RECONSTRUCTION OF THE PHASE FRONT IN A BASIS OF ORTHOGONAL FUNCTIONS FROM MEASUREMENTS WITH A HARTMAN TYPE SENSOR

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A numerical implementation of an algorithm for reconstructing a phase front in the form of an expansion in a system of orthogonal polynomials from the results of measurements of the partial derivatives of the phase front at the points of the aperture of a Hartman sensor is proposed. An example of the implementation of the algorithm using for the orthogonal basis Zernike polynomials in a Cartesian coordinate system is given.

The main element of adaptive optical systems (AOSs) for phase conjugation is the wavefront sensor. With its help the phase is measured at different points of the aperture of the optical system, after which the measurements are "joined together" and the distribution of the phase of the wavefront over the entire pupil is formed. Because of the specific nature of square-law detection, sensors of the interference and Hartman types[1,2] are most often employed in optics. Such sensors make it possible to measure the phase difference between neighboring sections of the aperture of local slopes of the phase front, which are proportional to quantities of the type $k^{-1}\cdot\dfrac{d\varphi(x, y)}{dx_{ij}}$ and $k^{-1}\cdot\dfrac{d\varphi(x, y)}{dy_{ij}}$, where k is the wave number and $\varphi(x, y)$ is a function describing the distribution of the phase on the aperture.

There exists an algorithm[3,4] for reconstructing the phase front from measurements of the partial derivatives at points of the aperture. In this algorithm, iIn processing the results of measurements from m by n subapertures, a system of $(m + 1)$ by $(n + 1)$ linear algebraic equations is solved. A recurrence procedure cannot be used to solve this system. For large m and n this results in larger amounts of computer time and limits the application of the algorithm indicated in real-time, while decreasing m and n increases the error of the approximation of the phase front. In the last few years interest in the application of flexible mirrors with response functions close to the orthogonal Zernicke polynomials as phase-front correctors has Increased in the technology of adaptive optics.[5] Here there arises the important problem of calculating on modern computers with minimum computer time the controlling signals for the mirrors.

In this paper we examine a method for reconstructing the phase front in the form of an expansion in a system of orthogonal functions based on measurements performed with a sensor of the Hartman type.

We shall study the following problem. Assume that on a rectangular aperture S with the dimensions $(a, b)\times(c, d)$, consisting of $m\times n$ subapertures, the Hartman sensor measures the values of the partial derivatives $\dfrac{d\varphi(x_i, y_j)}{dx}$ and $\dfrac{d\varphi(x_i, y_j)}{dy}$ at the center of each subaperture S_{ij}. We give on the intervals (a, b) and (c, d) two systems of orthogonal functions $\{\mu_k(x)\}$ and $\{\lambda_k(y)\}$, where $k = 0, \ldots, N$, from the space $C'(S)$, which satisfy the scalar products of the form

$$(\mu_k,\ \mu_r) = \int_a^b \mu_k(x)\mu_r(x)\rho(x)\ dx = 0 \quad \text{for } r \neq k,$$

$$(\lambda_k,\ \lambda_r) = \int_c^d \lambda_k(y)\lambda_r(y)\rho(y)\ dy = 0 \quad \text{for } r \neq k \tag{1}$$

and differential equations of the form[6]

$$\frac{d}{dx}[\sigma(x)\rho(x)] = \tau(x)\rho(x);$$

$$\frac{d}{dy}[\sigma_1(y)\rho_1(y)] = \tau_1(y)\rho_1(y) \tag{2}$$

under the conditions

$$x^m\sigma(x)\rho(x)\Big|_{x=a,b} = 0;\quad y^m\sigma_1(y)\rho_1(y)\Big|_{y=c,d} = 0,$$

$$(m = 0,\ 1, \ldots), \tag{3}$$

where

$$\sigma(x) = (x - a)(b - x);\qquad \sigma_1(y) = (y - c)(d - y);$$

$$\rho(x) = (b - x)^\alpha(x - a)^\beta;$$

$$\rho_1(y) = (c - y)^\gamma(y - d)^\eta;$$

$$\tau(x) = -(\alpha + \beta + 2)x + a + b + \alpha a + \beta b;$$

$$\tau_1(y) = -(\gamma + \eta + 2)y + c + d + \gamma c + \eta d.$$

The distorted phase front can be represented In the form[3,4]

$$\varphi(x,\ y) = \sum_{k=0}^{N} a_k\psi_k(x,\ y), \tag{4}$$

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where

$$\psi_k(x, y) = \mu_k(x)\lambda_k(y). \tag{5}$$

$$\frac{d\varphi(x, y)}{dx} = \sum_{k=0}^{N} a_k \frac{d\psi_k(x, y)}{dx};$$

The problem is to synthesize an algorithm for calculating the coefficients a_k an expansion of the form (4) from measurements of the partial derivatives of the phase front at points of the subaperture.

