ON THE POTENTIAL RESOLUTION OF PASSIVE IMAGE-FORMING METHODS THROUGH TURBULENT ATMOSPHERE. II. SPECKLE-INTERFEROMETRY IN SYNTHESIZED TELESCOPES

P.A. Bakut, I.A. Rozhkov, and A.D. Ryakhin

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An improved statistical model for the spatial spectrum of short exposure images distorted by the atmosphere is suggested. A more precise dependence of the resolution on observation conditions is derived for the case of a nonredundant synthesized telescope and small size targets. Quantitative estimates of the resolution are obtained for typical values of the parameters.

The previous study[1] considered classical speckle-interferometry (SI) designed to form the image in traditional telescopes with continuous apertures. It was demonstrated, in particular, that small-size target observations in reflected solar rays can provide resolutions up to $2 \cdot 10^{-8}$ rad, which corresponds to an aperture diameter of $D = 25$ m. For the analysis we employed a statistical model of images of short-exposure time (SEI) based on the so-called "5/3" – power law for the structural function of atmospheric distortions.[2] This law is known to be justified only in the case $D < L_a$, where L_a is the external turbulence scale; for $D \geq L_a$ the structure function is saturated. From this viewpoint the value of L_a approximately 10 m (Ref. 6).

The current state of the art limits the largest possible diameter of solid telescope mirrors to 10 m (Ref. 5). Quasisolid mirrors now being developed, which will consist of numerous separate segments, will have the same maximum diameter because of the difficulties of constructing a system to control the position of such segments within an accuracy of the order of the wavelength.[6] Therefore, the only practical way to achieve resolutions of 10^{-8} rad and less consists of using multi-aperture synthesized telescopes (MST). They comprise several separated independent receiver mirrors (subapertures) of moderate size. Operating in the regime of the coherent addition of light beams from separate subapertures, the MST opens up a possibility to achieve resolutions corresponding to the maximum separation distance between the mirrors. Several MST projects are now entering their practical implementation stage, in particular, the VLT system with a 104-m base.[7] MST's are characterized by poor filling of their synthesized apertures. From the point of view of forming images of maximum possible resolution the most promising among MST systems are those with low redundancy, in which the distances between their separate subapertures are not duplicated. It is well known that the diffraction response functions of strongly rarefied apertures have complex distributions with numerous interference maxima,[8,9] and that their random realizations distorted by the atmosphere differ appreciably from the ordinary SEI.[10] All these features necessitate refining of the results of the previous analysis. To do this we now transfer into the spatial frequency domain.

A description of SEI, mathematically equivalent to its description by the intensity distribution $J(\Theta)$, is the one employing the spatial spectrum (SS) $\tilde{J}(\vec{f})$, defined by a Fourier-transformation of the form

$$\tilde{J}(\vec{f}) = \int d\Theta J(\Theta) \exp\{2\pi i \vec{f}\Theta\} \quad (1)$$

Assuming isoplanar properties of the "atmosphere-telescope" system (ATS) the following representation of SS will be true:

$$\tilde{J}(\vec{f}) = E \cdot O(\vec{f}) \cdot H_0(\vec{f}) \quad (2)$$

where $E = \int d\Theta \cdot J(\Theta)$ is the SS energy; $O(\vec{f})$ is the SS of the desired target image to be estimated ($O(0) = 1$),

$$H_0(\vec{f}) = \frac{1}{T \cdot S_{ap} \cdot \Delta\lambda} \cdot \int_0^T dt \int_{\lambda_0-\Delta\lambda/2}^{\lambda_0+\Delta\lambda/2} d\lambda \cdot H(\lambda, t, \vec{f}) \quad (3)$$

is the ATS optical transfer function (OTF), and

$$H(\lambda, t, \vec{f}) = \int d\vec{v}\, W(\vec{v}) W(\vec{v} - \lambda\vec{f}) \times$$
$$\times \exp\{i[\varphi(\lambda, t, \vec{v}) - \varphi(\lambda, t, \vec{v} - \lambda\vec{f})]\} \quad (4)$$

is the "instantaneous" OTF. Here T and $\Delta\lambda$ are the temporal and spatial intervals of SEI recording; λ_0 is the average wavelength; S_{ap} is the total area of the receiving aperture; $W(\vec{v})$ is the aperture function, equal to unit within it, and to zero – outside the

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aperture; and $\varphi(\lambda, t, \vec{v})$ is a realization of atmospheric phase distortions at point $\vec{v}$ of the aperture, at instant of time t, and wavelength λ.

