A STABLE ALGORITHM FOR PROCESSING MULTIFREQUENCY DATA

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Reasons for the divergence of the profiles which have been reconstructed by solving the lidar equations are discussed. A stable algorithm for processing of multifrequency sounding of the atmosphere is proposed. The problem of estimating the optical parameters at the reference point is analyzed and a procedure for solving this problem is proposed. The possibilities of the procedure and algorithm are illustrated by numerical simulation .

At present, the method of multifrequency laser sounding has found application in the solution of a wide range of problems in atmospheric optics and physics.[1–2] Increasing the number of operating wavelengths of the lidar enables one not only to determine the profiles of the optical characteristics of the medium, but also to reconstruct the microphysical parameters, which are important for different practical applications.[3,4] In the single scattering approximation the relation between the returns $P(\lambda_i, z)$ recorded by the receiving lidar system and the optical parameters of the atmosphere is described by the system of multifrequency sounding equations[3]

$$P(\lambda_i, z) = A(\lambda_i, z)P_0(\lambda_i)z^{-2}\beta_\pi(\lambda_i, z)\exp\left(-2\int\limits_0^z \sigma(\lambda_i, z')\mathrm{d}z'\right), \quad (1)$$

where z is the coordinate along the sounding path, λ_i is the wavelength, $A(\lambda_i, z)$ is the instrument function, $P_0(\lambda_i)$ is the energy of the sounding pulse, and $\beta_\pi(\lambda_i, z)$ and $\sigma(\lambda_i, z)$ are the profiles of the backscattering and extinction indices.

In order to extend the relation between $\beta_\pi(\lambda_i, z)$ and $\sigma(\lambda_i, z)$, the formula

$$\sigma(\lambda_i, z) = \sum_j C_{ij}\beta_\pi(\lambda_i, z), \quad (2)$$

may be used, where C_{ij} is the coefficient matrix which generally depends on the distance z. This matrix is calculated by the modified method of linear estimates[5] or is assumed to be a matrix analog of the operator W recommended in Ref. 3. Moreover, it is assumed that an absolute calibration of the lidar can be performed in the course of the experimental investigations, i.e., the instrument function of this lidar is assumed to be well–known and the contribution of molecular scattering to be negligible. Then without loss of generality Eq. (1) taking account of Eq (2) reduces to the following system of equations:

$$S(\lambda_i, z) = \beta_\pi(\lambda_i, z)\exp\left(-2\int\limits_0^z \sum_j C_{ij}\beta_\pi(\lambda_i, z')\mathrm{d}z'\right), \quad (3)$$

where $S(\lambda_i, z) = P(\lambda_i, z)z^2/A(\lambda_i, z)P_0(\lambda_i)$.

By solving Eq. (3) it is possible to reconstruct the profiles of the optical parameters $\beta_\pi(\lambda_i, z)$ and $\sigma(\lambda_i, z)$ and susequently to calculate the necessary microphysical characteristics.[3–5] However, as experience in processing field data has shown, the solution of Eqs. (1) and (3) using the algorithm described in Ref. 3 is frequently rather difficult because of some disadvantages typical of the techniques used for solving the single frequency lidar equation, namely, instability and divergence of the reconstructed profiles of the optical parameters. The properties of the transcendental equation (or a system of transcendental equations for multifrequency sounding) are responsible for this effect. The profiles of the recorded lidar returns, as a rule, include the typical measurement errors. Moreover, in reconstructing the parameters $\beta_\pi(\lambda_i, z)$ and $\sigma(\lambda_i, z)$, the need usually arises of emploing *a priori* information (for example, of the form (2)), which also includes such errors. The salient feature of the lidar equation and of the system of lidar equations consists in the fact that, in their solution, the coefficient of error amplification $\alpha(z_p, z)$ in the first approximation is proportional to the value

$$\alpha(z_p, z) \approx T^2(z_p, z) = \exp\left\{2\int\limits_{z_p}^{z} \sigma(z')\mathrm{d}z'\right\}, \quad (4)$$

where z_p is the reference point. It can be seen from Eq. (4) that when $z > z_p \alpha(z_p, z) > 1$ and at large optical thickness within the altitude range $[z_p, z]$ the coefficient of error amplification may reach values which will result in divergence of the solutions. That is why the technique for solving the single–frequency equation proposed by Klett[6] is widely used. It consists in choosing the reference point z_p at the end of the sounding path. Then for $z_p < z$ $\alpha(z_p, z) < 1$ and with increase of the optical thickness within the altitude range $[z_p, z]$ $\alpha(z_p, z) \to 0$. This makes it possible to eliminate the instability of the obtained solutions, however, the problem of an *a priori* estimate of the optical thickness of the sounding path within the altitude range $[0, z_p]$ now arises.

