Consider again reflection of radiation from a layer. Let us solve the matrix differential equation (12) with the boundary condition (5) by means of the Laplace transform. The peculiarities of application of the Laplace transform to matrix differential equations are presented in Ref. 5. After transformations, we obtain the solution of Eqs. (12) and (13) in the form of convolution of the matrix exponents:

$$\overleftrightarrow{R_1}(\tau) = \int_0^\tau \exp\left[\left(\frac{\Lambda}{2}\overleftrightarrow{x_m^+}\overleftrightarrow{S}-\overleftrightarrow{E}\right)\overleftrightarrow{\psi}t\right]\times$$
$$\times\frac{\Lambda}{4}\overleftrightarrow{x_m^-}\exp\left(\overleftrightarrow{\psi}\left(\frac{\Lambda}{2}\overleftrightarrow{S}\overleftrightarrow{x_m^+}-\overleftrightarrow{E}\right)t\right)\mathrm{d}t, \qquad (18)$$

$$\overleftrightarrow{R_{2n+1}}(\tau) = \int_0^\tau \exp\left(\left(\frac{\Lambda}{2}\overleftrightarrow{x_m^+}\overleftrightarrow{S}-\overleftrightarrow{E}\right)\overleftrightarrow{\psi}t\right)\times$$
$$\times\int_0^t \overleftrightarrow{F_n}(t')\exp\left(\overleftrightarrow{\psi}\left(\frac{\Lambda}{2}\overleftrightarrow{S}\overleftrightarrow{x_m^+}-\overleftrightarrow{E}\right)(t-t')\right)\mathrm{d}t'\mathrm{d}t. \qquad (19)$$

Equations (15) and (18) describe the reflection with one "strong" scattering. The form of solution for the brightness coefficient of the reflected radiation [Eq. (18)] in approximation of "strong" single scattering is similar to solution of Eq. (1) obtained by means of the quasi-single approximation. But, contrary to Eq. (1), the small-angle approximation was not used in deriving Eqs. (15) and (18).

The results of calculation of the brightness coefficients of reflected radiation are shown in Fig. 2: from semi-infinite turbid medium and from the layer $\tau = 5$. Calculations were performed using Eq. (15), the modified method of spherical harmonics[6] and the Monte Carlo method (program SPIM-L). The possibility is realized in the program SPIM-L to follow the multiplicity of the change of the direction of motion relative to normal to the surface. The calculations and the results of simulation were compared for the sums of multiplicities of "strong" scattering up to the multiplicity $k$, inclusively. The value $\Lambda = 0.98$ and the Heyney–Greenstein scattering phase function with $g = 0.95$ were used in calculation and simulation. Comparison with the results of calculations according to Ref. 6 shows that the contribution of multiplicities $k > 19$ is inessential at these parameters.

