EFFECT OF SCATTERING BY RAINFALL ON THE FLUCTUATIONS OF AN OPTICAL IMAGE

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This paper presents the results of theoretical study of the random displacement of an optical image in a turbulent atmospheric layer under precipitation conditions. The variance of the image displacement is calculated on the basis of a solution of the equation for the fourth-order field coherence function using a perturbation technique. The conditions under which the contribution of atmospheric precipitation to the fluctuations of the displacement of the optical image centroid exceeds that of the atmospheric turbulence have been found. It is shown that atmospheric precipitation has a significant effect on the fluctuations of the displacement of the IR image centroid for virtually any size of the receiving aperture.

The phenomenon of "jitter" in the image of an optical source observed through the turbulent atmosphere was studied in detail in Refs. 1–3. Image jitter in the sharp image plane of the receiving lens is known to be due to phase fluctuations of the optical wave in a turbulent atmospheric layer.[1,2] However, phase fluctuations also occur in the case of wave scattering by an ensemble of discrete particles.[4,5] It follows from theoretical calculations[4] that the greatest distorting influence on the phase fluctuations of optical radiation is exerted by the largest atmospheric scatterers-hydrometeors (raindrops, in particular). A comparison of the experimental data with theoretical calculations has shown[4–13] that the process of optical propagation through rainfall can be adequately described by representing raindrops as "optically soft" spherical particles uniformly distributed in space. The present paper presents a theoretical study of random displacements of the image of a light beam passing through a turbulent atmospheric layer in rainfall.

Consider the propagation of a light beam with Gaussian distribution of the wave field in the plane $x' = 0$

$$u(0,\ \rho) = u_0(\rho) = u_0 \exp\left\{-\frac{\rho^2}{2a_0^2} - \frac{ik}{2F}\rho^2\right\}, \tag{1}$$

where u_0, a_0, and F are the field amplitude, the beam radius, and the radius of curvature of the beam wavefront in the center of the radiating aperture; $\rho = \sqrt{y^2 + z^2}$, $k = 2\pi/\lambda$, and λ is the wavelength. An optical beam traverses a path length x and is received by a telescope lens with amplitude transmission coefficient $T(\rho)$ and focal length F_t, to describe which we use the Gaussian approximation

$$T(\rho) = T_0 \exp\left[-\frac{\rho^2}{2a_t^2}\right],$$

where T_0 is the transmission coefficient of the telescope along the optical axis of the system and a_t is the radius of the receiving lens.

Defining the random displacements of the optical image of the wave beam as changes in the coordinates of the intensity distribution centroid behind the lens (see Ref. 2), we obtain the following expression for the variance of the image displacement:

$$\sigma^2 = -\frac{l^2}{k^2}\int d^2\rho_2 \int d^2\rho_4 T(\rho_2)T(\rho_4) \exp\left[\frac{ik}{2l}\left(1 - \frac{l}{F_t}\right)\times\right.$$
$$\times (\rho_2^2 + \rho_4^2)\, \nabla_{\rho_2}\nabla_{\rho_4}\left\{T(\rho_2)T(\rho_4)\Gamma_4(x,\ \underline{\rho_4}) \times\right.$$
$$\left.\left.\times \exp\left[-\frac{ik}{2l}\left(1 - \frac{l}{F_t}\right)(\rho_2^2 + \rho_4^2)\right]\right\}\right|_{\substack{\rho_1 = \rho_2,\\ \rho_3 = \rho_4}} \times$$
$$\times \left\{\int d^2\rho_2 \int d^2\rho_4 T^2(\rho_2)T^2(\rho_4)\Gamma_4(x,\ \underline{\rho_4})\Big|_{\substack{\rho_1 = \rho_2,\\ \rho_3 = \rho_4}}\right\}, \tag{2}$$

where l is the distance from the lens plane to the observation plane,

$$\Gamma_4(x,\ \underline{\rho_4}) = \Gamma_4(x,\ \rho_1,\ \rho_2,\ \rho_3,\ \rho_4)$$
$$= \left\langle \prod_{l=1}^{2} u(x,\ \rho_{2l-1})u^*(x,\ \rho_{2l})\right\rangle,$$

is the fourth-order moment of the field, $U(x, \rho)$ is the complex amplitude of the wave field incident upon the receiving lens, which satisfies the parabolic equation[2,10] for a wave propagating through a medium with continuous and discrete inhomogeneities. An expression for $\Gamma_4(x, \rho_4)$ can be derived from the solution of the relevant equation for the fourth-order moment of the optical field[10–12]

