SPATIAL FILTRATION IN PROBLEMS ON WAVEFRONT MEASUREMENTS BY THE HARTMANN METHOD

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A problem of wavefront measurement has been studied from the viewpoint of the linear systems theory. The transfer function of the measuring device is given. Formulas for estimating the measurement error and the optimal parameters of the measuring device have been derived.

Wavefront measurement In the study of various physical processes and quality control technical objects is an urgent problem. Recently there has been an increase in interest in wavefront measuring devices in conjunction, with the development of wavefront control systems.[1] In practice the measured wavefront deviates from the actual one. This is caused by the nonideal character of the measuring device, noise associated with the small-scale component of the wavefront in linear adaptive systems which compensate for the low-frequency wavefront component, etc.

It was previously noted[2] that the diaphragm (screen) used in the measurements in the Hartmann method acts as a filter of the wavefront spatial frequencies. Let us now consider the problem of optimization of the measurement process within the framework of the above-indicated approach.

Let the difference $\Delta\varphi(\rho)$ between the actual and measured values of the phase be represented in the form[3]

$$\Delta\varphi(\rho) = \int\left\{\left[1 - K(\omega)\right]S(\omega) - \xi(\omega)K(\omega)\right\}\exp(-i\omega\rho)\,d^2\omega,$$

where $S(\omega)$ and $\xi(\omega)$ are the spatial Fourier spectra of actual wavefront $S(\rho)$ and additive noise wavefront $\xi(\rho)$, $\rho = r/R$, r is the radius vector of a point on the wavefront, R is the beam radius, and $K(\omega)$ is the transfer function of the wavefront measuring device. Let $S(\rho)$ and $\xi(\rho)$ be statistically uncorrelated random fields with zeros expectations $\overline{S(\rho)}$ and $\overline{\xi(\rho)}$ (the bar above the symbol denotes statistical averaging) and spectral power densities $G_s(\omega)$ and $G_\xi(\omega)$, respectively. The formula for the rms measurement error is obtained by squaring and subsequent statistical averaging

$$\sigma^2 = \int\left\{|1 - K(\omega)|^2 G_s + |K(\omega)|^2 G_\xi\right\}d^2\omega. \tag{1}$$

The Hartmann method makes it possible to measure local wavefront tilts. In this case we are interested in the mean square wavefront tilt

$$g^2 = \frac{1}{P}\int_P \left(\text{grad}\,\Delta\varphi(\rho)\right)^2 d^2\rho,$$

where P is the beam area expressed as a function of ρ. An expression for g^2 can be obtained from the formula for $\Delta\varphi$ by differentiation

$$g^2 = \int\omega^2\left\{|1 - K(\omega)|^2 G_s + |K(\omega)|^2 G_\xi\right\}d^2\omega. \tag{2}$$

The Wiener–Hopf theorem identifies the optimal transfer function which minimizes the error as

$$K_{opt}(\omega) = G_s(\omega)/\left[G_s(\omega) + G_\xi(\omega)\right]. \tag{3}$$

In practice the function given by Eq. (3) is approximated by realizable functions with the goal of constructing a quasioptimal filter.

Let us now consider the transfer function of a real wavefront measuring device. In measuring with the Hartmann sensor the information about the phase (more correctly, about the local wavefront tilt) is attributed to a small but finite element of the area from which the energy necessary for the measurement is collected. Employing the spectral representation of the wavefront it is not difficult to derive the following expression for the transfer function of the measuring device:

$$K_0(\omega) = \frac{\int F(\rho)\exp(i\omega\rho)\,d^2\rho}{\int F(\rho)\,d^2\rho},$$

where $F(\rho)$ is the transmission function of the averaging aperture (hole). For a circular aperture with radius r_0

$$K_0(\omega) = 2J_1(\omega\rho_0)/(\omega\rho_0). \tag{4}$$

Here $\rho_0 = r_0/R$ and J_1 is the first-order Bessel function.[5]

Formulas (1) and (2) make it possible to estimate the measurement error on the basis of known statistics

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of the signal and noise. With proper choice of the size of the averaging aperture ρ_0 the transmission function (4) can be made to approach the optimal one. If the size of the averaging aperture is fixed, we can construct a quasioptimal transfer function with the help of an additional spatial filter,[4] a diagram of which is shown in Fig. 1. The lens *2* is used for the Fourier transform of the wavefront, which is measured in the plane *1*. The lens *4* performs the Fourier transform for the spatial frequencies transmitted by the filter *3*. The filter shown in Fig. 1 is characterized by the following transmission function for small phase aberrations:

$$K_1(\omega) = \begin{cases} 1, & \omega \in P_1, \\ 0, & \omega \notin P_1, \end{cases} \tag{5}$$

where $w = kRr/f$, k is the wave number, r is the radius vector of a point on the diaphragm *3*, P_1 is the set of points making up the circular hole in the diaphragm *3*. The system "filter-Hartmann sensor" has the transmission function $K_0(w)K_1(w)$.

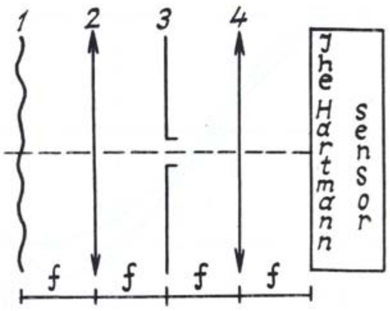

FIG. 1. Filtration diaphragm: f is the focal length of lenses 2 and 4; 1 is the wavefront being measured, and 3 is the spatial filter.

