# Optimization method of the phase front gauge topology

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The new optimization method of the phase front gauge topology is developed, using the basis of Zernike polynomials and the polar coordinate system. The optimization criteria of the topology of phase front gauge are offered allowing essential enhancement of the accuracy characteristics of phase front restoration.

## Introduction

One of the most effective (sometimes in combination with others) ways to attenuate the disturbing atmospheric effect on operation of an optical system is the use of adaptive methods and systems. Ideas providing the basis for development of adaptive systems have been supposed recently.[1]

Phase front gauges are used as measuring instruments in adaptive optical phase-conjugation systems at compensation of nonsteady phase distortions arising at radiation propagation in an optically inhomogeneous medium.[3–7] They are key elements in many modern radiation management and correction systems, and their capabilities mainly determine parameters of the systems and range of solved problems.

When developing adaptive optical phase-conjugation systems, indirect measurements of phase distribution are usually carried out at the aperture of an adaptive optical system.[5–7] Then these measurements are converted into the basis of elastic mirror response functions by a numerical method. Each of known algorithms has advantages and disadvantages; however, final specifications of phase front gauges, developed on the basis of the algorithms, depend on specific implementation of gauges. So, further improvement of such devices is of interest.

In this work, we describe an optimization method of the topology of a phase front gauge using the basis of Zernike polynomials for approximation.

## 1. Phase front restoration algorithm

The well-known Hartmann test[1] first suggested to control telescopic optics was later used for adaptive optics. Now it is the most common type of phase front gauge. An image of entrance pupil is projected to the lens matrix. All images are formed at one photoreceiver, usually CCD matrix. When an arriving wave front is plane, all images are situated in a regular grid, determined by the lens matrix geometry. When the wave front distorts, images shift from their true positions.

Note that the Karhunen–Loeve expansion is a universal expansion answering a number of optimum conditions.[1] It is characterized by the following properties causing its optimality: minimal mean square error when keeping a preset number of terms in infinite expansion, maximum obtained information on a function defined by a truncated series at any number of keeping terms in comparison with any other expansion, and noncorrelatedness of expansion coefficients, which simplifies further use of expansion results and their analysis. But this analytical expansion is hardly representable; therefore, the system of Zernike polynomials, sufficiently close to it, is usually[1] used in practice.

To approximate the response function of flexible adaptive mirror, use the system of Zernike polynomials orthogonal (orthonormalized) inside a unit circle or a circle of $R$ in radius and representable in polar coordinates $r$, $\theta$ [Refs. 1, 2, and 7]:

$$Z_j(r,\theta)= \begin{cases} \sqrt{n+1}R_n^m(r)\sqrt{2}\cos m\theta \text{ for even polinomials and } m\neq 0, \\ \sqrt{n+1}R_n^m(r)\sqrt{2}\sin m\theta \text{ for odd polinomials and } m\neq 0, \\ \sqrt{n+1}R_n^0(r) \quad \text{for } m=0, \end{cases} \tag{1}$$

where

$$\int_0^{2\pi}\int_0^1 Z_j(r,\theta)Z_{j'}(r,\theta)\mathrm{d}r\mathrm{d}\theta = \delta_{jj'}R_n^m(r) = \sum_{s=0}^{(n-m)/2}\frac{(-1)^s(n-s)!r^{n-2s}}{s![(n+m)/2-s]![(n-m)/2-s]!}.$$

Variables $n$ and $m$ are always integer and satisfy the condition $n \le m$, $n - |m|$ is even. The subscript $j$ is the sequence mode number and depends on $n$ and

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$m$. The orthogonality condition in a circle of unit radius has the form

$$W(r) = \begin{cases} 1/\pi & \text{at } |r| \le 1, \\ 0 & \text{at } |r| > 1, \end{cases} \quad (2)$$

where $\delta_{j'}$ is the Kronecker delta.