$$\left[\frac{\partial}{\partial x'} - \frac{i}{2k}(\Delta_1 - \Delta_2 + \Delta_3 - \Delta_4) + \frac{\pi k^2}{4} f(\rho_4)\right] \times \Gamma_4(x', \rho_4) = 0,$$

where Δ_j is the Laplacian in the plane normal to the direction of propagation with coordinates y_j and z_j of the point ρ_j, $j = 1, 2, 3, 4$;

$$f(\rho_4) = f(\rho_1, \rho_2, \rho_3, \rho_4) = H(\rho_1 - \rho_2) + H(\rho_1 - \rho_4) + H(\rho_3 - \rho_2) + H(\rho_3 - \rho_4) - H_2(\rho_1 - \rho_3) - H_2(\rho_2 - \rho_4);$$

$$H_{1,2}(\rho) = 2\int d^2\varkappa(1 - \cos \varkappa\rho)\, \Phi_\varepsilon(\varkappa) = \frac{2}{\pi} 0.73 C_\varepsilon^2 \rho^{5/3} \pm \frac{4m_0}{k^2}\int_0^\infty da\, p(a)\, a^2\left[1 - \exp\left(-\frac{\rho^2}{a^2}\right)\right],$$

C_ε^2 is the structure constant of fluctuations in permittivity describing the optical strength of the atmospheric turbulence, m_0 is the mean concentration of scattering particles (raindrops), $p(a)$ is the particle size distribution function, and $\bar{a}$ is the mean radius of the raindrops.

For weak fluctuations in the optical intensity $\beta_0^2 + \tau < 1$, (where $\beta_0^2 = 0.31 C_\varepsilon^2 k^{7/6} x^{11/6}$ is the variance of the intensity fluctuations of a plane wave in a turbulent atmosphere and $\tau = 2\pi m_0\ \bar{a}^2 x$ is the optical thickness of the scattering layer) the equation for the fourth-order moment of the optical field can be solved using a perturbation technique.[2,3] The variance of the random displacements of the optical image centroid (hereafter referred to as the variance) can be represented in the form of a sum of two terms

$$\sigma^2 = \sigma_t^2 + \sigma_d^2, \tag{3}$$

due to the turbulent and discrete inhomogeneities of the medium, respectively:

$$\sigma_t^2 = \frac{1}{2}\sigma_{t_0}^2 \frac{l^2}{F_t^2}\int_0^1 d\xi \times \left[PP^*NN^*\beta_1^{-1/6} + \mathrm{Re}(P^2N^2\beta_2^{-1/6})\right],$$

$$\sigma_d^2 = 4\pi\frac{m_0 x l^2}{k^2}\int_0^1 d\xi \int_0^\infty da\, p(a) \times \left[PP^*NN^*\beta_3^{-2} + \mathrm{Re}(P^2N^2\beta_4^{-2})\right],$$

$$P = \xi - \frac{i\Omega(1-\xi)}{m}, \quad N = 1 - \frac{i\alpha\Omega_t}{M},$$

$$M = 1 - \frac{\Omega\,\Omega_t}{m\,m^*}, \quad m = 1 - i\Omega\left[1 - \frac{x}{F}\right],$$

$$\alpha = 1 + \frac{x}{l}\left[1 - \frac{l}{F_t}\right] - \frac{\Omega^2\left[1 - \frac{x}{F}\right]}{m\,m^*},$$

$$\beta_1 = \frac{1}{2M}\left[P^2 + P^{*2}\right] + 2\frac{(1-\xi)^2\Omega}{m\,m^*\Omega_t},$$

$$\beta_2 = \frac{1}{M}P^2 + 2\frac{i(1-\xi)P}{\Omega_t},$$

$$\beta_3 = 1 + \frac{(P^2 + P^{*2})}{M}\left[\frac{a_t}{a}\right]^2 + 4\frac{(1-\xi)^2}{m\,m^*}\left[\frac{a_0}{a}\right]^2,$$

$$\beta_4 = 1 + 2\frac{P^2}{M}\left[\frac{a_t}{a}\right]^2 + 4id(1-\xi)P,$$

$\sigma_{t_0}^2 = \dfrac{\pi^2}{2}A_0\Gamma(1/6)C_\varepsilon^2 \times F_t^2(a_t^2/2)^{-1/6}$ is the variance of the displacements of the centroid of the image of a plane wave in a turbulent atmosphere, $A_0 = 0.033$, $\Omega = ka_0^2/x$, $\Omega_t = ka_t^2/x$ are the Fresnel numbers for the transmitting and receiving apertures, respectively, $d = x/(ka^2)$ is the wave parameter describing the near and far fields for a raindrop with radius a, and $d_0 = x/(k\bar{a}^2)$.