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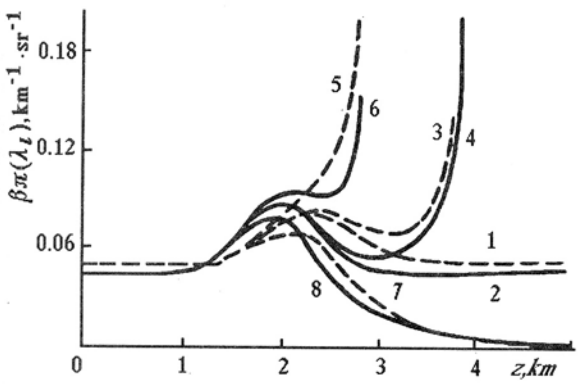

FIG. 1. Examples of the instability of the solutions caused by the error $\Delta\beta(\lambda_i, 0)$ at the reference point $z_p = 0$. Curves 1 and 2 represent model profiles of $\beta(\lambda_i, z)$ and 3 and 4 represent the reconstructed profiles. $\Delta\beta(\lambda_i, 0) = 0.5$ (3 and 4), 2 (5 and 6), and -2% (7 and 8).

The complicated relation between the optical characteristics that enter into the system of the multifrequency sounding equations makes it impossible to give a comprehensive theoretical analysis of the reasons for the instability of the solutions even for two operating wavelengths (with the exception of several trivial particular cases). Therefore, let us now illustrate the foregoing discussion by the results of numerical experiment which are shown in Fig. 1.

Here curves 1 and 2 indicate the model profiles of the backscattering index $\beta_\pi(\lambda_i, z)$ at $\lambda_1 = 0.532$ nm and $\lambda_2 = 1.064$ nm, respectively. Based on these profiles, the S–functions $S(\lambda_i, z)$ were calculated from Eq. (3) and subsequently used for reconstructing the profiles of the optical parameters. With the aim of studying the performance of the algorithm, no random errors were introduced into $S(\lambda_i, z)$. The spacing of the values of z at which this function was determined was sufficiently fine and the matrix C_{ij} was assumed to be constant along the sounding path both for the direct and inverse problems. Curves 3 and 4, 5 and 6, and 7 and 8 indicate the pairs of profiles $\beta_\pi(\lambda_1, z)$ and $\beta_\pi(\lambda_2, z)$ obtained by solving system of equations (3) at $z_p = 0$. The only reason for the divergence of the profiles was the error of approximation at the reference point $z_p = 0$, which was about +0.5, +2, and −2%, respectively. These small random deviations at the point $z_p = 0$ were enough to cause the divergence of the reconstructed profiles as a result of the properties of the transcendental equations and operating at large optical thickness. The amplification of the random errors in determining $S(\lambda_i, z)$ and the errors in assigning an *a priori* information occurs in the same way.

The purpose of this paper is to develop a stable algorithm for processing data of multifrequency laser sounding of the atmosphere. In addition, this algorithm must be applicable both for large optical thicknesses and for a weakly burdened atmosphere, that is, we must have a criterion for an objective estimate of the optical thickness of the sounding path at the operating wavelengths within the altitude range $[0, z_p]$. It is natural that starting from the foregoing discussion the algorithm is constructed according to the principle of choosing the reference point at the end of the investigated path.