A gauge-measured phase front has the following form in the Zernike basis:

$$\Phi_{\text{mes}}(r,\theta) = \sum_{j=1}^{N} Z_j(r,\theta)c_j, \quad (3)$$

where $c_j$ is the coefficients phase front expansion in terms of Zernike polinomials (signals from the phase front gauge output). The number of polynomials $N$ in the expansion is determined using the equation[1]

$$N = \left[ \left( \frac{-0.2944(D/r_0)^{5/3}}{\ln \text{St}} \right)^{2/\sqrt{3}} \right], \quad (4)$$

where $D$ is the aperture diameter, $r_0$ is the correlation radius; St is the Strehl number.

In known devices, local slopes of a phase front at the aperture point, proportional to quantities of types

$$\frac{\partial \Phi(x_i, y_j)}{\partial x} \text{ and } \frac{\partial \Phi(x_i, y_j)}{\partial t},$$

are to be measured with quadrant photoreceivers.[3–7] Here $i = \overline{1, L}$; $j = \overline{1, K}$; $L \times K$ is the number of quadrant photoreceivers of a gauge; $\Phi(x_i, y_j)$ is the phase value at the gauge aperture. Then these measured values are converted to the phase values $\Phi(x_i, y_j)$ or coefficients $a_j$ and used to manage the feedback loop of an adaptive optics system.

In contrast to known methods, where local slopes are to be measured in $x$ and $y$ planes proportional to the corresponding derivatives, here one measures tangential local and radial slopes proportional to quantities of the types $\partial\Phi(r_i, \theta_i)_{\text{mes}}/\partial\theta$ and $\partial\Phi(r_i, \theta_i)_{\text{mes}}/\partial r$, $i = \overline{1, M}$, respectively.

To measure these quantities, we propose to use dual-element photoreceivers arranged at aperture points on concentric circles, and the boundary between dual-element photoreceivers coincides with the radius of a corresponding circle or normal to it (Fig. 1).

Again, the solution of wave front restoration problem can be considered in the following statement. Let the Hartmann sensor measures the local slopes of phase front $\partial\Phi(r_i, \theta_i)_{\text{mes}}/\partial\theta$ at points with the coordinates $r_i$, $\theta_i$, which can be chosen arbitrary. For definiteness, consider points arranged, e.g., as in Fig. 1*a*.

For phase restoration, apply the least square method (LSM). In this case, the corresponding quadratic form of LSM is

$$J_1 = \sum_{i=0}^{M} \left( \frac{\partial \Phi(r_i,\theta_i)_{\text{mes}}}{\partial \theta} - \frac{\partial \Phi(r_i,\theta_i)}{\partial \theta} \right) \times \left( \frac{\partial \Phi(r_i,\theta_i)_{\text{mes}}}{\partial \theta} - \frac{\partial \Phi(r_i,\theta_i)}{\partial \theta} \right)^{\text{T}}, \quad (5)$$

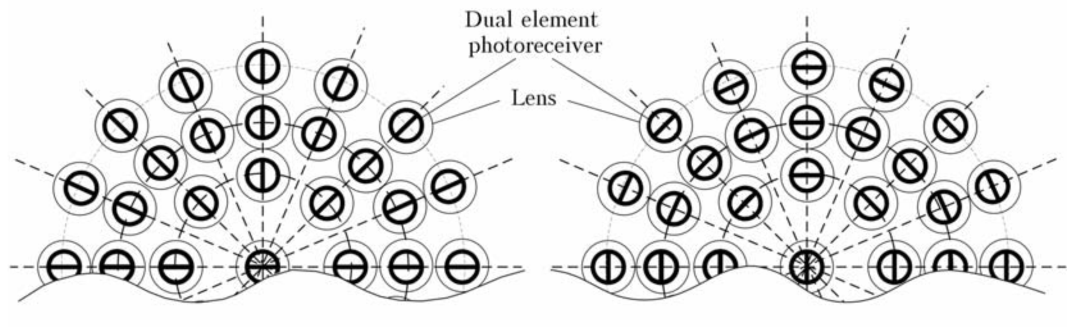
where $M + 1$ is the number of phase-front measuring points (photoreceivers).