Essentially, SI processing is reduced to forming M recorded SEI estimates of the correlation functions from the SS having the form

$$\tilde{J}(\vec{f})\tilde{J}^*(\vec{f} + \Delta\vec{f})\rangle = E^2\cdot O(\vec{f})\cdot O^* \times$$

$$\times(\vec{f} + \Delta\vec{f})\cdot\langle H(\vec{f})H(f + \Delta\vec{f})\rangle \qquad (5)$$

The ensuing reconstruction of SS $O(\vec{f})$ from them is separated into various subtasks of reconstructing both modulus and phase, different in terms of realizations but having comparable resulting accuracy.[11,12] For simplicity we will focus on the first of these subtasks. The solution of this problem consists of normalizing the estimate of the averaged squared modulus

$$\langle|\tilde{J}(\vec{f})|^2\rangle = |O(\vec{f})|^2\cdot\langle|H\langle\vec{f}\rangle|^2\rangle \qquad (6)$$

to the independently measured or calculated transfer function $\left\langle\left|H(\vec{f})\right|^2\right\rangle$. The accuracy of this solution can naturally be characterized by the ratio Q of the true value of this square $\left|O(\vec{f})\right|^2$ to the mean error of its estimate. As noted in Ref. 1, only the quantum noises of recording has to be considered for as the principal error source when we determine the potential resolution. Now, on the one hand, if the mean area S_j of the correlation range for the SS $\tilde{J}(\vec{f})$ is noticeably larger than the similar area S_r for the SS $N(\vec{f})$ of the recording noises, then a preliminary smoothing of $\tilde{J}(\vec{f})$ over their correlation areas before averaging (6) becomes advisable; this procedure increases the accuracy Q by a factor of S_j/S_r. On the other hand, if the area S_0 of the correlation range for the SS $O(\vec{f})$ is significantly larger than S_j, it becomes advisable to smooth the estimate $\left|O(\vec{f})\right|^2$ instead; this procedure increases the accuracy by a factor of $\sqrt{S_0/S_j}$. Taking account of the mutual statistical independence of M realizations of the recording noise and denoting their variance $\left\langle\left|N(\vec{f})\right|^2\right\rangle$ as σ_r^2, and also considering the obvious inequality $S_r \le S_j \le S_0$, we obtain the following general expression for Q:

$$Q(\vec{f}) = \sqrt{M}\cdot\frac{E^2\cdot\sqrt{S_0\cdot S_j}}{\sigma_r^2\cdot S_r}\cdot|O(\vec{f})|^2\cdot\langle|H(\vec{f})|^2\rangle. \qquad (7)$$

From it we can evaluate the limiting resolution Q_r as the inverse maximum frequency f_r, at which $Q(\vec{f}_r) \ge 5$. To do this we expand its separate factors from (7).

Envisaging our model target as a square with a side Θ_0 with a constant distribution of reflected radiation intensity, we have for its SS

$$O(\vec{f}) = \mathrm{sinc}(\pi\cdot f_x\cdot\Theta_0)\cdot\mathrm{sinc}(\pi\cdot f_y\cdot\Theta_0), \qquad (8)$$

where f_x and f_y are the coordinates of the vector $\vec{f}$. It is easy to see that the largest "weights" belong to spectral values in the axial frequency ranges, where either $f_x \simeq f$ or $f_y \simeq f$. It follows that at frequencies most interesting for us ($f > \Theta_0^{-1}$) an asymptotic expression of the form

$$|O(\vec{f})|^2 \approx \frac{1}{2\cdot(\pi\cdot f\cdot\Theta_0)^2}, \qquad (9)$$

is valid and the correlation ranges for the SS $O(\vec{f})$ look like as ellipses with axes $1/2\ \Theta_0$ and $1/\Theta_0$ and with an area of $S_0 = \pi/(8\Theta_0^2)$.