Let us examine σ_d^2 from Eq. (3). For a plane wave ($\Omega \to \infty$, $F \to \infty$) propagating through a monodisperse ($p(a) = \delta(a - \bar{a})$) discrete scattering medium, the variance in the sharp image plane (in the Gaussian plane), i.e., $\alpha = 0$, assumes the form

$$\sigma_{d,pl}^2 \simeq 4\pi\frac{m_0 x F_t^2}{k^2}\left\{\frac{2\left[1 + 2\left[a_t/\bar{a}\right]^2\right]^2 + 16d_0^2}{\left[1 + 2(a_t/\bar{a})^2\right]^2} \times \frac{1}{\left[(1 + 2(a_t/\bar{a})^2)^2 + 16d_0^2\right]}\right\}.$$

In the case of a light wave propagating through rainfall where $\bar{a} \sim 10$ m, the parameter d_0 is much greater than unity starting with path lengths $x > 10$ m. Therefore, for small receiving apertures ($a_t < \bar{a}$) the variance for a plane wave passing through rainfall is determined by the simple analytical expression

$$\sigma^2_{d,pl} \simeq 4\pi \frac{m_0 x F_t^2}{k^2} . \tag{4}$$

For large apertures ($a_t > \bar{a}$) we have

$$\sigma^2_{d,pl} \simeq 4\pi \frac{m_0 x F_t^2}{2} \beta \left[1 + 2(a_t/\bar{a})^2 \right]^{-2} , \tag{5}$$

where

$$\beta = \begin{cases} 1 & \text{for } \Omega_t < 1 , \\ 2 & \text{for } \Omega_t > 1 . \end{cases}$$

It can be shown that for $x = 10^2$ m and $F_t = 1$ m for moderate rainfall ($\bar{a} = 10^{-3}$ m, $m_0 \sim 10^{-3}$), for $\lambda = 0.63$ μm, $\sigma^2_{d,pl}$ is equal to 10^{-8} m^2 ($a_t < \bar{a}$), 10^{-13} m^2 ($a_t = 10\bar{a}$) and 10^{-17} m^2 ($a_t = 100\bar{a}$), respectively. For infrared radiation propagating through the atmosphere (e.g., at $\lambda = 10.6$ μm) discrete scatterers will have a stronger effect on image jitter: $\sigma^2_{d,pl} \sim 10^{-5}$ m^2 for $a_t < \bar{a}$, $\sigma^2_{d,pl} \sim 10^{-10}$ m^2 for $a_t = 10\bar{a}$, and $\sigma^2_{d,pl} = 10^{-14}$ m^2 for $a_t = 100\bar{a}$.

It should be noted that under these conditions the variance component attributed to the turbulent inhomogeneities is approximately equal to 10^{-15}–10^{-11} m^2 for $C_\varepsilon^2 \sim 10^{-17}$–$10^{-13}$ m$^{-2/3}$. Thus, for receiving apertures $a_t \le 10$ cm, the fluctuations in the image centroid of an IR source during rainfall are related only to the discrete scatterers (raindrops). For large apertures ($a_t > 10$ cm), the main contribution to the jitter of the optical image comes from the atmospheric turbulence. Discrete scatterers do affect the fluctuations in the optical image centroid in the visible range ($\lambda < 10^{-6}$ m) but only for receiving apertures $a_t < 1$ cm.

If the observation plane is displaced from the paraxial image plane, the variance $\sigma^2_{d,pl}(l)$ changes according to the law

$$\frac{\sigma^2_{d,pl}(l)}{\sigma^2_{d,pl}} = \begin{cases} \frac{l^2}{F_t^2}\left[1 + \frac{x}{2l}\left(1 - \frac{l}{F_t}\right)\Omega_t d_0^{-1} \frac{x^2}{l^2}\left(1 - \frac{l}{F_t}\right)^2 \Omega_t^2\right] & \text{if } a_t < \bar{a} \\ \frac{l^2}{F_t^2}\left[1 + \frac{x}{l}\left(1 - \frac{l}{F_t}\right)\Omega_t^2 + \frac{x^2}{l^2}\left(1 - \frac{l}{F_t}\right)^2 \Omega_t^2\right] & \text{if } a_t > \bar{a},\ \Omega_t < 1, \\ \frac{l^2}{F_t^2}\left[1 + \frac{2x}{l}\left(1 - \frac{l}{F_t}\right) + \frac{2x^2}{l^2}\left(1 - \frac{l}{F_t}\right)^2\right] & \text{if } a_t > \bar{a},\ \Omega_t > 1. \end{cases}$$

The larger the distance from the sharp image plane, the larger is the variance.