The true values of phase gradient can be written as

$$\frac{\partial \Phi(r_i,\theta_i)}{\partial \theta} = \frac{\partial}{\partial \theta} \sum_{j=1}^{N} a_j Z_j(r_i,\theta_i) = \sum_{j=1}^{N} a_j \frac{dZ_j(r_i,\theta_i)}{dr}. \quad (6)$$

To calculate Eq. (6) in an explicit form, equation (1) is to be used. Substituting Eq. (6) in Eq. (5), obtain

$$J_1 = \sum_{i=0}^{M} \left( \frac{\partial \Phi(r_i,\theta_i)_{\text{mes}}}{\partial \theta} - \sum_{j=1}^{N} a_j \frac{\partial Z_j(r_i,\theta_i)}{\partial \theta} \right) \times \left( \frac{\partial \Phi(r_i,\theta_i)_{\text{mes}}}{\partial \theta} - \sum_{k=1}^{N} a_k \frac{\partial Z_j(r_i,\theta_i)}{\partial \theta} \right)^{\text{T}}. \quad (7)$$




*a* *b*

**Fig. 1.** Tangential (*a*) and radial (*b*) arrangement of photoreceivers at the aperture of phase front gauge.

Find the coefficients $a_j$ from $M + 1$ linear equations equating the quadratic partial derivatives $J_1$ with respect to $a_j$ with zero:

$$\frac{\partial J_1}{\partial a_j} = 0; \;\; BA = C, \tag{8}$$

where $B$ is the matrix with the coefficients

$$b_{k,j} = \sum_{i=0}^{M} \frac{\partial Z_j(r_i,\theta_i)}{\partial\theta} \frac{\partial Z_k(r_i,\theta_i)}{\partial\theta};$$

$A$ is the row-vector of target coefficients of Zernike polynomials $a_j$; $C$ is the column-vector of the right part

$$c_j = \sum_{i=0}^{M} \frac{\partial \Phi(r_i,\theta_i)_{\text{mes}}}{\partial\theta} \frac{\partial Z_j(r_i,\theta_i)}{\partial\theta}, \;\; k, j = 1, N.$$

The solution of system (8) takes the form

$$A = B^{-1}C. \tag{9}$$

As has been shown in analysis, the structure of matrix $B^{-1}$ (positions of zero and non-zero elements) remains invariable at arbitrary chosen arrangement of dual-element photoreceiver points; only values of these elements vary.

Reasoning similarly, a phase front gauge can be built on the base of radial derivatives (Fig. 1*b*).

The corresponding quadratic form of LSM is the following in this case:

$$J_2 = \sum_{i=0}^{M} \left( \frac{\partial \Phi(r_i,\theta_i)_{\text{mes}}}{\partial r} - \sum_{j=1}^{N} a_j \frac{\partial Z_j(r_i,\theta_i)}{\partial r} \right) \times \left( \frac{\partial \Phi(r_i,\theta_i)_{\text{mes}}}{\partial r} - \sum_{k=1}^{N} a_k \frac{\partial Z_j(r_i,\theta_i)}{\partial r} \right)^{\text{T}}, \tag{10}$$

where $M+1$ is the number of phase front measuring points.

True values of phase gradient can be presented as

$$\frac{\partial \Phi(r_i,\theta_i)}{\partial r} = \frac{\partial}{\partial r} \sum_{j=1}^{N} a_j Z_j(r_i,\theta_j) = \sum_{j=1}^{N} a_j \frac{dZ_j(r_i,\theta_i)}{dr}. \tag{11}$$

To calculate Eq. (11) in an explicit form, equation (1) is to be used. Substituting Eq. (11) in Eq. (10), obtain

$$J_2 = \sum_{i=0}^{M} \left( \frac{\partial \Phi(r_i,\theta_i)_{\text{mes}}}{\partial r} - \sum_{j=1}^{N} a_j \frac{\partial Z_j(r_i,\theta_i)}{\partial r} \right) G_i \times \left( \frac{\partial \Phi(r_i,\theta_i)_{\text{mes}}}{\partial r} - \sum_{k=1}^{N} a_j \frac{\partial Z_j(r_i,\theta_i)}{\partial r} \right)^{\text{T}}. \tag{12}$$