Without loss of generality we shall consider sounding paths which are uniformly range–gated with interval Δz (nonuniform range gating only results in a dependence of the quadrature formulas on the distance). In this case, instead of the system of functions $S(\lambda_i, z)$, the matrix S_{ik}, where $i = 1, \ldots, n$, $k = 0, \ldots, m$, where i is the current number of the wavelength and k is the current number of the strobe, is obtained from the experimental results. Approximating the integrals by the trapezoidal rule, system of equations (3) then assumes the following form:

$$S_{ik} = \beta_{ik}\exp\left\{-\Delta z\left(\sum_j C_{ij}\beta_{j0} + \sum_{\nu=1}^{k-1}\sum_j 2C_{ij}\,\beta_{jm} + \sum_j C_{ij}\beta_{jk}\right)\right\},\ k > 0, \quad (5)$$

here $\beta_{ik} = \beta_\pi(\lambda_i, z_k)$. (Equation (5) and the algorithm described below also remain unchanged in the case in which the coefficients C_{ij} depend on the distance z).

In order to simplify the discussion of the solution technique for solving Eq. (5), let us introduce the following notation:

$$\tau_{ik} = \int_0^{z_p} \sigma(\lambda_i, z')\mathrm{d}z' = \int_0^{z_k} \sum_j C_{ij}\,\beta_\pi(\lambda_i, z')\mathrm{d}z'$$

is the optical thickness within the altitude range $[0, z_k]$ at the wavelength λ_i. In this case the relation

$$S_{ik} = \beta_{ik}\exp(-2\tau_{ik}) \quad (6)$$

is valid. Moreover, let us denote

$$a_i(z_{k-1}) = S_{ik-1}\exp\left(2\tau_{ik} - \Delta z\sum_j C_{ij}\,\beta_{jk}\right).$$

The parameters $a_i(z_{k-1})$ can be calculated if S_{ik-1} are measured and the values of τ_{ik} and β_{ik} are assumed to be calculated or assigned *a priori*. It immediately follows from Eq. (5) taking Eq. (7) into account that

$$\beta_{ik-1} = a_i(z_{k-1})\exp\left(-\Delta z\sum_j C_{ij}\,\beta_{jk-1}\right). \quad (8)$$

Equation (8) can be used to calculate an characteristics β_{ik-1} based on the iterative algorithm. That is, an approximation $\beta_{ik-1}^{(0)}$ is assumed to be available, which is then substituted into the right side of Eq. (8). After that $\beta_{ik-1}^{(1)}$ is calculated and subsequently used to calculate $\beta_{ik-1}^{(2)}$. The procedure is repeated untill the difference between the values $\beta_{ik-1}^{(p)}$ and $\beta_{ik-1}^{(p-1)}$ becomes negligible and then $\beta_{ik-1} = \beta_{ik-1}^{(p)}$. It is usually possible to obtain the solution of Eq. (8) within a few iterations. For the iterative algorithm to converge, it is necessary that the first derivatives of the right side of the equation with respect to β_{ik-1} be less than unity.[3] This is equivalent to the requirement that the optical thickness of the interval of range–gating Δz remain neglegible.

The main point of the algorithm for processing multifrequency sounding data consists in the following: the values of the backscattering indices β_{ip} at the reference point $z_p = z_m$ are assigned *a priori*. Taking $\beta_{im} = \beta_{ip}$ with an account of Eq. (6), the optical thicknesses τ_{im} are determined. The parameters $a_i(z_{m-1})$ are then calculated from Eq. (7) and the values of β_{im-1} are determined on the basis of the iterative algorithm (8), which allows to calculate $\tau_i(z_{m-1})$ taking into account Eq. (6). Given that the optical characteristics β_{im-1}, $\tau_i(z_{m-1})$, and S_{im-2} are available, we may proceed to the calculation of β_{im-2} and $\tau_i(z_{m-2})$ in an analogous way. The above procedure is repeated for all z_k in the reverse sense with k starting from $z_k = z_m = z_p$ and ending at $z_k = z_0$.

In the described algorithm, the question of an objective estimate of β_{ip} and $\tau_i(z_p)$ remains open. It is natural that the result of reconstructing profiles of the optical characteristics over the entire sounding path depends on the *a priori* choice of the indices of the backscattering indices β_{ip} at the reference point z_p. The absolute calibration of the lidar system permits us to eliminate the indicated disadvantage. In this case the condition $S_{i0} = \beta_{i0}$, which is used for the correction β_{ip}, should be satisfied with an accuracy not worse than the measurement accuracy.

