ON THE THEORY OF TOMOGRAPHIC SOUNDING OF THE ATMOSPHERE USING TWO LIDARS

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A theoretical analysis of the problem of interpretation of data on laser sounding of the atmosphere using two stationary lidars with angular scanning of the region under study has been carried out. An analytical solution describing the two–dimensional spatial distribution of the attenuation and backscattering coefficients without using additional information about the functional dependence between the optical characteristics has been obtained. The proposed solution can be used with the aim of constructing new numerical algorithms for laser tomographic sounding with angular scanning.

1. Introduction. The starting information in the study of the structure of a medium with the help of tomographic methods consists of a collection of projections including a family of integrals of the characteristics sought taken along different directions. As is well known, the lidar signal depends on the optical thickness of the atmosphere between the lidar and the scattering volume, which is also determined by integrating the volume attenuation coefficient along the sounding path. The integral dependence of the recorded signals on an unknown function sought after is a manifestation of a similarity of the methods of transmission tomography and laser sounding.

In contrast to the conventional transmission tomography, the lidar signal is determined by the optical thickness of the atmosphere from the radiation source to the scattering volume as a function of the depth of penetration of the sounding pulse into the medium under study. This information could have been quite sufficient to reconstruct the spatial distribution of the attenuation coefficient without carrying out measurements along different directions, if the lidar signal had been multiplicatively independent of another unknown function, i.e., the volume backscattering coefficient. For this reason, in the case under consideration the problem consists in reconstructing the spatial distribution of two optical characteristics of the medium, i.e., the volume attenuation and backscattering coefficients, from data on monostatic laser sounding. As a rule, when solving such problems additional *a priori* information about the functional relation between the optical characteristics sought after is used or simplifying assumptions regarding their spatial structure are introduced. Now an advanced theory for solving the inverse problems of laser sounding of the atmosphere (see, e.g., Ref. 1) has been created.

The tomographic approach to the problem of laser sounding of the atmosphere associated with retrieval of the data on the volume under study from the lidar signals recorded from different directions, was originally proposed by Weinman[2] for observations made with the use of an airborne lidar. The reason for which the method of tomographic lidar sounding was originally considered for airborne lidars is the possibility of operative motion of the lidar relative to the spatial region under study. Moreover, now a sufficient experience in the use of the lidars for airborne study of the atmosphere has been already accumulated.

The finite–difference algorithms for tomographic processing of the lidar signals in airborne sounding along two or three directions have been described in Refs. 2 and 3. Solutions of integral equations for lidar tomographic airborne sounding in an analytical form have been obtained in Ref. 4. The logarithmic derivative method employed in Ref. 4 makes it possible to formulate and solve other problems of lidar tomographic sounding which do not require airborne measurements. In this paper the analytical solution of the problem of tomographic sounding for lidar configuration, which does not require motion of the lidar and is based on employment of two stationary lidars with angular scanning in some specified region, is presented.

2. Formulation of the problem. Let us consider a mathematical formulation of the problem of two–lidar tomographic sounding with angular scanning.

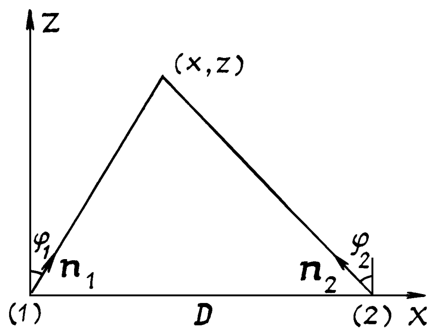

FIG. 1. Scheme of sounding.

Figure 1 shows the scheme of sounding. The lidars are located at the points *1* and *2* separated at the distance D. Sounding is carried out from the points *1* and *2* in the plane which contains the straight line joining these points and the sounding directions $\mathbf{n}_1$ and $\mathbf{n}_2$. Let us specify in the sounding plane the Cartesian coordinate system (x, z), whose origin is at the point of location of the first lidar while the x axis is oriented along the line joining the points *1* and *2*. Now the echo signal received by the ith lidar from the scattering volume located at the point $\mathbf{r} = (x, z)$ in a single–scattering approximation is given by the formula