Find the coefficients $a_j$ from $M$ linear equations equating the quadratic partial derivatives $J_2$ with respect to $a_j$ with zero:

$$\frac{\partial J_2}{\partial a_j} = 0; \;\; BA = C, \tag{13}$$

where $B$ is the matrix with the coefficients

$$b_{k,j} = \sum_{i=0}^{M} \frac{\partial Z_j(r_i,\theta_i)}{\partial r} \frac{\partial Z_k(r_i,\theta_i)}{\partial r};$$

$A$ is the row-vector of target coefficients of Zernike polynomials $a_j$; $C$ is the column-vector of the right part

$$c_j = \sum_{i=0}^{M} \frac{\partial \Phi(r_i,\theta_i)_{\text{mes}}}{\partial r} \frac{\partial Z_j(r_i,\theta_i)}{\partial r}, \;\; k, j = 1, N.$$

The solution of system (13) has the form

$$A = B_1^{-1}C. \tag{14}$$

Thus, matrices $B^{-1}$ or $B_1^{-1}$ for a preset arrangement of points can be calculated in advance, while the algorithm for calculation of expansion coefficients in the Zernike basis of the vector $A$ reduces to calculation of the vector of right part of $C$ and matrix multiplication by the matrix $B^{-1}$ or $B_1^{-1}$.

Based on the considered phase front restoration method, the structure of a corresponding gauge can be built. This structure is shown in Fig. 2 and realizes the algorithm in the following way. A distorted wave front is focused by lens matrix *1* to matrix of dual-element photoreceivers *2*. In this case, in any local zone, restricted by the lens aperture, the focused spot shifts relative to the optical axis depending on the local phase front slope, proportional to quantities of the form $\frac{\partial \Phi(r_i,\theta_i)_{\text{mes}}}{\partial\theta}$ at points with the coordinates $r_i$, $\theta_i$.

Presence of phase front distortions results in difference signals at the output of photoreceivers *2*, which are amplified by differential amplifiers *3* output signals of which are proportional to the quantities $\frac{\partial \Phi(r_i,\theta_i)_{\text{mes}}}{\partial\theta}$. Then, signals from the output of differential amplifiers *3* feed to the summation units, where signals are calculated proportional to the elements of column-vector of the right part $c_j$.

In this case, the amplification constants $K_{\text{a}}$ of the scale amplifiers of summation unit are calculated according to the equation