We differentiate the expression (4) with respect to x and y:

$$\frac{d\varphi(x, y)}{dx} = \sum_{k=0}^{N} a_k \frac{d\psi_k(x, y)}{dx};$$

$$\frac{d\varphi(x, y)}{dy} = \sum_{k=0}^{N} a_k \frac{d\psi_k(x, y)}{dy}. \tag{6}$$

Then the values of the partial derivatives of the phase front at the points of the aperture can be represented as

$$\frac{d\varphi(x_i, y_j)}{dx} = \sum_{k=0}^{N} a_k \frac{d\psi_k(x_i, y_j)}{dx};$$

$$\frac{d\varphi(x_i, y_j)}{dy} = \sum_{k=0}^{N} a_k \frac{d\psi_k(x_i, y_j)}{dy};$$

$$i = \overline{1, m}, \quad j = \overline{1, n}. \tag{7}$$

We obtain the coefficients a_k by minimizing a functional of the form

$$Q(a_0, a_1, \ldots, a_N) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{d\varphi(x_i, y_j)}{dx} - \sum_{k=0}^{N} a_k \frac{\psi_k(x_i, y_j)}{dx} \right]^2 + \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{d\varphi(x_i, y_j)}{dy} - \sum_{k=0}^{N} a_k \frac{\psi_k(x_i, y_j)}{dy} \right]^2, \quad i = \overline{1,m}; \quad j = \overline{1,n}. \tag{8}$$

Differentiating the expression (8) with respect to a_1 and equating the values of partial derivatives of the type $\dfrac{dQ(a_0, a_1, \ldots, a_N)}{da_1}$ to zero, we obtain a system of $N + 1$ linear equations:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{d\varphi(x_i, y_j)}{dy} \cdot \frac{d\psi_l(x_i, y_j)}{dy} + \frac{d\varphi(x_i, y_j)}{dx} \cdot \frac{d\psi_l(x_i, y_j)}{dx} \right] =$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\sum_{k=0}^{N} a_k \left[\frac{d\varphi_k(x_i, y_j)}{dx} \cdot \frac{d\psi_l(x_i, y_j)}{dx} + \frac{d\varphi_k(x_i, y_j)}{dy} \frac{d\varphi_l(x_i, y_j)}{dy} \right]\right], \quad N + 1 < m;$$

$$N + 1 < n; \quad l = \overline{0, N}; \quad i = \overline{1, m}; \quad j = \overline{1, n}. \tag{9}$$

We introduced the following notation:

$$\frac{d\varphi_k(x_i, y_j)}{dx} = P_{kij}; \quad \frac{d\psi_k(x_i, y_j)}{dx} = R_{kij};$$

$$\frac{d\varphi_k(x_i, y_j)}{dy} = L_{kij}; \quad \frac{d\psi_k(x_i, y_j)}{dy} = M_{kij}.$$

Taking into account the orthogonality of the derivatives of the polynomials satisfying conditions (1), (2), and (3), we write the solution of the system (9) in the form

$$a_k = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{L_{kij} M_{kij} + P_{kij} R_{kij}}{R_{kij}^2 + M_{kij}^2}. \tag{10}$$

In accordance with Rodrigues' formula[6] the partial derivatives of orthogonal polynomials can be represented by the following relations:

$$\frac{d\varphi_k(x, y)}{dx} = \frac{d\mu_k(x)\lambda_k(y)}{dx} = A_k A_k^1 \frac{d}{dx} \left[\frac{1}{\rho(x)} \cdot \frac{d^k}{dx^k} \times (\sigma^k(x)\rho(x)) \right] \cdot \left[\frac{1}{\rho_1(y)} \cdot \frac{d^k}{dy^k} \cdot (\sigma_1^k(y)\rho_1(y)) \right]; \tag{11}$$

$$\frac{d\varphi_k(x, y)}{dy} = \frac{d\lambda_k(y)\mu_k(x)}{dy} = A_k A_k^1 \frac{d}{dy} \left[\frac{1}{\rho_1(y)} \cdot \frac{d^k}{dy^k} \times (\sigma^k(y)\rho_1(y)) \right] \cdot \left[\frac{1}{\rho(x)} \frac{d^k}{dx^k} (\sigma^k(x)\rho(x)) \right]; \tag{12}$$

where A_k and A_k^1 are constants which depend on the normalization and are determined by the method presented in Ref. 6. The values of the partial derivatives $\dfrac{d\varphi_k(x_i, y_j)}{dy}$ and $\dfrac{d\varphi_k(x_i, y_j)}{dx}$ for all values of i, j, and k can be calculated beforehand.

To reconstruct the phase front using Eq. (10) $P = 3Nmn$ operations are required. The response functions of real mirrors may not satisfy the conditions (2) and (3). Even in this case, however, it is possible to construct an algorithm for reconstructing the phase front.