It is just as easy to obtain an approximation for the correlation function of the SS of the recording noises:

$$\langle N(\vec{f})N^*(\vec{f} + \Delta\vec{f})\rangle = \int d\Theta\langle J(\Theta)\rangle\ \exp\{2\pi i\Delta\vec{f}\Theta\}. \qquad (10)$$

It follows in particular that the variance σ_r^2 is equal to energy E, and the area S_r of the correlation region is the inverse of angular area of the SEI. The total energy E of SEI, determined from the average number of recorded photons, is expressed as

$$E = \rho_0\cdot\Delta\lambda\cdot T\cdot S_{ap}\cdot\Theta_0^2\cdot\xi\cdot\eta. \qquad (11)$$

Here ξ is the transmittance for the MST optics; η is the quantum efficiency of the recorder used; and ρ_0 is the number of photons of light recorded from the target within unit solid angle, in a unit time interval per unit aperture area within a unit spectral bandwidth. The angular area of SEI is evaluated as $(\Theta_0 + 4\lambda_0/\pi\cdot r_{ef})^2$, where the quantity $4\lambda_0/\pi\cdot r_{ef}$ characterizes the angular size of the ATS response function. Parameter r_{ef} is similar to the Fried parameter r_0 (Ref. 13); it determines the effective average correlation region for atmospheric distortions of the light field in the MST aperture plane. However, in contrast to r_0, the new parameter takes account of the finite diameter D_T of the telescope subaperture and of the effect of compensating random, atmospherically induced wavefront declinations at the separate subapertures. While an equality of the form

$$\frac{\pi\cdot r_0^2}{4} = \int_{-\infty}^{\infty} d\vec{v}\ \langle\exp\{i[\varphi(\vec{v}_1)-\varphi(\vec{v}_1+\vec{v})]\}\rangle \qquad (12)$$

is valid for r_0 (Ref. 2), a different one holds for r_{ef}

$$\frac{\pi \cdot r_{ef}^2}{4} = \frac{1}{S_{ap}} \cdot \iint\limits_{-\infty}^{\infty} d\vec{v}_1 d\vec{v}_2 \cdot W(\vec{v}_1) W(\vec{v}_2) \cdot \langle \exp\{i[\psi(\vec{v}_1) - \psi(\vec{v}_2)]\} \rangle = \int d\vec{v} \cdot B(\vec{v}) \cdot \langle \exp\{i[\psi(\vec{v}_1) - \psi(\vec{v}_2)]\} \rangle. \tag{13}$$

Here $\psi(\vec{v}) = \varphi(\vec{v}) - \vec{a}\vec{v}$ decribes the phase distortions for the case of compensated declination $\vec{d}$ (vector $\vec{d}$ is different for each subaperture), and the weighting function

$$B(\vec{v}) = \frac{1}{S_{ap}} \cdot \int d\vec{v}_1 \cdot W(\vec{v}_1) W(\vec{v}_1 - \vec{v}) \tag{14}$$

determines the frequency of occurrence for the difference vector $\vec{v}$ between aperture points. Figures 1 and 2 present the dependences of r_{ef} on D_T and r_0 for separated subapertures (i.e., such that their individual distortions are practically independent of each other), following from the relationship (13). It can be seen that for small D_T/r_0 the value of r_{ef} coincides with D_T and for large values with r_0. In the intermediate interval the value of r_{ef} increases monotonically at first, reaching a maximum at $D_T/r_0 = 3.8$ ($r_{ef} = 1.9 \cdot r_0 = 0.5 \cdot D_T$), and then decreases monotonically, following the approximate relationship:

$$r_{ef} = r_0 \cdot \left[1 + 1.43 \cdot \sqrt{r_0/D_T}\right] \tag{15}$$

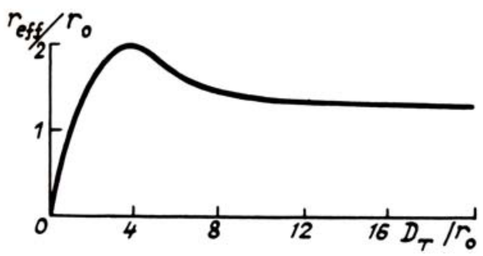

FIG. 1. The dependence of the ratio r_{eff}/r_0 on the ratio D_T/r_0.

