OPTICAL RADIATIVE TRANSFER TROUGH A BOUNDED SCATTERING VOLUME

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A solution of the problem of radiative transfer through bounded scattering media, based on the exact solution of the transfer equation for a one-dimensional medium, and on the parameterization of the scattering phase function by six integral parameters is proposed. The accuracy of asymptotic formulas for the reflection coefficient is estimated.

INTRODUCTION

The transfer of optical radiation through scattering media is accompanied by its complex transformation into a scattered field. In general the attenuation of the radiation incident upon the medium and the formation of the scattered radiation field are described by the transfer equation, analytical solutions of which are presently available only for individual, particular cases. Therefore, a systematic analysis of radiative transfer through scattering media based on the existing solutions of this equation appears to be ineffective: recall the wide range of scattering properties of various natural media and the range of optical depths of practical interest.

Particular difficulties are encountered in analyzing the propagation of optical radiation through scattering media when the spatial boundaries of the beam, or the scattering volume, or both have to be accounted for. Even the first experiments in propagation of weakly divergent laser beams of small diameter through fogs and hazes demonstrated[1–3] that the extinction of brightness for such beams is described by Bouguer's law down to unexpectedly large optical thickness. Similar experimental results were obtained for optical beams from sources of thermal radiation.[4,S] Further studies made it possible to qualitatively interpret this effect as resulting from a considerably depleted brightness of the multiply scattered background radiation for narrow beams, as compared with wide ones. The effect of conservation of the observed contrast between the brightnesses of the total and forward scattered laser radiation at large optical depths stimulated the design and construction of laser navigation devices for aircraft landing and ship navigation under conditions of low visibility.[6] However the physical interpretation of this effect cannot be considered complete. In particular, so far we lack either relationships or sufficiently effective algorithms to calculate the background multiply scattered radiation at large optical thicknesses for complex boundary conditions, which take place when narrow beams propagate through spatially bounded volumes of scattering media.

Attempts were undertaken many times to account for finite dimensions, shape, and other boundary conditions of the scattering volume when solving the transfer equation. The most complete (though restricted to a limited range of optical depths) account of the boundary conditions for the scattering medium and different optical beams was achieved in radiative transfer calculations which applied statistical techniques.[7] Analytic efforts to account for the shape of the scattering volume at high orders of scattering are also known.[8]

Below, one of the heuristic techniques is discussed which claim to provide a general approach to the account of spatial limitations of the scattering volume in the solution of the problem of optical beam propagation. This approach serves as a basis for a generalized semi-analytical technique for calculating radiative fluxes in scattering media, which yields an acceptable level of accuracy in the solution of the transfer problem for different geometrical schemes and boundary conditions.

The idea of this approach consists in taking the known analytical solutions for the one-dimensional case as the basis for the desired solution. In the course of generalizing these solutions the individual parameters in these analytical formulas acquire an additional physical meaning and new computational algorithms are designed for the three-dimensional case. The new computational algorithms are based on a consistent account of the transmission. reflection, and absorption of radiation by the individual scattering layers which comprise the whole scattering volume. The advantage of this technique, as compared with the statistical approach, in which the trajectories of individual photons are tracked, consists in the consistent tracking of the trajectories of whole fluxes of multiply scattered radiation, which simplifies the computational procedures and considerably increases their efficiency.

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EXTENSION OF THE EXACT SOLUTION OF THE RADIATIVE TRANSFER EQUATION FOR THE ONE-DIMENSIONAL CASE

Regard an elementary scattering volume with volume extinction coefficient $\alpha = \sigma + \kappa$. Here σ is the volume scattering coefficient and κ is the volume absorption coefficient. We shall characterize the scattering properties of an elementary volume by the integral parameters of the scattering phase function, to be determined below.