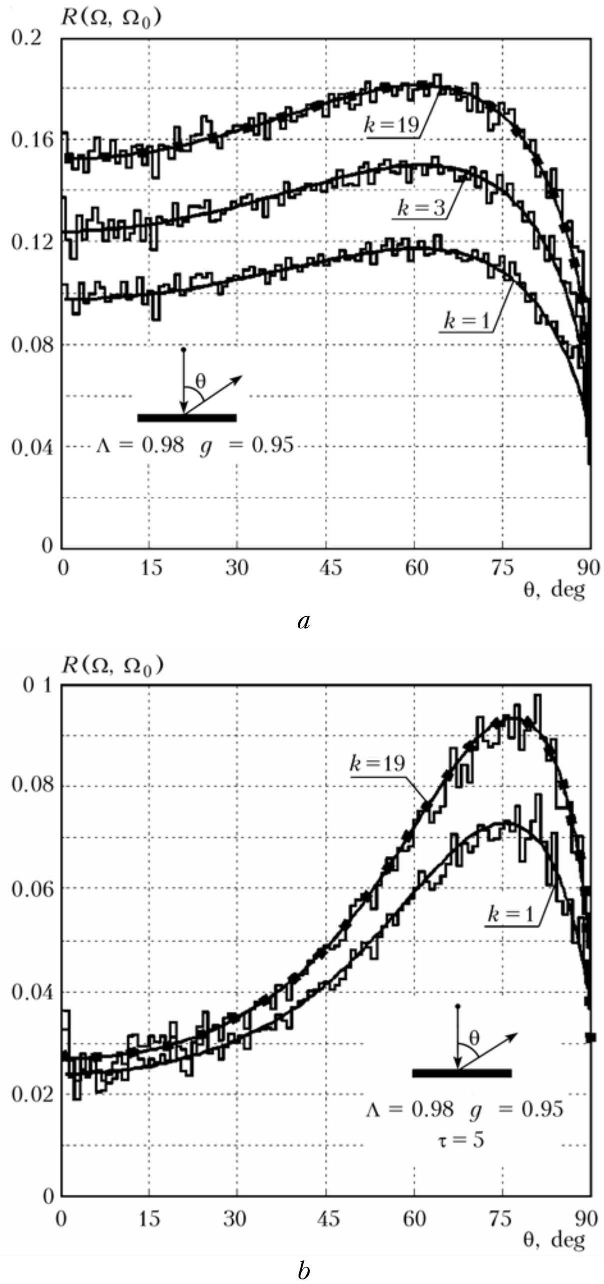


**Fig. 2.** The brightness coefficients of the reflected radiation: reflection from semi-infinite turbid medium (*a*), reflection from the layer of the optical thickness $\tau = 5$ (*b*). Incidence angle is normal. Numbers denote the multiplicity of "strong" scattering $k$. Solid line shows the results of calculation by Eq. (15). Points indicate calculation by the modified method of spherical harmonics[6] and histogram shows the results of statistical simulation with the program SPIM-L.

One can obtain the condition from analysis of Eqs. (15) and (16), when the reflection with one "strong" scattering determines the whole reflected radiation flux. To do this, it is necessary that $F_0 \gg F_1$, or

$$\frac{\Lambda}{4}\overleftrightarrow{x^-} \gg \Lambda\overrightarrow{R_1}\,\overleftrightarrow{\psi}\overleftrightarrow{S}\overleftrightarrow{x_m^-}\overleftrightarrow{S}\overleftrightarrow{\psi}\overrightarrow{R_1}. \tag{20}$$

Let us integrate the inequality (20) with respect to all cosines of the incidence and detection angles, that corresponds to multiplication from the left and from the right to the matrix $\overleftrightarrow{S}$:

$$\frac{\Lambda}{4}\overleftrightarrow{S}\overleftrightarrow{x^-}\overleftrightarrow{S} \gg \Lambda\overleftrightarrow{S}\overrightarrow{R_1}\,\overleftrightarrow{\psi}\overleftrightarrow{S}\overleftrightarrow{x_m^-}\overleftrightarrow{S}\overleftrightarrow{\psi}\overrightarrow{R_1}\overleftrightarrow{S} > \Lambda\left(\frac{1}{2}\overleftrightarrow{S}\overrightarrow{R_1}\overleftrightarrow{S}\right)^2\overleftrightarrow{S}\overleftrightarrow{x_m^-}\overleftrightarrow{S}. \tag{21}$$

If to use the estimates

$$\overleftrightarrow{S}\overleftrightarrow{x^-}\overleftrightarrow{S} \le b,$$

$$\overleftrightarrow{S}\overrightarrow{R_1}\overleftrightarrow{S} < \overleftrightarrow{S}\int_0^\infty \exp\left[-(1-\Lambda)\overleftrightarrow{\psi}t\right]\frac{\Lambda}{4}\overleftrightarrow{x^-}\times$$

$$\times\exp\left[-(1-\Lambda)\overleftrightarrow{\psi}'t\right]\mathrm{d}t\overleftrightarrow{S} \le \frac{\Lambda}{1-\Lambda}b, \tag{22}$$

we obtain from inequalities (21) and (22)