$$K_{\text{a}_j} = \frac{\partial Z_j(r_i,\theta_i)}{\partial\theta}. \tag{15}$$

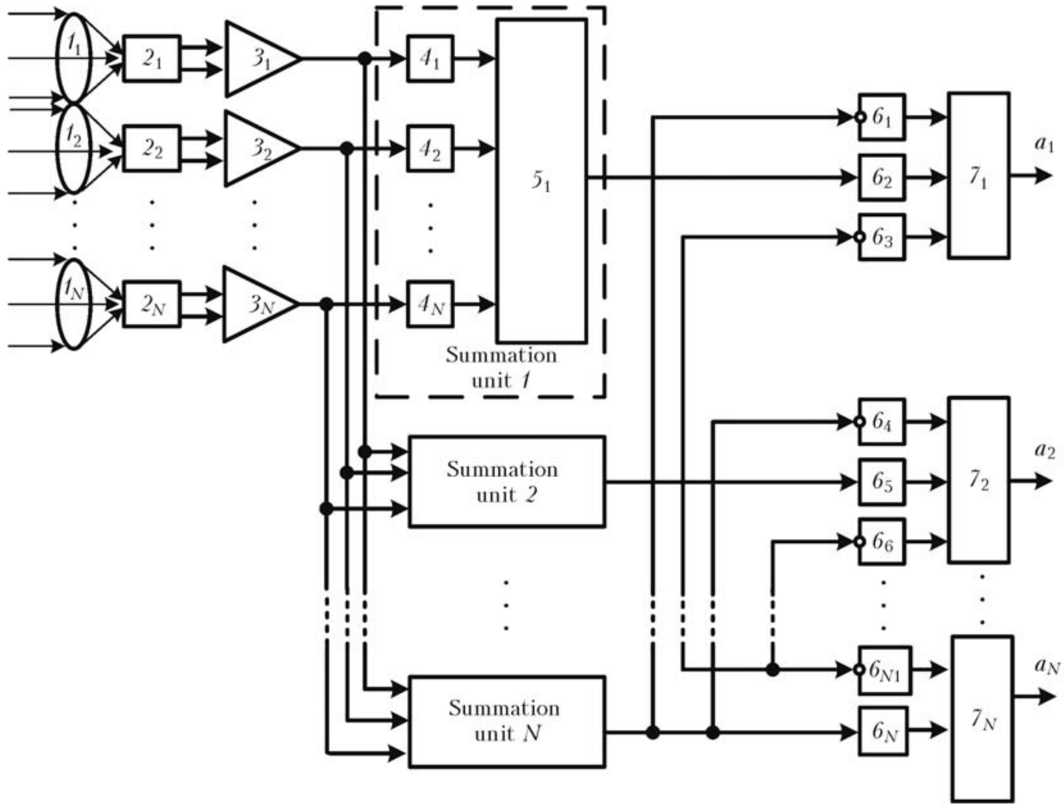



**Fig. 2.** Block-diagram of a gauge of tangential type: lens (*1*); matrix of dual-element photoreceivers (*2*); differential amplifiers (*3*); scale amplifiers of the summation unit (*4*); summators of the summation unit (*5*); groups of scale amplifiers (*6*); second group of summators (*7*).

Output signals of summation units *5*, proportional to $C_j$, arrive to the inputs of a group of scale amplifiers, coefficients of amplification of which are calculated as follows:

$$K_{kj} = b_{k,j}^{-1}. \quad (16)$$

Only nonzero elements are considered in this case. Output signals of the second scale amplifiers *6* arrive to the inputs of summators *7*, from outputs of which signals $a_j$, proportional to the expansion coefficients in the Zernike basis, are output.

Output signals can be used directly for feed to the input of the flexible piezoelectric mirror of adaptive optical system, which simplifies its design essentially.

## 2. Optimization of the phase front gauge topology

When implementing a phase front gauge and at arbitrary choice of photoreceiver positions, the weight coefficients of amplification are maximal in modulus. This results in much smaller signal-to-noise ratio in these channels in comparison with others, which causes essential loss in phase front gauge performance. According to investigations, these weight coefficients depend on arrangement of photoreceivers. An optimal arrangement is to be chosen with accounting for the maximum of the following criterion:

$$J_3 = \max \sum_{i=1}^{N} \left( \frac{\partial Z_i(r,\theta)}{\partial \theta} \right)^2 \quad (17)$$

for a tangential (see Fig. 4) and

$$J_4 = \max \sum_{i=1}^{N} \left( \frac{\partial Z_i(r,\theta)}{\partial r} \right)^2 \quad (18)$$

for radial phase front gauge.

Consider an example of optimization of the phase front gauge topology.

Find the number of Zernike polynomials by Eq. (4). Thus, e.g., at $D = 0.05$ m and $L_t = 5$ km, maximum of $N$ does not exceed 12–16 according to Eq. (4). Let $N = 16$ in our case. To choose coordinates of optimal arrangement of dual-element photoreceivers, located at aperture points on a concentric circle of $r$ in radius, use criterion (17). The points $a$, $b$, $c$, and $d$ in Fig. 3 are the optimized polar coordinates of gauge photoreceivers on a circle with $r_1 = 0.2$; the points $e$, $f$, $g$, $h$, $i$, $j$, $k$, $l$, $m$, and $n$ are the coordinates of gauge photoreceivers on a circle with $r_4 = 0.8$. Again, the optimized coordinates of quadrant photoreceivers on circles with $r_2 = 0.4$, $r_3 = 0.6$, and $r_5 = 1$ are chosen in a similar way. Finally, we obtain an optimized topology of a tangential phase front gauge.