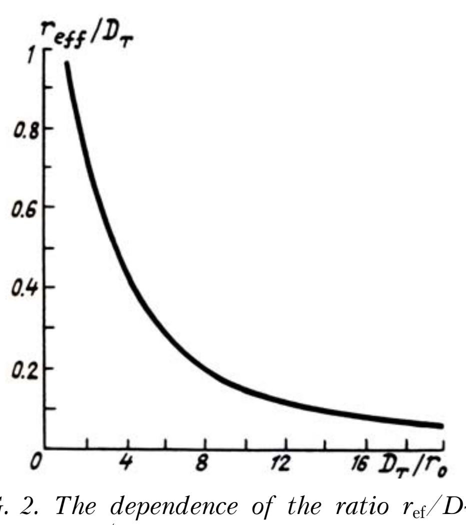

FIG. 2. The dependence of the ratio r_{ef}/D_T on the ratio D_T/r_0.

As for the characteristics of the OTF, following the technique employed in Refs. 2, 14, and 15, we obtain its approximate expression in the form

$$\langle H(\lambda_1, t_1, \vec{f}_1) H^*(\lambda_2, t_2, \vec{f}_2) \rangle = 0.34 \cdot \frac{r_{ef}^2}{S} \cdot B(\lambda \cdot \vec{f}) \times \exp\left\{-\pi \left[\frac{t_1 - t_2}{T_c}\right]^2\right\} \cdot \exp\left\{-\frac{1}{2} \cdot \left[\frac{\lambda_1 - \lambda_2}{\lambda_0}\right]^2\right. \tag{16}$$

where $D_a(\vec{v}) = \langle [\varphi(\vec{v}_1) - \varphi(\vec{v}_1 + \vec{v})]^2 \rangle$ is the structure function of atmospheric phase distortions; T_c is the time interval of SEI correlation. The function $B(\lambda \cdot \vec{f})$ has the meaning of the MST OTF. The first exponential factor here describes the decrease in correlation of the instrument OTF's due to their temporal incoherence; the second reflects the average effect of the difference in beam paths between subaperture points separated by the vector $\lambda\vec{f}$ from each other; the third accounts for the finite spatial area of correlation between the phase distortions. Since the third factor outweighs the second one in terms of limiting the spectral range, integrating over t and λ yields the OTF variance (3)

$$\langle |H(\vec{f})|^2 \rangle \approx 0.34 \frac{r_{ef}^2}{S_{ap}} \cdot B(\lambda_0 \vec{f}) \cdot \frac{T_k}{T + T_c} \cdot \frac{\Delta\lambda_k}{\Delta\lambda + \Delta\lambda_c}, \tag{17}$$

so that its correlation function is approximated by

$$\langle H(\vec{f}) H^*(\vec{f} + \Delta\vec{f}) \rangle = \langle |H(\vec{f})|^2 \rangle \cdot \exp\left\{-\frac{1}{2} \cdot \left[\frac{\Delta f_1}{f}\right]^2 \times D_a(\lambda f)\right\} \cdot \exp\left\{-1.72 \left[\frac{\lambda \Delta f_2}{r_{ef}}\right]^2\right\} \cdot Y(\vec{f}, \Delta f_1, \lambda, \Delta\lambda), \tag{18}$$

Here $\Delta\lambda_c = r_{ef} f$ is the spectral interval of correlation; Δf_1 is the value of the projection of $\Delta\vec{f}$ upon the direction normal to $\vec{f}$, and the function Y is described as

$$Y(\vec{f}, \Delta f_1, \lambda, \Delta\lambda) = \begin{cases} 1, & \text{if } |\Delta f_1| \le \frac{\Delta\lambda}{\lambda} \cdot f + \frac{r_{ef}}{\lambda}; \\ \text{else } 0 & \end{cases} \tag{19}$$

Analyzing the above expression we see that the correlation range of the OTF is elliptical in shape, with one of its axes parallel and the other perpendicular to the vector $\vec{f}$; the length of the first axis $\Delta f_\parallel$ is given by

$$\Delta f_\parallel = \min\left\{\Delta f_{eq}, \frac{\Delta\lambda + \Delta\lambda_c}{\lambda} \cdot f\right\}. \tag{20}$$