To describe the transfer of monochromatic radiation through a scattering medium we shall use the exact solution of the transfer equation for the one-dimensional case. One can then introduce the variables η and β which characterize the fraction of the radiation scattered forward and backward, respectively, by an element of optical length $d\tau = \alpha dx$. For radiation propagating along the x axis the values I_1 (forward direction) and I_2 (backward direction) are introduced. If an element of the medium of length $d\tau$ does not emit radiation itself, the equation of radiative transfer for it consists of a system of two differential equations[9]

$$dI_1(\tau) = -I_1(\tau) + \Lambda[\eta I_1(\tau) + \beta I_2(\tau)],$$

$$-dI_1(\tau) = -I_2(\tau) + \Lambda[\eta I_2(\tau) + \beta I_1(\tau)], \quad (1)$$

where the photon survival probability is $\Lambda = \sigma/(\sigma + \kappa)$. The solutions of Eq. (1) have the form[9]

$$I_1 = I_0 \frac{(1 - r^2)\exp(-k\tau_x)}{1 - r^2\exp(-2k\tau_x)},$$

$$I_2 = I_0 \frac{r[1 - \exp(-2k\tau_x)]}{1 - r^2\exp(-2k\tau_x)},$$

$$I_3 = I_0 \frac{(1 - r)[1 - \exp(-k\tau_x)]}{1 + r\exp(-k\tau_x)}, \quad (2)$$

where I_1 is the transmitted radiation, I_2 is the reflected radiation, and I_3 is the radiation absorbed by the layer. Note that the parameters k and r stand for the following expressions:

$$k = \sqrt{(1 - \Lambda)[1 - \Lambda(\eta - \beta)]},$$

$$r = [k - (1 - \Lambda)]/[k + (1 - \Lambda)]. \quad (3)$$

The principal step in extending solutions (2) for the one-dimensional case consists in accounting for the radiation leaving the transfer process not only as a result of absorption and reflection, but also for that exiting through the sides of the propagation channel. One of the most evident physical reasons for such an effect is sideway scattering. The representation of the scattering phase function via its integral parameters in the form of two scattered fluxes — one forward and the other backward — is insufficient for such an extension. A next-in-accuracy representation of the scattering phase function would in this case be a six-flux approximation in which the integral parameters satisfy the normalization condition[10]

$$\eta + \beta + \mu_1 + \mu_2 + \mu_3 + \mu_4 = 1$$

These are described for an axially symmetric phase function by the relations:

$$\eta = \frac{|F_{+x}|}{F}, \quad \beta = \frac{|F_{-x}|}{F},$$

$$\mu_{1,2} = \frac{|F_{\pm y}|}{F}, \quad \mu_{3,4} = \frac{|F_{\pm z}|}{F},$$

$$F_{+x} = 2\pi \int_0^{\pi/2} \chi(\theta)\sin\theta\cos\theta\, d\theta,$$

$$F_{-x} = 2\pi \int_{\pi/2}^{\pi} \chi(\theta)\sin\theta\cos\theta\, d\theta,$$

$$F_{\pm y} = F_{\pm z} = 2\int_0^{\pi} \chi(\theta)\sin^2\theta\, d\theta,$$

$$F = |F_{+x}| + |F_{-x}| + 2|F_{\pm y}| + 2|F_{\pm z}|. \quad (4)$$

It is apparent from Eqs. (4) that the integral parameters of the scattering phase function thus introduced determine the fraction of radiation scattered in the direction of the x axis (η), the apposite direction (η); the direction of the y axis (μ_1), the opposite direction (μ_2); the direction of the z axis (μ_3). and the opposite direction (μ_4). The normalization condition introduced above for the integral parameters eliminates the seeming contradiction of integrating the scattering phase function for each parameter over overlapping scattering angles. To account for losses due to radiation exiting through the sides of the volume ("sideway losses") we introduce, similar to the notion of the photon survival probability Λ, the photon extinction probability. After thus accounting for sideway losses we may write for the value of $p_1 = 1 - \Lambda = \kappa/\alpha$ in the case of radiation propagating through a scattering medium

$$p = \frac{\sum_i \chi_i}{\sigma + \sum_i \chi_i} = \frac{\sum_i \chi_i}{\alpha} = \sum_i p_i. \quad (5)$$