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$$S_i(\rho_i, \mathbf{n}_i) = \beta(\mathbf{r}) \exp\left\{-2\int_0^{\rho_i} a\left[\mathbf{r} - \left(\rho_i - \rho'\right)\mathbf{n}_i\right]d\rho'\right\},$$
$$i = 1, 2, \qquad (1)$$

where ρ_i is the distance from the point $\mathbf{r}$ of the scattering volume to the ith lidar; $S_i(\rho_i, \mathbf{n}_i) = P_i(\rho_i, \mathbf{n}_i)/(P_{0i}A_i)$, where P_{0i} and $P_i(\rho_i, \mathbf{n}_i)$ are the powers of the transmitted and received signals and A_i is the instrumental constant for the ith lidar, respectively; $\alpha(\mathbf{r})$ and $\beta(\mathbf{r})$ are the volume coefficients of attenuation and backscattering at the point $\mathbf{r}$. The sounding directions $\mathbf{n}_i$ $(i = 1, 2)$ are characterized by the polar angles z_1 and z_2 counted off from the z axis so that

$$\mathbf{n}_1 = \left(\sin\varphi_1, \cos\varphi_1\right), \quad \mathbf{n}_2 = \left(-\sin\varphi_2, \cos\varphi_2\right). \qquad (2)$$

The position of each point $\mathbf{r} = (x, z)$ of the volume under study is uniquely related to the polar angles

$$\varphi_1 = \mathrm{arcot}(x/z), \quad \varphi_2 = \mathrm{arcot}\left(\frac{D - x}{z}\right). \qquad (3)$$

The starting data for solving the inverse problem are a collection of lidar returns received from all of the points inside the region under study for two sounding directions. As a region under study, some bounded volume located between the lidars may be considered. The problem is to reconstruct the spatial distribution of the fields of two optical characteristics, i.e., the attenuation and backscattering coefficients $\alpha(\mathbf{r})$ and $\beta(\mathbf{r})$, from the collection of lidar returns for the two lidars.

3. Procedure for constructing the solution. In order to determine the unknown functions $\beta(\mathbf{r})$ and $\alpha(\mathbf{r})$ from system (1) in an analogy with Ref. 4, let us take the logarithmic derivatives of both sides of the first equation with respect to the direction $\mathbf{n}_1$ and of both sides of the second equation with respect to the direction $\mathbf{n}_2$

$$\frac{\partial L(x, z)}{\partial x}\sin\varphi_1 + \frac{\partial L(x, z)}{\partial z}\cos\varphi_1 - 2a(x, z) = \frac{\partial G_1(x, z)}{\partial \mathbf{n}_1},$$

$$-\frac{\partial L(x, z)}{\partial x}\sin j_2 + \frac{\partial L(x, z)}{\partial z}\cos\varphi_2 - 2\alpha(x, z) = \frac{\partial G_2(x, z)}{\partial \mathbf{n}_2}, \quad (4)$$

where $L(x, z) = \ln \beta(x, z)$ and $G_i(x, z) = \ln S_i$ $(i = 1, 2)$. For the function $L(x, z)$, each of Eqs. (4) is a partial differential equation, whose free term is determined by the lidar return and the other unknown function $\alpha(x, z)$.

Before proceeding to the solution of system (4), let us make several remarks. As regards the writing form, system (4) is identical to system (4) of Ref. 4 which was obtained for the problem of tomographic sounding with a dual–beam airborne lidar. At the same time, typical internal distinctions, which are responsible for different procedures for constructing the solutions, are inherent to the two above–indicated systems. First, this concerns the right sides of system (4). The functions $G_i(x, z)$ $(i = 1, 2)$ in Ref. 4 describe the lidar returns arriving from the point $\mathbf{r} = (x, z)$ from two different fixed directions $\mathbf{n}_i$, which do not change when the lidar moves. In the present paper, as has already been indicated, the functions $G_i(x, z)$ determine the lidar returns from the point r, which are recorded by two lidars located at two different points $i = 1, 2$, while the scanning is due to varying the polar angles φ_1 and φ_2. Second, the coefficients at the partial derivatives in the equations of system (4), in contrast to Ref. 4, are not fixed and are dependent of the position of the point $\mathbf{r}$. The form of these dependences is defined explicitly by the relations

$$\sin\varphi_1 = x/\rho_1, \ \cos\varphi_1 = z/\rho_1, \ \sin\varphi_2 = (D - x)/\rho_2, \ \cos\varphi_2 = z/\rho_2, \qquad (5)$$

where

$$\rho_1 = \left[x^2 + z^2\right]^{1/2}, \ \rho_2 = \left[(D - x)^2 + z^2\right]^{1/2}.$$