It is necessary to account for the effect of the polydisperse character of a discrete scattering media (e.g., the turbid atmosphere) on the variance only for $a_t > \bar{a}$, in which case Eq. (5) is replaced by the relation

$$\sigma^2_{d,pl} \simeq \pi \frac{m_0 x F_t^2}{k^2} \beta \frac{(\gamma + 3)(\gamma + 2)(\gamma + 1)}{\gamma^3} \left[\frac{\bar{a}}{a_t} \right]^4 .$$

Here γ is the parameter of the Γ-distribution[4] ($\gamma = 1, 2, 3$). It is evident that this substitution does not lead to any fundamental changes in the obtained results.

The variance of the image jitter can be roughly estimated from simple qualitative considerations. Tatarskiĭ[1] has derived the following expression for the variance:

$$\sigma^2 = \theta \frac{D_s(2a_t) F_t^2}{k^2 (2 a_t)^2} ,$$

where θ is a constant and $D_s(\rho)$ is the structure function of the wave phase fluctuations. For a monodisperse discrete scattering medium the phase fluctuations of a plane wave are determined by the relation[14]

$$S(x, \rho) = \frac{1}{x} \cos\left[\frac{k \rho^2}{2x}\right] \mathrm{Im} \left\{ f_0\left[k\bar{a} \frac{\rho^2}{x^2}\right] \right\} ,$$

where $f_0\left(k\bar{a}\frac{\rho^2}{x^2}\right)$ is the optical wave scattering function for the wave number k and particle radius $\bar{a}$. Therefore, it is easy to show for a plane wave that

$$D_s(\rho) = 2 m_0 \int_0^I \frac{dx}{x} \iint_{-\infty}^{\infty} d\rho \times$$
$$\times \left\{ 1 - \cos\left[\frac{k \rho'^2}{2x}\right] \cos\left[\frac{k(\rho' - \rho)^2}{2x}\right] \right\} \times$$
$$\times f_0\left[k\bar{a} \frac{\rho'^2}{x^2}\right] f_0^*\left[k\bar{a} \frac{(\rho' - \rho)}{x}\right] .$$

For small values of ρ ($\rho < \bar{a}$) $D_s(\rho) \simeq 2\pi \cdot m_0 \cdot x \cdot \rho^2$, i.e., for $a_t < \bar{a}$, the variance for a plane wave has the form

$$\sigma^2 \simeq 2\pi\theta \frac{m_0 x F_t^2}{k^2} .$$

Comparison with Eq. (4) shows that heuristic and rigorous calculations agree to within the constant θ ($\theta = 2$) for $a_t < \bar{a}$.

For a spherical wave ($\Omega = 0$), the variance in the Gaussian plane ($\alpha = 0$) of the receiving lens is written as

$$\sigma^2_{d,sph} \simeq 4\pi \frac{m_0 x F_t^2}{k^2} \int_0^1 d\xi \left\{ \frac{\xi^2}{[1 + 2(a_t/\bar{a})^2 \xi^2]^2} + \right.$$

$$+\frac{[1+2(a_t/\bar{a})^2\xi^2]^2\xi^2-16d_0^2(1-\xi)^2\xi^4}{[(1+2(a_t/\bar{a})^2\xi^2)^2+16d_0^2(1-\xi)^2\xi^2]^2}\Big\}.$$

For $a_t < \bar{a}$

$$\sigma^2_{d,sph} = \frac{4\pi\ m_0 x l_t^2}{3\cdot k^2}, \qquad (6)$$

and for $a_t > \bar{a}$

$$\sigma^2_{d,sph} \simeq \frac{\pi^2\beta}{2\sqrt{2}}\cdot\frac{m_0 x l_t^2}{k^2}\left[\frac{a_t}{\bar{a}}\right]^{-3}. \qquad (7)$$

Thus, the variance for a plane wave is three times that for a point source ($a_t < \bar{a}$). On the other hand, as a_t increases ($a_t > \bar{a}$), $\sigma^2_{d,sph}$ decays slower than $\sigma^2_{d,pl}$. The two variances decrease as a_t^{-3} and a_t^{-4}, respectively.