The index "i" here points to the physical process causing the radiation to leave the propagation channel. The probability of a photon leaving the volume due to absorption is $p_1 = 1 - \Lambda$ and the respective probability of radiation leaving the propagation channel due to scattering in the various directions is $p_2 = \mu_1\Lambda$, $p_3 = \mu_2\Lambda$, $p_4 = \mu_3\Lambda$, $p_5 = \mu_4\Lambda$. The total probability of a photon leaving the process due to sideway losses, as a result of scattering only, looks as follows:

$$p = \sum_{i=1}^{5} p_i = (1 - \Lambda) + \Lambda(\mu_1 + \mu_2 + \mu_3 + \mu_4). \tag{6}$$

The solution of the radiative transfer equation for the one-dimensional case can now be extended and formally written in the same form as formulas (2), where the parameters k and r are replaced by K and R

$$K = \sqrt{p[1 - \Lambda(\eta - \beta)]} = \sqrt{[(1 - \Lambda) + \Lambda(\mu_1 + \mu_2 + \mu_3 + \mu_4)][1 - \Lambda(\eta - \beta)]}, \tag{7}$$

$$R = \frac{K - p}{K + p} = \frac{\sqrt{1 - \Lambda(\eta - \beta)} - \sqrt{(1 - \Lambda) + \Lambda(\mu_1 + \mu_2 + \mu_3 + \mu_4)}}{\sqrt{1 - \Lambda(\eta - \beta)} + \sqrt{(1 - \Lambda) + \Lambda(\mu_1 + \mu_2 + \mu_3 + \mu_4)}}.$$

The form of formulas (7) has a heuristic character and can be justified by the need to account for "sideway losses" from a one-dimensional column. This need is actual even when such a column is surrounded by a medium with identical scattering properties. Note that I_3 in formulas (2) will then determine the total radiation flux both absorbed by the column and transferred through the column sides. According to Eq. (6) the fraction of the total radiation flux absorbed by the medium is given by the value p_1/p and the respective fractions exiting through the column sides — by the values p_1/p (for $i = 2$, 3 — in the direction of y axis and in the opposite direction, and for $i = 4$, 5 — in the direction of the z axis, and opposite).

The following step in extending the solution (2) of the one-dimensional case is dictated by the need to broaden its range of applicability; the latter should include the case in which the radiation source is contained within the propagation channel (i.e., between the top and bottom of the scattering column). Such an extension can be based on the successive application of the multiple reflection method (the layer adding method).

The essence of the multiple reflection method becomes transparent if we consider two layers homogeneous in their scattering properties of optical thicknesses τ_1 and τ_2. We will denote their transmittances as A_1 and A_2, their reflectances as B_1 and B_2, and their absorptances as C_1 and C_2, and their combined transmittance, reflectance, and absorptance as A_{12}, B_{12}, C_{12}, respectively. Then, according to the available solutions[9,11] the formulas for A_{12}, B_{12}, and C_{12} have the form

$$A_{12} = \frac{A_1A_2}{1 - B_1B_2}, \quad B_{12} = B_1 + \frac{A_1^2 + B_2}{1 - B_1B_2},$$

$$C_{12} = C_1 + \frac{A_1(B_2C_1 + C_2)}{1 - B_1B_2}. \tag{8}$$

Let the source of optical radiation be placed at the boundary between the scattering layers with optical thicknesses τ_1 and τ_2 so that it uniformly illuminates each of the layers with a parallel beam of intensity $I_0/2$ along the x axis (the same intensity goes into the opposite direction). Accounting for all the orders of scattering covered by formulas (8) for the radiation intensities exiting the combined layer of thickness $\tau_0 = \tau_1 + \tau_2$ in the direction of the x axis (I_{+x}), we obtain[11]

$$I_{+x} = \frac{I_0A_2(1 + B_1)}{2(1 - B_1B_2)}, \quad I_{-x} = \frac{I_0A_1(1 + B_2)}{2(1 - B_1B_2)},$$