$$\Lambda < \Lambda_g \approx \frac{1}{1+b/\varepsilon}. \tag{23}$$

Here $b = \frac{1}{2}\int_{-1}^{0} x(\eta)\mathrm{d}\eta$ determines the probability of "strong" scattering; $\varepsilon$ is the value much less than 1. At decreasing the degree of asymmetry of the scattering phase function, the contribution of "strong" scattering of higher multiplicities increases. On the contrary, at decreasing the value of the single scattering albedo, the contribution of single "strong" scattering increases. When reaching some critical value $\Lambda_g$, the reflection is formed mainly by single "strong" scattering. The parameter $\Lambda_g$ depends on the scattering phase function. One can propose the estimate of the value $\Lambda_g$ for the Heyney–Greenstein scattering phase function:

$$\Lambda_g \approx \frac{1}{1+[3(1-g)]^{3/2}}. \tag{24}$$

If parameters of the medium have satisfied the condition $\Lambda < \Lambda_g$, it is possible to apply the quasi-single scattering approximation to calculation of the brightness coefficients of radiation, reflected from semi-infinite turbid medium.

In the case of reflection of radiation from real medium of the $\tau$ thickness, let us use the ratio $\rho_1(\tau,\theta_0)/\rho_\Sigma(\tau,\theta_0)$ for estimation of the application range of the quasi-single scattering approximation. Here

$$\rho_1(\tau,\theta_0) = \int R_1(\tau,\Omega,\Omega_0)\mathrm{d}\Omega$$

is the total brightness coefficient of the reflected radiation with single "strong" scattering;

$$\rho_\Sigma(\tau,\theta_0) = \int R(\tau,\Omega,\Omega_0)\mathrm{d}\Omega$$

is the total brightness coefficient of the reflected radiation.

The dependence of the ratio $\rho_1(\tau, \theta_0)/\rho_\Sigma(\tau, \theta_0)$ on the optical thickness is shown in Fig. 3. The incidence angle is normal.

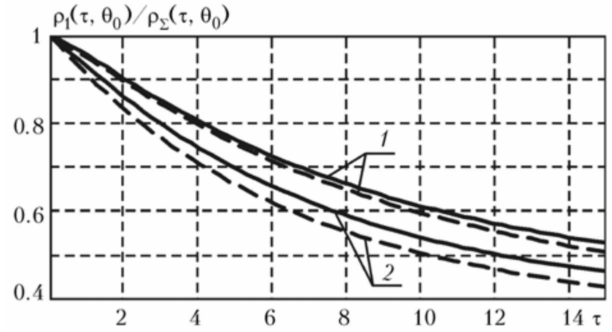


**Fig. 3.** The dependence of the ratio $\rho_1(\tau, \theta_0)/\rho_\Sigma(\tau, \theta_0)$ on the optical thickness: curves (*1*) are for cloud cover *C*1, and (*2*) is for haze *L*. Incidence angle is normal. Solid line shows the results of calculations for a wavelength of 300 nm and dotted line is for a wavelength of 700 nm.

Calculation of $\rho_1(\tau, \theta_0)$ was performed by Eq. (18), and $\rho\Sigma(\tau, \theta_0)$ was calculated by Eq. (19) and formulas from Ref. 6. The Deirmenjian parameters[7] were used in calculating the scattering phase function. The range of optical thickness was taken with a reserve for determination of the application range of the quasi-single scattering approximation. The effect of absorption by ozone was not taken into account in calculations, because ozone is variable gas component of the atmosphere depending on season, day, and geographic place. So to take into account this factor, additional data are necessary, which do not affect the technique. However, the appearance of absorption leads to decrease of the single scattering albedo, and under condition that $\Lambda < \Lambda_g$, the error of quasi-single approximation decreases.

The dependence of $\rho_1(\tau, \theta_0)/\rho_\Sigma(\tau, \theta_0)$ on the thickness and the type of turbid medium is shown in the Table. One can conclude from analysis of the Table that the contribution of multiple "strong" scattering at reflection from the optical thickness $\tau > 2$ into the brightness coefficient is more than 10% for all types of clouds and hazes. Hence, the magnitude of $\rho_1(\tau, \theta_0)/\rho_\Sigma(\tau, \theta_0)$ determines the minimum error at the use of the quasi-single scattering approximation.