Assume that ψ_k in the expression (4) does not satisfy Eqs. (2) and (3) and is a function of the response of the flexible mirror. Then, introducing the notation

$$\frac{d\varphi(x_i,y_j)}{dy}\cdot\frac{d\psi_1(x_i,y_j)}{dy}+\frac{d\varphi(x_i,y_j)}{dx}\cdot\frac{d\psi_1(x_i,y_j)}{dx}=b_{1ij};$$

$$\frac{d\psi_k(x_i,y_j)}{dx}\,\frac{d\psi_1(x_i,y_j)}{dx}+\frac{d\psi_k(x_i,y_j)}{dy}\,\frac{d\psi_1(x_i,y_j)}{dy}=c_{k1ij}$$

the system (9) can be written as

$$\sum_{k=1}^{N}\sum_{i=1}^{m}\sum_{j=1}^{n}a_k c_{k1ij}=\sum_{i=1}^{m}\sum_{j=1}^{n}b_{1ij},\quad l=\overline{1,N}, \tag{13}$$

or in the matrix form

$$Da=F, \tag{14}$$

where D is the matrix on the right side of the system of linear equations (13) with the elements $d_{k1}=\sum_{i=1}^{m}\sum_{j=1}^{n}c_{k1ij}$; a is the ρ vector of the coefficients sought; F is the column vector of the right side of the system with the elements

$$f_1=\sum_{i=1}^{m}\sum_{j=1}^{n}b_{1ij};$$

The solution of the system (14) can be written

$$a=D^{-1}F. \tag{15}$$

The matrix D^{-1} for a AOS is calculated beforehand, since its elements do not depend on the local slopes of the phase front on the subapertures measured by a sensor of the Hartman type. Thus the processing of the measurements of the phase front in real time reduces to calculating the elements F in accordance with Eq. (14) and multiplying the matrix D^{-1} by F. The number of calculations required in so doing is equal to $P_1 = N(2N + 3mn - 1)$.

Example. Let the response function of the corrector correspond to the polynomials.[4] The first polynomial can beneglected, since it describes the important, for an AOS, average phase on the aperture.[4] Table I gives the values of the derivatives of the Zernicke polynomials for $N = 4$, written in a Cartesian coordinate system.

Then the matrix D assumes the form

$$\begin{vmatrix} \Sigma 4 & \Sigma 0 & \Sigma 8\sqrt{3}x & \Sigma 4\sqrt{6}y \\ \Sigma 0 & \Sigma 4 & \Sigma 8\sqrt{5}y & \Sigma 4\sqrt{6}x \\ \Sigma 8\sqrt{3}x & \Sigma 8\sqrt{3}y & \Sigma 54(x^2+y^2) & \Sigma 12\sqrt{2}xy \\ \Sigma 4\sqrt{6}y & \Sigma 4\sqrt{6}x & \Sigma 12\sqrt{2}yx & \Sigma 24(y^2+x^2) \end{vmatrix}$$

where the symbol Σ denotes the double sum $\sum_{i=1}^{m}\sum_{j=1}^{n}f(x_1, y_1)$, and x_1 and y_1 with $i=\overline{1,m}$, $j=\overline{1,n}$ are the given coordinates of the subapertures of the Hartman sensor on S. The matrix D is inverted once, before the calculations start. Further processing reduces to calculating the vector on the right side in accordance with Eq. (14) and solving the system Eq. (15).

TABLE I

	ψ_k	$\frac{d\psi_k}{dx}$	$\frac{d\psi_k}{dy}$
2	$2x$	2	0
3	$2y$	0	2
4	$\sqrt{3}(2x^2 + 2y + 1)$	$\sqrt{3}\,4x$	$\sqrt{3}\,4y$
5	$2\sqrt{6}\,xy$	$2\sqrt{6}\,y$	$2\sqrt{6}\,x$

The controlling programs implementing the algorithms (10) and (15) were written in the PL-1 language for the ES series computers.

Conclusions. In choosing the response functions of flexible mirrors the preference must be given to functions whose derivatives satisfy the conditions of orthogonality (2) and (3) or functions that are close to them. For such mirrors the volume of the calculations is reduced to a minimum. If this is impossible, then the measurements must be processed in accordance with Eq. (15). In this case the volume of calculations will also be reduced compared with the well-known algorithms,[3,4] since to reconstruct the phase front by the methods proposed in Refs. 3 and 4 with $n = m$, for example, not less than $2/3(n + 1)^6$ operations is required. The proposed algorithms can be easily implemented on a computer. The speed of operations of the AOS can be further increased by using a parallel computational scheme. For flexible mirrors with response functions close to the Zernicke basis the matrix D^{-1} is symmetric. This makes it unnecessary to calculate and store in the computer memory all $(N + 1)^2$ elements of the matrix; only $N(N + 1)/2$ elements need be stored.

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