$$I_\Lambda = \frac{I_0[C_1(1 + B_2) + C_2(1 + B_1)]}{2(1 - B_1B_2)} \tag{9}$$

After introducing the values of intensities (2) and (3) we obtain

$$I_{+x} = I_0\,\frac{(1 + R)[1 + R\exp(-2K\tau)]\exp(-K(\tau_0 - \tau))}{2[1 - R^2\exp(-2K\tau_0)]}.$$

$$I_{-x} = I_0\,\frac{(1 + R)[1 - R\exp(-2K(\tau_0 - \tau))]\exp(-K\tau)}{2[1 - R^2\exp(-2K\tau_0)]}.$$

$$I_\Lambda = I_0\left\{1 - \frac{(1 + R)[1 + \exp(-K\tau)]\exp(-K\tau)}{2[1 - R^2\exp(-K\tau_0)]}\right\} \tag{10}$$

where K and R are given by Eqs. (7) and, consequently, are uniquely determined by the integral parameters of the scattering phase function.

3. APPROXIMATE SOLUTION OF THE RADIATION TRANSFER PROBLEM FOR THE THREE-DIMENSIONAL CASE OF A SPATIALLY BOUNDED SCATTERING VOLUME

To retrieve the three-dimensional solution of the radiative transfer equation from the exact analytical one-dimensional solution we do as follows. First we treat the radiative transfer through elementary volumes in each of three mutually independent directions step by step and then combine the general solution from these. A similar approach is also implemented to solve the radiative transfer problem by the Monte-Carlo technique, in which the independent photon trajectories are modeled step by step as separate acts of interaction with scatterers, and the photons exiting the volume are summed in different combinations. In contrast to direct modeling (i.e., to the Monte-Carlo technique), using the one-dimensional solution to solve the three-dimensional problem opens up an attractive possibility to model the transfer process as whole chains of interaction acts. For each independent direction the exact three-dimensional solution can be envisaged as exactly such a separate and independent interaction chain.

Since the formulas for the one-dimensional case are written out (see the above section) in a form which accounts for the sideway losses, this approach can hopefully result in an analytical account of all the orders of scattering in any arbitrary direction.

Another principal factor in the obtaining of such three-dimensional solutions is that we use the well-known technique of multiple reflections. It is that particular technique that justifies the layer-by-layer summation of all the components of the direct and scattered radiation. Consistent application of this technique is what principally distinguishes our approach from the six-stream approximation and the four-parameter two-stream approximation. While in the latter case we not only have to resort to an approximate description of the scattering phase function, but must also content ourselves with approximate solutions for the various intensity components, in the proposed approach the accuracy of the solution is only limited by the approximate (six-stream) representation of the scattering phase function. Note that analytic solutions are used for the radiation intensities in the different directions.

To simplify the subsequent discussion we take as our scattering volume a rectangular parallelepiped with arbitrary side lengths τ_x, τ_y, and τ_z. Such a restriction of the shape of the volume is of no fundamental importance since application of the layer summing technique makes possible the retrieval of the necessary relations for a scattering volume of any shape. The final solution of the radiative transfer equation in the three-dimensional case can then be expressed by writing out the solutions for the elementary scattering volumes in each of the three cardinal directions. Combination of the one-dimensional solutions in three mutually perpendicular directions yields the solution of the three-dimensional problem. Because of the bulky form and complicated character of the Intermediate expressions for the radiation fluxes in the three cardinal directions we only present a simplified example, considering only one of these three directions (along the x axis).

Let an elementary scattering volume of the medium be uniformly illuminated along the x axis by a flux I_0. The flux transmitted in the x direction may be thought of as consisting of three components: the direct flux attenuated by the layer $d\tau$, the forward single-scattered flux $I_0 \eta \Lambda J_{+x} d\tau$, and the forward multiply scattered flux $I_0 4 \mu J_{+x} d\tau$, where J_{+x} is the fraction of the multiply scattered radiation in the x direction. The flux reflected from the elementary volume $d\tau$ is composed of two components: the backward singly scattered component $I_0 \beta \Lambda d\tau$, and the backward multiply scattered component $I_0 4 \mu J_{-x} d\tau$, where J_{-x} is the fraction of multiply scattered radiation in the $-x$ direction. The multiply scattered fluxes leaving the elementary scattering volume $d\tau$ in the y and z directions will be equal to $I_0 4 \mu J_{\mp y} d\tau$ and $I_0 4 \mu J_{\mp z} d\tau$ respectively. Finally the flux absorbed by the elementary layer will be equal to $I_0[(\eta + \beta)(1 - \Lambda) + 4 \mu J_\Lambda] d\tau$ where J_Λ is the fraction of the multiply scattered flux absorbed by the layer $d\tau$.