After these remarks, let us proceed to the solution of system (4). A particular case should be noted preliminary, which follows from system (4) at $z = 0$. Among the variants, which are possible here, let us separate out the scheme, for which $\varphi_1 = \varphi_2 = \pi/2$ (sounding from the opposite directions). In this case, the coefficients at $\partial L/\partial z$ vanish and the system (4) is reduced to a form

$$\frac{\partial L}{\partial x} \pm 2\alpha = \frac{\partial G_i}{\partial x}, \qquad (6)$$

where the sign plus corresponds to $i = 2$ and minus to $i = 1$. The solution of Eq. (6) with the boundary condition $\beta(0, 0) = \beta_0 = S_1(0, 0)$ is given by the relations

$$\alpha(x, z = 0) = \frac{1}{4}\frac{\partial}{\partial x}\left[G_2(x, 0) - G_1(x, 0)\right],$$

$$\beta(x, z = 0) = R\left[S_1(x, 0)\, S_2(x, 0)\right]^{1/2},$$

$$R = \left[S_1(0, 0)/S_2(0, 0)\right]^{1/2}. \qquad (7)$$

The scheme and the solution for the problem of two–lidar sounding from the opposite directions were proposed in Ref. 5.

Let us turn our attention to the solution of system (4) in general form. Let us change the variables in system (4)

$$u(x, z) = \sin\varphi_1, \ v(x, z) = \sin\varphi_2. \qquad (8)$$

In so doing, only the partial derivatives with respect to one of the variables will be preserved in each equation of system (4)

$$c_1(\varphi_1, \varphi_2)\frac{\partial L}{\partial v} + 2\alpha = -\frac{\partial G_1}{\partial \mathbf{n}_1},$$

$$c_2(\varphi_1, \varphi_2)\frac{\partial L}{\partial u} + 2\alpha = -\frac{\partial G_2}{\partial \mathbf{n}_2}, \qquad (9)$$

where

$$c_1 = D^{-1}\frac{\cos\varphi_2}{\cos\varphi_1}\sin^2(\varphi_1 + \varphi_2), \ c_2 = c_1\frac{\cos^2\varphi_2}{\cos^2\varphi_1}.$$

The right sides in system (9) have the forms

$$\frac{\partial G_1}{\partial \mathbf{n}_1} = -c_1\frac{\partial G_1}{\partial v}, \ \frac{\partial G_2}{\partial \mathbf{n}_2} = -c_2\frac{\partial G_2}{\partial u}, \qquad (10)$$

Formulas (9) and (10) are valid when the inequalities $\cos \varphi_1 \neq 0$ and $\cos \varphi_2 \neq 0$ are satisfied. The case in which $\cos \varphi_1 = 0$ and $\cos \varphi_2 = 0$ corresponds to the condition $z = 0$ we considered previously (see formulas (6) and (7)). Eliminating the function $\alpha(\mathbf{r})$ from system (9), we obtain

$$\frac{\partial L}{\partial u}\cos^2\varphi_1 - \frac{\partial L}{\partial v}\cos^2\varphi_2 = A\left[\frac{\partial G_1}{\partial \mathbf{n}_1} - \frac{\partial G_2}{\partial \mathbf{n}_2}\right], \qquad (11)$$

where $A = \cos^2\varphi_2/c_1$. The solution of Eq. (11) takes the simplest form in the (μ, η) coordinate system, where

$$\mu = \ln\frac{1+u}{1-u}\ ;\ \eta = \ln\frac{1+v}{1-v}\ . \qquad (12)$$

After substituting (μ, η) for the variables (u, v) Eq. (11) is reduced to a form

$$\frac{\partial L}{\partial \mu} - \frac{\partial L}{\partial \eta} = Q(\mu, \eta)\ , \qquad (13)$$

where

$$Q(\mu, \eta) = \frac{A(\mu, \eta)}{2}\left[\frac{\partial G_1}{\partial \mathbf{n}_1} - \frac{\partial G_2}{\partial \mathbf{n}_2}\right] = \frac{\partial G_2}{\partial \mu} - \frac{\partial G_1}{\partial \eta}\ , \qquad (14)$$

$$A(\mu, \eta) = D\,\frac{\mathrm{ch}(\mu/2)\,\mathrm{ch}(\eta/2)}{\left[\mathrm{sh}(\mu/2) + \mathrm{sh}(\eta/2)\right]^2}\ .$$