Let the profiles β_{ik} be calculated based on the proposed algorithm for some $\beta_{ik}^{(0)}$. The deviation of the parameters $\gamma_i = S_{i0}/\beta_{i0}$ from unity may charactirize the accuracy of the choice of the characteristics $\beta^{(0)}{}_{ip}$. If the condition

$$\sum_i |\gamma_i - 1| < \varepsilon \tag{9}$$

is satisfied, where ε is some small preset value, i.e., the calculated results agree with the available experimental data to within a preset error, then there is no reason to change the parameters $\beta_{ip}^{(0)}$ and the calculated profiles β_{ik} yield the final result of solving system (3). If condition (9) is not satisfied, $\beta_{ip}^{(0)}$ must be corrected according to the formula

$$\beta_{ip}^{(t)} = \gamma_i \beta_{ip}^{(t-1)} \tag{10}$$

for $t = 1$. The new values of $\beta_{ip}^{(1)}$ are then used for repeated calculation of the profiles of the optical parameters β_{ip} in accordance with the algorithm described above. The calculations by the iteration loop over t, where t is the iteration number, with $\beta_{ip}^{(t)}$ being corrected according to formula (10) are repeated until condition (9) is satisfied.

For the case of single–frequency sounding, if we neglect the errors in the measurement of $S(\lambda, z)$ and the error in assigning the *a priori* information, it is possible to show that the deviations $\Delta\beta_0$ of the solutions obtained from the true values of β_0 increase monotonically when the errors $\Delta\beta_p$ introduced into β_p vary from $-\beta_p$ to ∞. A numerical analysis of the dependence of $\Delta\beta_{i0}$ on the errors $\Delta\beta_{ip}$ introduced into β_{ip} for the case of multifrequency sounding also shows that the function $\Delta\beta_{i0} = F(\Delta\beta_{ip})$ is smooth and $\Delta\beta_{i0} = 0$ if $\Delta\beta_{ip} = 0$ for all i. Therefore, the iterative algorithm (10) for correcting β_{ip} converges rather rapidly .

Here it is necessary fo note the following important remark. The value of the parameter ε in Eq. (9) cannot be chosen infinitely small. In the general case of single–frequency sounding it can be rigorously shown that as $\varepsilon \to 0$, the iterative algorithm for calculating the corrections for β_{ip} is equivalent to the choice of the reference point at $z = 0$ with all the undesirable consequences discussed above. The situation is similar for multifrequency sounding. The value of the parameter ε is determined by the number of operating wavelengths and by the level of the measurement errors. The main idea of correction (10) consists exactly in performing the calculations on the basis of the available experimental data taking into account the possible random and systematic errors.

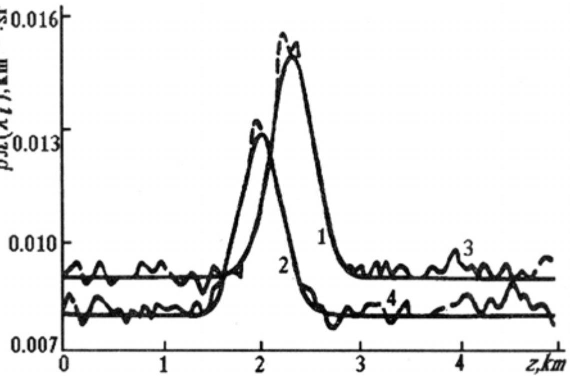

FIG. 2. Numerical simulation of processing of bifrequency sounding data with the help of the proposed algorithm at $\tau \simeq 1$; 1), 2) model profiles of $\beta(\lambda_i, z)$, and 3), 4) the reconstructed profiles.

To illustrate the performance of the proposed algorithm, let us consider the results of some numerical experiments. Figure 2 shows the results for small optical thicknesses of the sounding path ($\tau < 1$). Curves 1 and 2 represent the model profiles of the backscattering index $\beta_\pi(\lambda_i, z)$ at the wavelengths $\lambda_1 = 0.532$ nm and $\lambda_2 = 1.064$ nm, respectively. Based on these profiles, the characteristics S_{ik} were calculated from Eq. (3) (the matrix C_{ij} was assumed to be constant on the sounding path for both the direct and the inverse problem). Relative errors with a 3% standard deviation were then introduced into S_{ik} with the help of a normally distributed random number generator (at some points the relative errors amounted to 8–10%). The thusly obtained matrices S_{ik}^*, were then used for reconstructing the profiles of the optical parameters β_{ik}. The solution of the inverse problem is shown in Fig. 2 by curves 3 and 4. The starting preset values of $\beta_{ik}^{(0)}$ were chosen to deliberately exceed (by a factor of 10) the values of $\beta_p(\lambda_i, z_p)$ derived from the model profiles ($\beta_{ip}^{(0)} = 0.08$, where $i = 1$ and 2). Nevertheless, a quite acceptable result was obtained for $\varepsilon = 0.01$ within 30 iterations. The deviations of curves 3 and 4 from the corresponding profiles 1 and 2 are due to the random errors introduced into S_{ik}.