Let us now study an example of the choice of the optimal parameters for the wavefront measuring device. Let the signal have the spectral power density

$$G_s(\omega) = \left(\sigma_s^2 G_s^2/4\pi\right)\exp\left(-\omega^2 G_s^2/4\right), \tag{6}$$

where $C_S = R_S/R$ is the correlation length of the signal scaled by the beam radius and σ_s is the phase variance. We shall assume that the spectral power density of the noise is analogous to expression (6) with correlation length C_ξ and variance σ_ξ^2. In the case of a circular hole in the diaphragm *3* in Fig. 1, on the basis of formula (2) we obtain for the rms measurement error of the local wavefront tilts

$$g^2 = \left(4\sigma_s^2/C_s^2\right)\left(1 + x_s\right)\exp(-x_s)\left(4\sigma_\xi^2/C_\xi^2\right)\left[1 - \left(1 + x_\xi\right)\exp(-x_\xi)\right], \tag{7}$$

$$x_s = \omega_1^2 C_s^2/4; \quad x_\xi = \omega_1^2 C_\xi^2/4, \quad \omega_1 = kr_1R/f,$$

where r_1 is the radius of the diaphragm *3* shown in Fig. 1.

The analogous expression for the transfer function (4) has the form

$$g^2 = \left(4\sigma_s^2/C_s^2\right)\left[1 + 2\exp(-U_s)I_1(U_s)/U_s - 2\exp(U_s/2)\right] + \left(8\sigma_\xi^2/C_\xi^2\right)\exp(-U_\xi)\cdot I_1(U_\xi)/U_\xi, \tag{8}$$

$$U_s = 2\rho_0^2/C_s^2; \quad U_\xi = 2\rho_0^2/C_\xi^2,$$

where I_1 is the modified Bessel function[5] (of imaginary argument).

Expression (7) exhibits a minimum at the point

$$\omega_{opt}^2 = 8\ln\left(\sigma_\xi C_\xi/\sigma_s C_s\right)/\left(C_\xi^2 - C_s^2\right), \tag{9}$$

which exists if $G_s/G_\xi > 1$ and $\sigma / \sigma_s\xi > G_\xi/G_s$. The optimal radius r of the diaphragm *3* (Fig. 1) is equal to fw_{opt}/kR. The optimal size of the aperture for the transfer function (4) can be found by means of an analysis of the dependence (8) or it can be estimated by approximating the transfer function (4) with the function (5). The optimal transfer function scaled to 1 is plotted in Fig. 2 (curve 1) as a function of the spatial frequency for $G_s = 2\exp(-2\omega^2)$ and $G_\xi = \exp(-\omega^2)$. The dashed line corresponds to expression (9) for w_{opt} and curve 2 represents the transfer function (4) at $\rho_0 = 2.26/\omega_{opt}$.

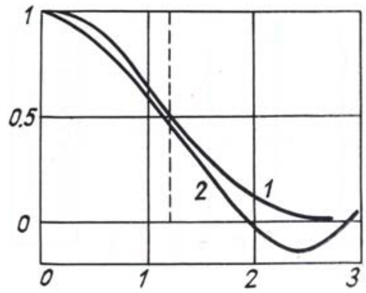

FIG. 2. Transfer functions vs spatial frequency. 1) for the optimal frequency and 2) the Hartmann sensor.

Two questions are associated with ω_1 – the spatial transmission frequency of the sensor. First, if the discrete signal is obtained with the help of a measuring device, then according to the Kotel'nikov theorem in order that the reconstruction of the continuous function be unique the sampling interval must satisfy the inequality $d < 1/2\omega_1$. Second, the number of modes in the representation of the wavefront by a finite series depends on ω_1. This dependence can be estimated in the simplest manner using, as an example, the Fourier expansion. The maximum spatial frequency for a series of N^2 functions (sines and cosines) must be of the order of the half-width of the wavefront spectrum ω_1, i.e., $N \sim 2\omega_1/\pi$. These estimates make it possible to minimize the measurement error at the stage at which the wavefront is reconstructed.

The description that has been developed here of wavefront measurement with the help of the Hartmann sensor as a process of spatial frequency filtration makes it possible to estimate the measurement error and to optimize the basic parameters of the measuring device. The results can also be applied to the solution of problems of wavefront reconstruction and control.

REFERENCES

1. V.P. Lukin, *Atmospheric Adaptive Optics* (Nauka, Novosibirsk, 1986), 248 pp.
2. E.A. Vitrichenko, Astron. Zh. **53**, No. 3, 660 (1986).
3. S.A. Alkhimov, Yu.E. D'yakov, and A.S. Chirkin, (Novosibirsk, 1981), 640 pp.
4. F.T.S. Yu, *Introduction to Diffraction Theory, Information Processing and Holography* [Russian translation] (Sov. Radio, Moscow, 1979), 304 pp.
5. E. Jahnke, F. Erode, and F. Losch, *Tables of Higher Functions* (McGraw-Hill, New York, 1960).