Integrating Eq. (13) by the method of characteristics with the boundary condition on the beam $(\mu = 0)$

$$L(\mu = 0, \eta) = L_0(\eta) \qquad (15)$$

gives the solution

$$L(\mu, \eta) = L_0(\mu + \eta) + \int_0^{\mu} Q(\mu - \omega, \eta + \omega)\,d\omega\ , \qquad (16)$$

and for the backscattering coefficient

$$\beta(\mu, \eta) = \beta_0(\mu + \eta)\exp\left[\int_0^{\mu} Q(\mu - \omega, \eta + \omega)\,d\omega\right]. \qquad (17)$$

A rechange from the variables (μ, η) to the variables (u, v) is performed based on the formulas

$$u = \mathrm{th}(\mu/2)\ ;\ v = \mathrm{th}(\eta/2)\ . \qquad (18)$$

Let us determine the attenuation coefficient $\alpha(\mu, \eta)$ from the first equation of system (9)

$$\alpha(\mu, \eta) = \frac{1}{A(\mu, \eta)}\left[\frac{\partial G_1}{\partial \eta} - \frac{\partial L}{\partial \eta}\right]. \qquad (19)$$

Substituting the derivative $\partial L/\partial \eta$ given by Eq. (16) into Eq. (19), we have finally

$$\alpha(\mu,\eta) = \frac{1}{A(\mu,\eta)}\left[\frac{\partial}{\partial \eta}\ln\frac{S_1(\mu,\eta)}{\beta_0(\mu+\eta)} - \int_0^{\mu}\frac{\partial}{\partial \eta}Q(\mu-\omega,\eta+\omega)d\omega\right]. \qquad (20)$$

Formulas (17) and (19) completely define the solution of the posed problem which describes the spatial distribution of the optical characteristics $\alpha(\mu, \eta)$ and $\beta(\mu, \eta)$ in some region in the form of functional dependence on the signals $S_1(\mu, \eta)$ and $S_2(\mu, \eta)$ recorded by the two lidars which are separated at the distance D.

4. Conclusion. Thus, as shown in this paper, recording of the lidar signals from each point of the volume under study from two directions with additional use of the second lidar in the laser monitoring system makes it possible to reconstruct simultaneously the spatial distribution of the attenuation and backscattering coefficients from the experimental data without using *a priori* information about the functional dependence between them. The analytical solution of the corresponding inverse problem based on integrating the system of the two first–order partial differential equations has been obtained. In principle the problem will be changed slightly if one uses only one lidar and, instead of the second lidar, uses a reflecting system irradiated by the first lidar. In contrast to the airborne tomographic lidar sounding, the proposed technique does not require any motion of the lidar relative to the volume under study and is based only on angular scanning. The main point in processing of the lidar returns according to the proposed scheme is the calculation of the logarithmic derivatives of these signals with respect to the sounding directions for each lidar at arbitrary points of the volume under study. Taking into account a discrete nature of the real experimental data as well as incorrectness of the problem of numerical differentiation it may be expected that an implementation of the proposed technique in practice will be efficient when using the spline method. A subsequent work to be done in this direction will be devoted to the problems of construction of the appropriate computing algorithms.

REFERENCES

1. V.E. Zuev, G.M. Krekov, and M.M. Krekova, in: *Remote Sensing of the Atmosphere* (Nauka, Novosibirsk, 1978), p. 3.
2. J.A. Weinman, Appl. Opt. **23**, No. 21, 3882 (1984).
3. V.V. Veretennikov, Atm. Opt. **2**, No. 8, 706 –711 (1989).
4. V.V. Veretennikov, Atm. Opt. **4**, No. 6, 466–496 (1991).
5. N.M. Sergeev, M.M. Kugeiko, and D.A. Ashkinadze, *Abstracts of Reports at the 6th All–Union Symposium on Laser and Acoustic Sounding of the Atmosphere*, Tomsk (1980), Vol. 1, p. 126.