AN OPTICAL TRANSFER OPERATOR OF THE ATMOSPHERE–NONORTHOTROPIC SURFACE SYSTEM

T.A. Sushkevich

M.V. Keldysh Institute of Applied Mathematics, Moscow
Received June 27, 1991

A technique is developed to solve the problems of radiation transfer in the "atmosphere–nonorthotropic surface" system based on the use of spatial frequency characteristics and influence functions. An optical transfer operator of this system is constructed, its kernel being the linear influence functions and spatial frequency characteristics.

From the point of view of ecological studies aimed at monitoring of the environment the methods of remote sensing of the atmosphere–underlying surface (land, water surface, and clouds) system including spaceborne techniques are of particular interest.

This paper formulates mathematical models which make it possible to study in detail the processes of formation of radiation fields in the atmosphere–nonorthotropic surface system (ANOSS) based on numerical experiments. The same can be done for the atmosphere–Lambertian surface system (ALSS) since it is a particular case of such a system. The technique based on the use of spatial frequency characteristics (SFC) and the influence functions (IF) is generalized and developed in application to the problems dealing with the nonorthotropic homogeneous and inhomogeneous boundaries.[1-3]

The mathematical apparatus for constructing the SFC and IF models is based on the expansion series of the perturbation theory and on the theory of generalized solutions of kinetic equations.[1,4-6] The problem of seeking for the complete solution of this problem allowing for the nonlinear approximations that take into account multiple reflections from the surface is reduced to seeking for the fundamental solution of a linearized problem and to calculating nonlinear functionals, whose kernels are the corresponding SFC's and IF's. As a result we can find an explicit relationship between the measured radiation characteristics and the parameters of a surface being sensed, which determine the optical transfer operator (OTO) of the system.

FORMULATION OF THE PROBLEM

The problem of solar radiation propagation in the system atmosphere–surface (SAS) can be divided into two classes of problems:

1) the class of problems dealing with the systems in which the land surface is described as an isotropically reflecting surface (Lambertian boundary) (ALSS); and,

2) the class of problems dealing with the systems in which either the land or oceanic surface is described as an anisotropically reflecting surface (ANOSS).

The direction of the radiation propagations is described by the vector $\mathbf{s} = \{\mu, \varphi\}$, where $\mu = \cos\vartheta$ and $\mu \in [-1, 1]$ on the unit sphere $\Omega = [-1, 1]\times[0, 2\pi]$, where $\vartheta \in [0, 180°]$ is the zenith angle, counted from the positive direction of the z axis, and $\varphi \in [0, 2\pi]$ is the azimuth angle. The direction $\varphi = 0$ is assumed to lie in the plane of the solar vertical circle, i.e., the solar radiation is incident on the boundary of the layer along the direction $s_0 = \{\mu_0, \varphi_0\}$, at the zenith angle $\vartheta_0 \in [0, 90°]$, $\mu_0 = \cos\vartheta_0$, and its azimuth being $\varphi_0 = 0$. For the downwelling transmitted radiation a hemisphere of the directions $\Omega^+ = \{(\mu, \varphi): \mu > 0\}$ is introduced and the other $\Omega^- = \{(\mu, \varphi): \mu < 0\}$ is introduced for the upwelling reflected radiation $\Omega = \Omega^+ U\, \Omega^-$. Just for clarity indices "+" and "–" are sometimes ascribed to the functions defined on the Ω^+ and Ω^- hemispheres, respectively.

Spatial coordinates within a plane layer are described by the radius–vector $\mathbf{r} = \{x, y, z\}$. For the horizontal plane it is $r_\perp = \{x, y\}$. A projection of the vector of the direction s onto the horizontal plane is given by $s_\perp = \{\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi\}$, the reference direction for the azimuth φ being the positive direction of the x axis.

The boundary conditions are written in terms of the following sets:

$$\Gamma_0 = \{(r, s): z = 0,\ s \in \Omega^+\} \text{ and } \Gamma_H = \{(r, s): z = H,\ s \in \Omega^-\}.$$

The law of reflection from the lower boundary of the system ($z = H$) is defined by the operator

$$\hat{R}\mathrm{F} = \frac{1}{\pi}\int_{\Omega^+} \Phi(r_\perp, z = H, s')\, \mu' \mathrm{d}s'$$

for the case of Lambertian surface, or by the operator

$$[\hat{R}_H\Phi](s) = \int_{\Omega^+} \Phi(r_\perp, z = H, s')\, \eta(s, s\Omega)\mathrm{d}s\Omega$$

for a nonorthotropic surface (e.g., the Fresnel).

The optical properties of the atmosphere are described by the vertical profiles of coefficients of extinction $\sigma_t(z) = \sigma_s(z) + \sigma_{abs}(z)$, absorption $\sigma_{abs}(z)$, and total scattering $\sigma_s(z) = \sigma_a(z) + \sigma_m(z)$ that includes both the aerosol $\sigma_a(z)$ and molecular $\sigma_m(z)$ scattering components. Also the total scattering phase function

$$\gamma(z, \chi) = \frac{\sigma_a(z)}{\sigma_s(z)}\gamma_a(z, \chi) + \frac{\sigma_m(z)}{\sigma_s(z)}\gamma_m(\chi)$$

is used to describe the optical properties of the atmosphere. In general, the latter value involves the aerosol $\gamma_a(z, \chi)$ and molecular $\gamma_m(\chi) = 3(1 - \cos^2\chi)/(16\pi)$ components.