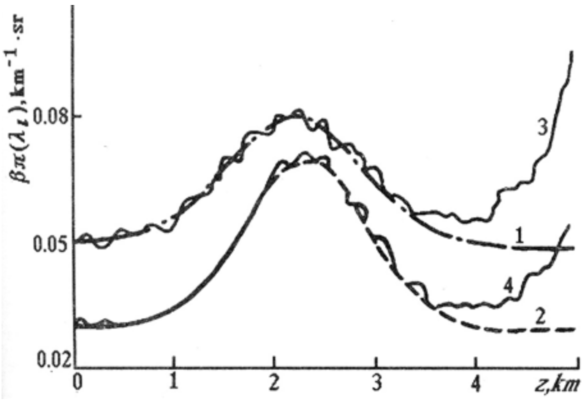

FIG. 3. Numerical simulation of processing of data of bifrequency sounding with the help of the algorithm at $\tau \simeq 5$*: 1), 2) model profiles* $\beta(\lambda_i, z)$ *and 3), 4) reconstructed profiles.*

An example of the performance of the algorithm at large optical thickness ($\tau \simeq 5$) of the sounded path is shown in Fig. 3. The technique of the numerical experiment was analogous to that shown in Fig. 2 (namely, $\beta_{ip}^{(0)} = 0.08$, where $i = 1$ and 2, and $\varepsilon = 0.01$). The only difference consisted in the values of the optical parameters derived from the model profiles 1 and 2. Curves 3 and 4 show the result of solving the inverse problem (condition (9) was satisfied already in the second step of the iteration algorithm over $\beta_{ip}^{(1)}$).

First of all, appreciable discrepancies between the reconstructed and model profiles manifest themselves at the end of the sounding path. These discrepancies are of a fundamental character and it is necessary to discuss them in detail. As has been already noted, for the case of single–frequency sounding when $z < z_p$ the coefficient of error amplification $\alpha(z_p, z)$ is less than 1 and vanishes as the optical thickness increases within the altitude range $[z_p, z]$, i.e., at large τ gross errors at the reference point lead to small errors at $z_k = z_0$ and condition (9) is satisfied automatically. As the results of numerical calculations show, a similar situation obtains in the case of multifrequency sounding. The difference consists in the fact that the relation between the values β_{ip} at different wavelengths also plays an important role, while $\alpha(z_p, z)$ decreases at a higher rate as τ increases. In the case shown in Fig. 3 condition (9) is already satisfied after the first correction according to formula (10), while the parameters γ_i become virtually equal to unity. As a result, the obtained solution of the inverse problem completely agrees with the "experimental data" and there are no objective grounds for correcting the reconstructed profiles of the optical parameters. Thus, in the processing of the sounding data at large τ a "dead zone" has arisen, in which the errors in estimating the values of the optical parameters are gross and cannot be eliminated based on the lidar data alone. In addition to the possibilities noted above, increasing the number of operating wavelengths and broadening the spectral range of sounding makes it possible to reduce "the dead zone" and thereby to improve the quality of the information. To summarize, we recommend that sounding of dispersed media be performed in as wide a spectral range as possible at more operating frequencies and that an attempt should be made to record the lidar returns from the farthest distances. The latter enables one to remove "the dead zone" thereby increasing the range of action.

In conclusion we note that the technique for processing multifrequency sounding data described here can be generalized, without any fundamental difficulties, to the case in which molecular scattering is taking into account in addition to aerosol scattering. The optical parameters of molecular scattering derived from the data of meteorological measurements and calculated according to the standard model of the atmosphere enter into both $\beta_\pi(\lambda_i, z)$ and $\sigma(\lambda_i, z)$ additively. Therefore, it is not difficult to take into account the molecular components in the above given formulas.

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