For a collimated beam ($F = \infty$) the expression for the variance in a discrete scattering medium (e.g., the turbid atmosphere) has a cumbersome form, therefore, we present here only asymptotic expressions for σ_d^2 ($F \to \infty$). For small apertures ($a_t < \bar{a}$) there are two subranges of the asymptotic behavior of the variance

1. $\{\Omega_t, \Omega/(1+\Omega^2)\} < d_0^{-1} < 1$

$$\sigma_d^2(F\to\infty) \simeq 4\pi\frac{m_0 x\ l^2}{k^2}\ \frac{1/3+\Omega^2}{1+\Omega^2}, \qquad (8)$$

2. $\Omega_t < d_0^{-1} < \Omega/(1+\Omega^2) < 1$, where

$$\sigma_d^2(F\to\infty) \simeq 4\pi\frac{m_0 x\ l^2}{k^2}\ \frac{\pi\sqrt{1+\Omega^2}}{8\sqrt{\Omega\ d}}. \qquad (9)$$

Asymptotic formula (8) goes over to the case of a spherical wave (6) as $\Omega \to 0$ and for a plane wave (4) as $\Omega \to \infty$.However, because of the condition $a_0/\sqrt{1+\Omega^2} < \bar{a} < \sqrt{x/k}$ Eq. (8) is not valid for $\Omega \sim 1$. As follows from Eq. (9), for $\Omega \sim 1$ the variance is found to decrease more significantly than $\sigma^2_{d,sph}$ or $\sigma^2_{d,pl}$. This is because of the fact that the optical fluctuations in discrete scattering, turbid medium are due to the screening of the receiver by raindrops[4–13] and, therefore, a decreased probability P of particles entering into the beam (P decreases as the scattering volume decreases) leads to lower optical fluctuations. For small receiving apertures σ_d^2 ($F = \infty$) (Eqs. (8) and (9)) is practically invariant with respect to the size of the receiving aperture, i.e., the "image jitter saturation effect" is observed. As the radius of the receiving aperture is increased ($a_t > \bar{a}$) the variance is dramatically decreased. In particular, for $d_0^{-1} < \Omega_t < \Omega/(1+\Omega^2) < 1$, σ_d^2 is described by formula (9), and for $\{d_0^{-1}, \Omega/(1+\Omega^2)\} < \Omega_t < 1$, goes over to the case of a spherical wave (7) for $\Omega < \bar{a}/a_t < 1$, or a plane wave (5) for $\Omega > \bar{a}/a_t$.

For a focused beam ($F = x$) and small receiving apertures ($a_t < \bar{a}$)

$$\sigma_d^2(F=x) = \frac{4\pi\ m_0 x\ l^2}{3\cdot k^2}(1+\Omega^2)\ \text{for}\ \{\Omega_t,\ \Omega\} < d_0^{-1},$$

$$\sigma_d^2(F=x) \simeq \frac{4\pi\ m_0 x\ l^2}{k^2}\ \frac{\pi}{8\sqrt{\Omega d}}\ \text{for}\ \Omega_t < d_0^{-1} < \Omega.$$

It should be noted that fluctuation averaging over the transmitting aperture results in a dramatic decrease in the image jitter for a focused beam. For large receiving apertures ($a_t > \bar{a}$), the variance is small. In particular, when $\{\Omega, d_0^{-1}\} < \Omega_t < 1$, for $\Omega < \bar{a}/a_t$ d_0^{-1} $(F = x) \simeq \sigma^2_{d,sph}$ and for $d_0^{-1} < \Omega_t < \{1, \Omega\}$, $\sigma_d^2(F=x)\sim\dfrac{1}{\sqrt{\Omega\cdot d_0}}$.

Finally, let us consider the case in which the contribution of rainfall to the fluctuations in the displacement of the optical image centroid exceeds that of atmospheric turbulence. For a plane wave this happens when

$$a_t \lesssim \left[\frac{\pi}{0.28}\ \frac{m_0 a^{-4}}{C_\varepsilon^2\ k^2}\right]^{3/11}.$$

Thus rainfall has a significant effect on the propagation of IR radiation through the atmosphere for any size of the receiving aperture, whether small or large. In the visible range its influence is important only for $a_t \lesssim 1$ cm. The relations derived in this work can be used in the development of remote sensing techniques for measuring optical characteristics of atmospheric turbulence and precipitations. For instance, the simultaneous reception of a light beam by a large aperture and a small one or the reception of two optical waves of different wavelengths by a single aperture should make it possible to measure the mean raindrop concentration and the structure parameter of the fluctuations of the air permittivity during rainfall.

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