PIECE-WISE LINEAR APPROXIMATION METHOD IN THE PROBLEM OF PHASE FRONT RECONSTRUCTION FOR ADAPTIVE OPTICAL SYSTEMS

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An efficient algorithm which allows one to decrease substantially the length of computations and to improve the accuracy of the phase front reconstruction is constructed based on the method of piece-wise linear approximation. Analytic expressions are derived for estimating the length of computations and error variance of the phase front reconstruetion as a function of the measurement error variance. Advantages of the proposed method are demonstrated.

At the present time phase–conjugation systems are being widely used for solving problems of compensation for nonstationary distortions of optical radiation propagating in the turbulent atnosphere.[1–2] One of the main questions pertaining to fabrication of such systems is the numerical calculation of the optimal controlling force vector for a flexible adaptive mirror in the real time and with a given accuracy. In so doing, the vector of the optimal controlling signals is calculated by minimization of the functional

$$\min_{U}\sigma^2 = \min_{U}\left\{\frac{1}{S}\int\left[\psi(\rho) - UF\right]^2 d^2\rho\right\}, \tag{7}$$

where where σ^2 is the square of phase perturbations averaged over the input aperture, U is the controlling force vector, F is the vector of response functions of the phase front corrector, ψ Is the function describing phase perturbation, $\rho = 2r/D$, r is the radius vector of points on the input aperture, and D is the aperture diameter of the form[3]

$$U_0 = M\Phi^{-1}, \tag{2}$$

where $M = \frac{1}{S}\int\psi F\, d^2\rho$ is the vector, and $\Phi = \left\| \frac{1}{S}\int F_P F_1\, d^2\rho \right\|$ is the square matrix.

Measurements with a sensor of the Hartmann type are generally used for determinating the phase front parameters ψ with an account of peculiarities of the optical radiation. Measurements of local tilts of the phase front at the aperture points of the form[1,2]

$$U_{ij} = k^{-1}\frac{\partial\psi(x_i,y_j)}{\partial x} \text{ and } V_{ij} = k^{-1}\frac{\partial\psi(x_i,y_j)}{\partial y} \tag{3}$$

are performed with this sensor, where k is the radiation wave number.

Up to now the modifications of the Hartmann sensor are many in number.[3] However, irrespective of physical principles used for measuring 1 ocal tilts of the phase front, reconstruction of the real phase front and numerical calculation of the optimal vector of controlling signals is performed in accordance with Eq. (2) based on the measurements of the form (3). Practical implementation of the algorithm (2) would entail lengthy computations thereby limiting the application of these algorithms in real time.

The authors of Refs. 3 and 4 proposed their version of the phase conjugation algorithm, which does not require any preliminary reconstruction of the phase front in explicit form. In this case the optimal phase surface is found as:

$$U_{\text{opt}} = PG^{-1} \tag{4}$$

where

$$P = \left\| \int \text{grad}\,\psi\ \text{grad}\,F_1\, d^2\rho \right\|$$

is the row matrix and

$$G = \left\| \int \text{grad}\,F_P\ \text{grad}\,F_1\, d^2\rho \right\|$$

is the square matrix. However, the practical implementation of such an algorithm for a wide class of response functions of flexible adaptive mirrors is extremely problematic since the matrix G may be nonexistent in some cases on account of the linear dependence of rows or columns. It should be also noted that the real response functions F_1 of flexible adaptive mirrors may deviate from theoretical or experimental functions that can also result in ill–conditionality of the matrix G.

This paper is concerned with the method of the phase front reconstruction with the help of the piece–wise linear approximation.

Let the square aperture be segmented of $M \times N$ identical subapertures and the local phase front tilts (3) be measured at the centers of these subapertures. Now we shall consider the jth ($j = \overline{1,M}$) wavefront cross section through the coordinate y (Fig. 1).

In the general case the wavefront cross sect ion is a random function of the coordinate x_1 ($I = \overline{1,M}$). Let us divide the wavefront cross section into M segments. On every segment the wavefront is approximated by a straight line segment

$$z_1 = a_1 + U_1\, x, \tag{5}$$

where a_1 is the where a is the phase shift on the ith segment and z_1 is the piece–wise linear approximation of the phase front. The subscripts j are omitted in Eq. (5) and hereafter.