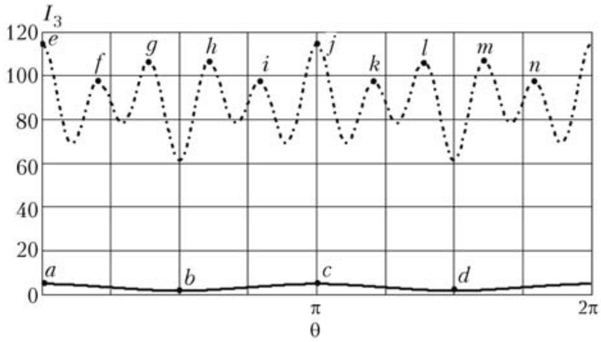



**Fig. 3.** Choice of optimal arrangement of photoreceivers.

Reasoning by analogy, we can optimize the topology of a radial phase front gauge with the use of Eq. (18).

## 3. Estimate of the noise error of phase front restoration

When estimating the efficiency of adaptive optical systems against Gaussian noises the arrangement of photoreceivers at aperture points on concentric circles of gauges (see Fig. 1) should be taken into account, as well as at optimized arrangement of photoreceivers (Fig. 4).

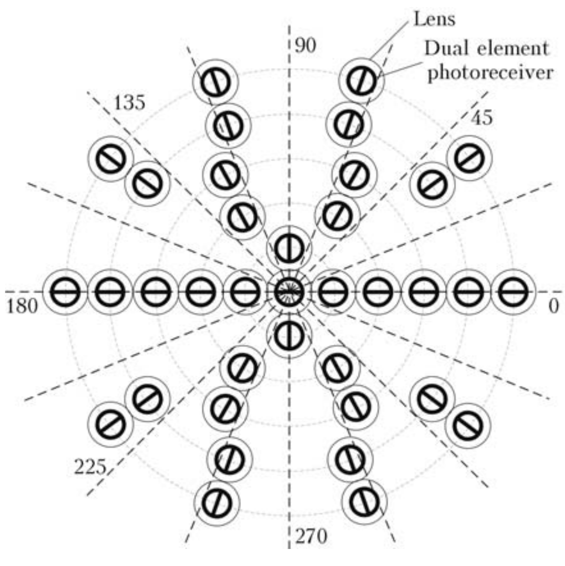



**Fig. 4.** Optimized arrangement of tangential photoreceivers at the aperture.

The dependence of variance of the phase front restoration error $\mathscr{D}_r$ on noise variance $\mathscr{D}_n$ has been also studied in gauge channels for the topology (Fig. 1) and optimized topology (Fig. 4) at $N = 16$. The study results are shown in Fig. 5.

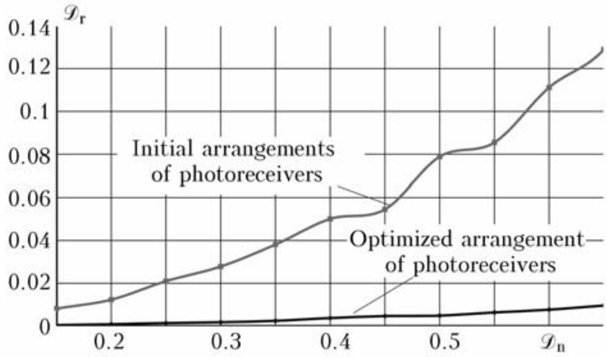



**Fig. 5.** Variance of phase front restoration.

Analysis of the results has shown that the suggested optimization of the phase front gauge topology allows an essential increase in accuracy of phase front restoration.

## Conclusion

A new algorithm is suggested for phase front restoration in the Zernike basis with the use of polar coordinates. A new structure scheme of phase front gauge is worked out, as well as an optimization method of the phase front gauge topology, which provides a possibility to enhance the accuracy characteristics of the gauge. Optimization criteria of the phase front gauge topology (17) and (18) are introduced, which allow an essential increase in accuracy of phase front restoration.

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