The main difficulty in the subsequent calculations consists in determining the fractions of multiply scattered radiation for each of the three cardinal directions. Such fractions for the fluxes J_{+x}, J_{-x}, and J_Λ at any optical thickness are given by formulas (2). As for the other directions the respective fractions of the multiply scattered radiation can be obtained from formulas analogous to (2), but after accounting for that fraction of radiation which exits the volume in the y and z directions, taken as the initial flux. In general such a step-by-step scheme of reasoning is quite cumbersome even for such a primitive shape as a parallelepiped.[11] Forgoing a treatment of this problem here, we note only that it is straightforward and simple, although cumbersome, and therefore possible to calculate for volumes of complex shape.[12–20] Also that the normalization condition for fractions of the multiply scattered radiation is obvious; it is derived from the law of energy conservation and has the form

$$I_0(J_\Lambda + J_{+x} + J_{-x} + J_{+y} + J_{-y} + J_{+z} + J_{-z}) = I_0. \quad (11)$$

We also stress that because of the finite size of the scattering volume in our example the fractions of radiation propagating along the y and z axes of the elementary layer $d\tau$ do not depend on the optical thicknesses τ_y and τ_z.

Assuming the fractions of multiply scattered fluxes to be known, let us determine the probability that a photon will exit our elementary layer. We denote the total probability that the photon will exit the volume as P_0; this value is equal to the sum of the components

$$P_0 = \sum_{i=1}^{5} P_{0i} , \quad (12)$$

Here $P_{01} = (\eta + \beta)(1 - \Lambda) + 4\mu J_\Lambda$ is the probability that the photon will leave the volume due to absorption;

$P_{02} = 4 \mu J_{+y} (\tau_y)$ is the same due to radiation exiting in the direction of the y axis;

$P_{03} = 4 \mu J_{-y} (\tau_y)$ – the same due to radiation exiting in the opposite direction;

$P_{04} = 4 \mu J_{+z}(\tau_z)$ – the same due to radiation exiting in the direction of the z axis;

$P_{05} = 4 \mu J_{-z} (\tau_z)$ – the same due to radiation exiting in the opposite direction.

Using the generalized one-dimensional solution of the radiative transfer equation we can again write the three-dimensional solution for an elementary layer in the form of Eqs. (2) with the parameters

$$K_0(\tau_y, \tau_z) = \sqrt{P_0(\tau_y, \tau_z)[1 - \Lambda(\eta - \beta)]},$$

$$R_0(\tau_y, \tau_z) = \frac{K_0(\tau_y, \tau_z) - P_0(\tau_y, \tau_z)}{K_0(\tau_y, \tau_z) + P_0(\tau_y, \tau_z)}. \quad (13)$$

In combination with the computational algorithms for calculating the respective variables relations (2) and (13) represent the desired semi-analytical three-dimensional solution of the problem of radiative transfer for a spatially bounded scattering volume. The principal advantage of the obtained solution is its high efficiency: computation of the intensity components along each direction can be performed using data which include only quite common and definite parameters of the medium and of the scattering volume as a whole.

The main source of possible errors in the these solutions stems from the approximate description of the scattering phase function in terms of its six-stream integral parameters. The accuracy of this solution can be estimated by comparing it to the exact solution for an unbounded scattering layer with a spherical scattering phase function.