where $\Delta f_{el} = \sqrt{2\pi f / D_a(\lambda f)}$, and the second axis $\Delta f_\perp$ is equal to r_{ef}/λ. If $\lambda f \ll L_a$ when $D_a(\lambda f) = 6.88 \cdot (\lambda f/r_0)^{5/3}$ (Ref. 13), we have $\Delta f_{el} \approx r_0/\lambda$, while for $\lambda f \geq L_a$, $\Delta f_{el} = \sqrt{2\pi f / D_a(\infty)}$. According to estimates[3] we have $D_a(\infty) = 6 \cdot 10^2$ rad^2 at $\lambda = 0.6$ μm, so that $\sqrt{2\pi / D_a(\infty)} \approx 0.1$. Thus, while the OTF correlation region for common telescopes is practically limited to a circle of diameter r_0/λ (in the case $D_T \ll L_a$), in MST's of large separation distances this region acquires its shape of a drastically elongated ellipse, with its minor axis $\Delta f_\perp = r_{ef}/\lambda$ being constant, and its major axis depending on both the width $\Delta\lambda$ and the frequency f. All this holds for $\lambda f \geq L_a$. The major axis is the shortest when $\Delta\lambda \ll \Delta\lambda_c$; it is then equal to r_{ef}/λ; when $\Delta\lambda_c \leq \Delta\lambda \leq \Delta\lambda_{eq}$, where $\Delta\lambda_{el} = \lambda\sqrt{2\pi / D_a(\infty)} \approx 0.1 \cdot \lambda$, its lengths increases as $\Delta\lambda \cdot f/\lambda$ with larger $\Delta\lambda$, and, finally, at $\Delta\lambda \geq \Delta\lambda_{eq}$ its maximum value $\Delta f_{el} \gg \Delta f_\perp$ is reached. When observing a small-size target $\Theta_0 \leq 10 \cdot \Theta_r$ (this case is of greatest interest), whose SS correlation region is larger than Δf_{el}, it becomes desirable to increase the value of $\Delta\lambda$ to $\Delta\lambda_{el} \gg \Delta\lambda_c$ or even larger. Then the area S_j of the SS correlation region of SEI is evaluated as

$$S_j = \frac{\pi}{4} \cdot (\Delta f_\perp \cdot \Delta f_{eq}) = f \cdot r_{ef} / 10\lambda.$$

Combining the obtained relationships, we have for the ratio Q at $T \gg T_c$ (when its maximum is reached)

$$Q = 5.10^{-3}\sqrt{M} \cdot T_c \rho_0 \xi \eta \; B(\lambda f) \cdot \left[\frac{\lambda_0 \cdot r_{ef}}{f} \right]^{3/2} (\theta f)^{-1} \tag{21}$$

where the OTF $B(\lambda f)$ for nonredundant MST of N_T subapertures is estimated as

$$B(\lambda \cdot f) = 1/N_T , \tag{22}$$

and for a traditional telescope with a solid (or quasisolid) aperture of diameter D as

$$B(\lambda \cdot f) \approx 1 - f/f_q , \tag{23}$$

where $f_q = D/\lambda$ is the cutoff frequency. Note that the size of the subapertures D_T for MST is usually appreciably less than their maximum separation distance L. To simplify the estimation let us assume that $D_T = 10 \cdot r_0$ and hence that $r_{ef} = 1.6 \cdot r_0$. At the same time for a traditional telescope of limiting resolution $D \gg r_0$, and $r_{ef} = r_0$.

Substituting the typical values of $\rho_0 = 2 \cdot 10^{26}$ m$^{-3} \cdot$ s$^{-1} \cdot$ sr^{-1}, $T_c = 10^{-2}$ s, $\xi = 0.5$, $\eta = 0.2$, $\lambda_0 = 6 \cdot 10^{-7}$ m, $r_0 = 0.1$ m into the relationship (21) and assuming $Q = 5$ for an MST with $N_T = 6$, we obtain the relationship for the limiting resolution Θ_r in the form

$$\theta_r = (10^{-6}/M^{1/3}) \text{ rad} \tag{24}$$

It follows from (24) that for $M = 10^3$, $Q_r = 10^{-7}$ rad, fnd for $M = 10^5$, $Q_r = 2 \cdot 10^{-8}$ rad. At the same time the resolution of a traditional telescope for $f_r = (5/6) \cdot f_q$ is defined as

$$Q_r = (1.6 \cdot 10^{-6}/M^{1/3}) \text{ rad},$$

which yields $Q_r = 1.6 \cdot 10^{-7}$ rad for $M = 10^3$, and $Q_r = 3 \cdot 10^{-8}$ rad for M = 10^5.

Therefore, a more rigorous analysis produces a new expression for the limiting resolution, differing in its stronger dependence on the number of recorded images.

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