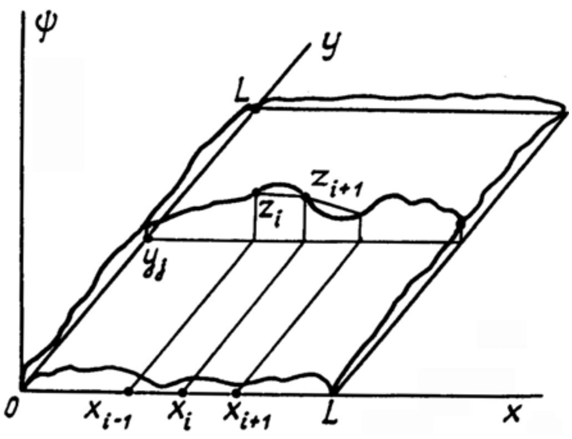

FIG. 1. The jth cross section of phase front.

To evaluate a_1, we will employ the procedure of joining the adjacent segments

$$a_i + U_i x_i = a_{i+1} + U_{i+1} x_i, \tag{6}$$

where x_1 is the coordinate x at the point i. In accordance with Eq. (6), a system of $(M - 1)$ linear equations can be written as

$$a_1 - a_2 = (U_2 - U_1)x_1,$$

$$a_2 - a_3 = (U_3 - U_2)x_2,$$

and (7)

$$a_{M-1} - a_M = (U_{M-1} - U_M)x_{M-1}.$$

One more equation to complete the system (7) will be derived from the zero average phase relation on the entire aperture

$$\sum_{i=1}^{M} \int_{i-1}^{x_1} (a_i + U_i x)dx = 0, \tag{8}$$

or

$$\sum_{i=1}^{M} (x_i - x_{i-1}) \left[a_i + \frac{U_i}{2}(x_i - x_{i-1}) \right] = 0. \tag{9}$$

Taking $x_1 = i\Delta x$ and $\Delta x = \dfrac{L}{M}$ into account, where L is the sensor aperture dimension, Eq. (9) can be written in the form

$$\sum_{i=1}^{M} a_i = -\frac{\Delta x}{2} \sum_{i=1}^{M} U_i (2i - 1). \tag{10}$$

Introducing the notation

$b_1 = (U_{i+1} - U_i)\, i\Delta x$

and

$$\alpha = -\frac{\Delta x}{2} \sum_{i=1}^{M} U_i (2i - 1), \tag{11}$$

we can write a system of linear equations

$$\begin{vmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & 1 & -1 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{vmatrix} \times \begin{vmatrix} a_1 \\ a_2 \\ .. \\ a_{M-1} \\ a_M \end{vmatrix} = \begin{vmatrix} b_1 \\ b_2 \\ .. \\ b_{M-1} \\ \alpha \end{vmatrix} \tag{12}$$

By solving the system (12) we will obtain the values of the phase shifts

$$a_i = \sum_{i=1}^{M-1} b_i - \sum_{i=1}^{M-1} \frac{b_i}{M} + \frac{\alpha}{M}. \tag{13}$$

Thus taking into account Eq. (11) the method of reconstruction of real phase front from the measurements of its local tilts is reduced to the following algorithm:

$$a_1 =$$

$$= \sum_{l=1}^{M-1} \Delta x l (U_{l+1} - U_1) - \sum_{l=1}^{M-1} \frac{\Delta x}{M} l^2 (U_{l+1} - U_l) - \sum_{l=1}^{M} \frac{\Delta x}{2M}(2l - 1)U_l. \tag{14}$$

Applying procedure (14) to all the subaperture matrix rows for U_{ij} and to all the subaperture matrix columns for V_{ij}, the value of the phase shift on the ijth subaperture can be written as

$$a_{ij} = (a_{ij}^x + a_{ij}^y)/2, \tag{15}$$

where a_{ij}^x is the piece–wise linear approximation for the subaperture matrix rows and a_{ij}^y is the piece–wise linear approximation for the subaperture matrix columns. In so doing, the phase front on the entire aperture is reconstructed in the form

$$\psi_{ij} = a_{ij} + U_{ij} x + V_{ij} y. \tag{16}$$

In this case the vector of optimal controlling signals for the adaptive optical system is sought according to Eq. (2).

Now we will consider the errors in the phase front reconstruction based on the proposed method. We will proceed from the assumption that errors in measuring the phase front tilts on adjacent subapertures are uncorrelated. Taking into account the superposition principle and by virtue of the linearity of the proposed reconstruction method instead of

$$U_{ij}^n = U_{ij} + n_{ij}^x \text{ and } V_{ij}^n = V_{ij} + n_{ij}^y \tag{17}$$

we can examine the case with

$$U_{ij}^n = n_{ij}^x \text{ and } U_{ij}^n = n_{ij}^y$$

at the input of the device by which the algorithm (14) is implemented. Here n_{ij} is the statistically independent measurement noise.