For a scattering layer with $\tau_y = \tau_z = \tau_x = \infty$ and an arbitrary scattering phase function the intensity of the reflected radiation is given within the framework of the proposed approximation by the formula[21]

$$P_\infty = \frac{(1-\Lambda)\left\{(1-\Lambda)^2 + 4\mu\Lambda[3(1-\Lambda) - 3\mu(1-4\Lambda)\right.}{(1-\Lambda)^2 + 4\mu\Lambda(2 - 2\Lambda - \mu + 4\mu\Lambda)} \times$$

$$\times \frac{\left. - 4\mu^2(1+4\Lambda)]\right\}}{} \quad (14)$$

In the particular case of the Rayleigh phase function we have $\eta = \beta = 1/4$, $\mu = 1/8$. Substituting the these values of η, β and μ into Eq. (14) we obtain

$$P_{\infty Ray} = \frac{(1-\Lambda)\left(32 - 23\Lambda + 4\Lambda^2\right)}{2\left(16 - 17\Lambda + 4\Lambda^2\right)}. \quad (15)$$

For the spherical phase function $\eta = \beta = 1/6$ and the Eq. (14) becomes

$$P_{\infty sph} = \frac{(1-\Lambda)\left(27 - 11\Lambda + \Lambda^2\right)}{3\left(9 - 7\Lambda + \Lambda^2\right)}. \quad (16)$$

Table I compares the values of the reflectance for a semi-infinite medium R_∞ as function of the photon survival probabilities, calculated using different computational formulas for R_∞, with the exact solution, which is known in the literature,[17] and also with the results of two-stream and six-stream approximate calculations.[18] As can be seen from Table I, tie obtained solution of the transfer equation yields acceptable accuracy. It is the most accurate among the other approximate solutions in this asymptotic case.

TABLE I

Λ	R_{sph} exact solution	R_{sph} from Eq. (16)	R_{sph} (six-stream)	R_{sph} (two-stream)	R_{Ray} from Eq. (15)
0.1	0.0164	0.0167	0.0091	0.0263	0.0173
0.2	0.0352	0.0356	0.0200	0.0557	0.0365
0.3	0.0572	0.0575	0.0334	0.0889	0.0584
0.4	0.0834	0.0834	0.0501	0.1270	0.0838
0.5	0.1152	0.1148	0.0718	0.1715	0.1146
0.6	0.1554	0.1545	0.1010	0.2251	0.1517
0.7	0.2087	0.2073	0.1429	0.2922	0.2013
0.8	0.2853	0.2840	0.2087	0.3820	0.2732
0.9	0.415	0.416	0.333	0.519	0.399
0.95	0.536	0.539	0.461	0.634	0.521
0.975	0.641	0.647	0.578	0.727	0.629
1.0	1.0	1.0	1.0	1.0	1.0

4. CONCLUSION

The present review paper discusses the principal prerequisites for obtaining an approximate solution of the radiative transfer problem for the case of a spatially bounded scattering volume. Comparison of it with other approximate solutions and the exact solution demonstrates that for a wide range of scientific and practical tasks the accuracy of the new solutions is quite acceptable. We expect that the main application of the solution will be to the applied spectroscopy of various scattering media, where quantitative data are usually obtained from measurement within spatially bounded scattering volumes. The approximate solutions described above also simplify the task of computing the planetary radiation budget as well as other radiative properties of spherical atmospheres.

The main advantage of the obtained approximate semi-analytical solutions consists in the fact that the computational algorithm for radiation scattered by a spatially bounded volume of arbitrary shape appears to be simple enough at an acceptable accuracy level and not too demanding in terms of computational time. Therefore, operational calculations of the transmitted, side-scattered and reflected radiation for the volume as a whole appear possible if the optical and geometrical parametersof the scattering volume are known. This advantage would seem important for radiative studies of the atmosphere both from ground-based and from airborne platforms.

Note also that the efforts to obtain an approximate solution of the transfer equation described in the present study were stimulated by experimental results on the propagation of narrow laser beams through scattering media. The obtained solutions present a reliable basis for both qualitative and quantitative studies of the brightness contrast transfer for laser beams propagating through the turbid atmosphere. However, a discussion of the result from such an analysis as well as a presentation of the solutions obtained for that case are outside the scope of the present study.

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