**Table. Dependence of $\rho_1(\tau, \theta_0)/\rho_\Sigma(\tau, \theta_0)$ on the thickness and type of turbid medium**

| τ | cloud *C*1 | | cloud *C*3 | | haze *M* | | haze *L* | | haze *H* | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 300 nm | 700 nm | 300 nm | 700 nm | 300 nm | 700 nm | 300 nm | 700 nm | 300 nm | 700 nm |
| 0.5 | 0.9814 | 0.9797 | 0.9798 | 0.9715 | 0.9591 | 0.9464 | 0.9662 | 0.9559 | 0.9490 | 0.9339 |
| 1.0 | 0.9578 | 0.9549 | 0.9552 | 0.9406 | 0.9176 | 0.8954 | 0.9302 | 0.9119 | 0.9009 | 0.8705 |
| 1.5 | 0.9325 | 0.9286 | 0.9290 | 0.9096 | 0.8786 | 0.8501 | 0.8950 | 0.8712 | 0.8581 | 0.8161 |
| 2.0 | 0.9069 | 0.9022 | 0.9026 | 0.8793 | 0.8422 | 0.8096 | 0.8614 | 0.8337 | 0.8196 | 0.7687 |
| 2.5 | 0.8817 | 0.8763 | 0.8768 | 0.8501 | 0.8083 | 0.7731 | 0.8296 | 0.7991 | 0.7846 | 0.7271 |
| 3.0 | 0.8572 | 0.8513 | 0.8518 | 0.8222 | 0.7768 | 0.7400 | 0.7997 | 0.7672 | 0.7526 | 0.6900 |
| 3.5 | 0.8338 | 0.8273 | 0.8279 | 0.7957 | 0.7477 | 0.7098 | 0.7716 | 0.7377 | 0.7234 | 0.6570 |
| 4.0 | 0.8114 | 0.8045 | 0.8051 | 0.7706 | 0.7206 | 0.6822 | 0.7454 | 0.7105 | 0.6965 | 0.6273 |
| 4.5 | 0.7902 | 0.7829 | 0.7835 | 0.7471 | 0.6956 | 0.6569 | 0.7209 | 0.6854 | 0.6717 | 0.6006 |
| 5.0 | 0.7700 | 0.7624 | 0.7630 | 0.7249 | 0.6723 | 0.6337 | 0.6981 | 0.6621 | 0.6489 | 0.5766 |
| 5.5 | 0.7509 | 0.7430 | 0.7436 | 0.7041 | 0.6508 | 0.6123 | 0.6769 | 0.6406 | 0.6277 | 0.5549 |
| 6.0 | 0.7329 | 0.7246 | 0.7253 | 0.6846 | 0.6309 | 0.5925 | 0.6570 | 0.6207 | 0.6082 | 0.5354 |
| 6.5 | 0.7158 | 0.7073 | 0.7080 | 0.6663 | 0.6124 | 0.5743 | 0.6386 | 0.6023 | 0.5900 | 0.5176 |
| 7.0 | 0.6997 | 0.6910 | 0.6917 | 0.6492 | 0.5952 | 0.5575 | 0.6215 | 0.5852 | 0.5731 | 0.5016 |
| 7.5 | 0.6845 | 0.6756 | 0.6763 | 0.6332 | 0.5793 | 0.5419 | 0.6055 | 0.5693 | 0.5575 | 0.4870 |
| 8.0 | 0.6702 | 0.6611 | 0.6618 | 0.6183 | 0.5645 | 0.5275 | 0.5906 | 0.5546 | 0.5429 | 0.4738 |
| 8.5 | 0.6566 | 0.6474 | 0.6482 | 0.6043 | 0.5508 | 0.5142 | 0.5768 | 0.5410 | 0.5294 | 0.4617 |
| 9.0 | 0.6438 | 0.6345 | 0.6353 | 0.5913 | 0.5381 | 0.5018 | 0.5639 | 0.5283 | 0.5168 | 0.4507 |
| 9.5 | 0.6318 | 0.6224 | 0.6231 | 0.5791 | 0.5263 | 0.4903 | 0.5519 | 0.5166 | 0.5050 | 0.4407 |
| 10.0 | 0.6204 | 0.6109 | 0.6117 | 0.5676 | 0.5152 | 0.4796 | 0.5407 | 0.5056 | 0.4940 | 0.4315 |

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