Such an approach to the analysis of the errors in the phase front reconstruction will essentially simplify further calculations and makes it possible to derive analytic expressions for the intercomparison of the proposed method with other known methods. The error variance of the phase front reconstruction will be written in the following form:

$$\left\langle n_{ij}^2 \right\rangle = \left\langle \left[\sum_{l=1}^{M-1} \Delta x l (n_{l+1} - n_l) - \sum_{l=1}^{M-1} \frac{\Delta x}{M} l^2 (n_{l+1} - n_l) - \right.\right.$$

$$\left.\left. - \sum_{l=1}^{M} \frac{\Delta x}{2M} (2l - 1) n_l \right]^2 \right\rangle. \qquad (19)$$

Taking into account the statistical independence of noise in different channels and with due regard to Eq. (14), Eq. (19) can be represented as

$$\sigma_{ri}^2 = \sigma^2 k_i, \qquad (20)$$

where σ_r^2 is the error variance of the phase front reconstruction and σ^2 is the variance of noise in measuring the local phase front tilts. The subscript i adjacent to the coefficient k_i means that σ_{ri}^2 depends on the subaperture position. For $i = 1$, taking into account Eq. (19), we have

$$k_i \approx \frac{M}{30} - \frac{1}{M^3} \approx \frac{M}{30}. \qquad (21)$$

A similar expression can be derived for a subaperture positioned in the middle of the row

$$k_{M+1/2} = \frac{M}{30} - \frac{16}{30M^2} \approx \frac{M}{30}. \qquad (22)$$

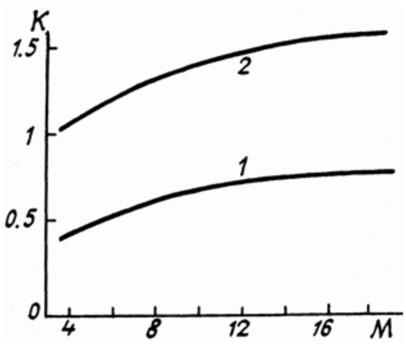

FIG. 2. Dependence of the coefficient k_i on the number of subapertures of the sensor of the Hartmann type: 1) piece-wise linear approximation, 2) well-known algorithm of Ref. 5.

Shown for comparison in Fig. 2 are computer calculations of the average coefficient κ_{av} together with the dependence of the identical coefficient on the number of subapertures M obtained with the use of the well-known algorithm for the phase front reconstruction.[5]

The time consumption for the synthesized algorithmic implementation can be determined from the formula

$$Q = 10\, M^3 + 2\, M^2. \qquad (23)$$

It should be noted that the implementation of the well-known algorithm of the phase front reconstruction[5] will require at least $2/3\,(M + 1)^6$ operations.

The method was realized on an EC–type computer in terms of PL/1.

Conclusions. The method of phase front reconstruction proposed in this paper makes it possible to decrease the length of computations and to increase the accuracy of reconstruction from measurements performed with the sensor of the Hartmann type. This method is versatile and can be realized not only with the help of the analog devices[6] but also with the help of current high-performance parallel computer systems. The representation of the phase front on every subaperture in the form (16) makes it possible to use the wavefront sensor[6] in an adaptive optical system with a segmented mirror, when control is fulfilled on the basis of the wavefront tilts and position. In this case the error in approximation of the phase front decreases by a factor of 2 in comparison with a segmented mirror.[4] Special emphasis should be laid on the fact that, in the process of reconstruction by the proposed method there is no need in the inversion of matrices of G–type. In addition the solution of the reconstruction problem always exists for a wide class of response functions of flexible adaptive mirrors.

REFERENCES

1. M.A. Vorontsov and V.I. Shmal'gausen, *Principles of Adaptive Optics* (Nauka, Moscow, 1985), 336 pp.
2. D.P. Luk'yanov, A.A. Kornienko, and B.E. Rudnitskii, *Optical Adaptive System*, (Radio i Svyaz', Moscow, 1989), 240 pp.
3. V.I. Kislov and V.G. Taranenko, Radiotekhn. Elektron. **31**, No. 11, 2187–2190 (1986).
4. V.G. Taranenko and O.I. Shanin, *Adaptive Optics* (Radio i Svyaz', Moscow, 1990), 112 pp.
5. N.D. Ustinov, I.N. Matveev, and V.V. Protopopov, *Methods of Processing Optical Fields Processing in Laser Detection and Ranging* (Nauka, Moscow, 1983), 272 pp.
6. Positive decision for the inventor's application. No. 4361099/25–10, *Wave Front Sensor*, by D.A. Bezuglov, E.N. Mishchenko, and V.L. Tyurikov, ICI G02B 27/00, Priority of January, 7, 1988.