0235-6880/91/11 755-10 $02.00

The integrodifferential operator of the kinetic equation $\hat{K} \equiv \hat{D} - \hat{S}$ involves the transfer operator $\hat{D} \equiv (s, \text{grad}) + \sigma_t(z)$ and the integral of collisions $\hat{S}\Phi \equiv \sigma_s(z) \int_\Omega \Phi\gamma ds'$.

In the case of a three–dimensional problem dealing with plane layers (when there are inhomogeneties in the horizontal planes) the transfer operator is written as follows:

$$\hat{D} \equiv \mu \frac{\partial}{\partial z} + \sin\vartheta \cos\varphi \frac{\partial}{\partial x} + \sin\vartheta \sin\varphi \frac{\partial}{\partial y} + \sigma_t(z) .$$

In the case of a spatially one–dimensional problem (when the medium consists of horizontally homogeneous layers) the transfer operator $\hat{D}$ is more simple

$$\hat{D}_z \equiv \mu \frac{\partial}{\partial z} + \sigma_t(z) ; \quad \hat{K}_z \equiv \hat{D}_z - \hat{S} .$$

The integral Fourier transform over the horizontal coordinates can be introduced as follows:

$$F[\Phi](p) = \int_{-\infty}^{+\infty} \Phi(r_\perp) \exp[i(p, r_\perp)] dr_\perp = \check{\Phi}(p) ,$$

where symbol "$^\vee$" denotes the resulting Fourier transform. When applied to the three–dimensional equation of radiation transfer through plane layers the Fourier transform results in a complex one–dimensional parametric equation of radiation transfer

$$F[\hat{K}\Phi](p) = \hat{L}(p)\check{\Phi}(z, p, s) ,$$

where

$$\hat{L}(p) \equiv \mu \frac{\partial}{\partial z} + \sigma_t(z) - i(p, s_\perp) - \hat{S} ,$$

$$(p, s_\perp) = p_x \sin\vartheta\cos\varphi + p_y \sin\vartheta\sin\varphi .$$

Here $p = \{p_x, p_y\}$ is the real parameter (the spatial frequency).

The boundary problem in modeling the propagation of solar radiation ($f_0 \equiv \pi S_\lambda \sigma(s - s_0)$) in the SAS can be written in the form

$$\{\hat{K}\Phi = 0 , \quad \Phi\big|_{\Gamma_0} = f_0 , \quad \Phi\big|_{\Gamma_H} = q(r_\perp)\hat{R}_H \Phi \tag{1}$$

with the albedo (or emissivity) being either
a) $q = \text{const}$ or b) $q(r_\perp) \neq \text{const}$.

This problem can be generalized to involve the case of radiation sources different from the sun, by introducing the functions F, f_0, and f_H

$$\begin{cases} \hat{K}\Phi = F(r,s), \\ \Phi\big|_{\Gamma_0} = f_0(r_\perp, s), \\ \Phi\big|_{\Gamma_H} = q\hat{R}_H \Phi + \hat{f}_H(r_\perp, s). \end{cases} \tag{2}$$

SEPARATION OF CONTRIBUTIONS FROM THE ATMOSPHERIC BACKGROUND AND FROM THE IRRADIANCE DUE TO REFLECTION FROM THE UNDERLYING SURFACE

Let us make use of the linearity of the boundary problem (2) with respect to radiation sources. For this let us represent the total radiation field in the system as a superposition $\Phi = \Phi^0 + \Phi_a + \Phi_{aR}$ with its components satisfying the following conditions.

The direct (attenuated) solar radiation Φ^0 is being sought for when solving the problem

$$\begin{cases} \hat{D}_z \Phi^0 = 0 , \\ \Phi^0\big|_{\Gamma_0} = [\pi S_\lambda \delta(s - s_0)] , \\ \Phi^0\big|_{\Gamma_H} = 0 \end{cases} \tag{3}$$

for the layer $z \in [0, H]$, and $\Phi^0 \neq 0$ for $s = s_0$ solely.

The atmospheric background radiation Φ_a is given by the solution of the problem with zero boundary conditions for the layer

$$\{\hat{K}_z \Phi_a = [\hat{S}\Phi^0] , \quad \Phi_a\big|_{\Gamma_0} = 0 , \quad \Phi_a\big|_{\Gamma_H} = 0 . \tag{4}$$

The atmospheric radiation reflected from the boundary is given by the solution of the boundary problem with a source at the boundary $z = H$.

$$\begin{cases} \hat{K}\Phi_{aR} = 0 , \\ \Phi_{aR}\big|_{\Gamma_0} = 0 , \\ \Phi_{aR}\big|_{\Gamma_H} = \hat{R}_H \Phi_{aR}^+ + [\hat{R}_H(\Phi^0 + \Phi_a^+)] , \end{cases} \tag{5}$$

which may be sought for in a more detailed form involving two components $\Phi_{aR} = \Phi_{aR}^0 + \Phi_{aR}^g$.

The component F_{aR}^0 here is the contribution from the atmospheric haze to the total flux produced due to scattering of directly attenuated solar radiation reflected from the boundary. It is determined by the solution of the problem

$$\begin{cases} \hat{K}\Phi_{aR}^0 = 0 , \\ \Phi_{aR}^0\big|_{\Gamma_0} = 0 , \\ \Phi_{aR}^0\big|_{\Gamma_H} = \hat{R}_H \Phi_{aR}^0 + [\hat{R}_H \Phi^0] . \end{cases} \tag{6}$$

The atmospheric scattering of the diffuse component of light scattered by the haze and reflected from the surfsce results in a component Φ_{aR}^g which is a solution of the problem

$$\begin{cases} \hat{K}\Phi_{aR}^g = 0 , \\ \Phi_{aR}^g\big|_{\Gamma_0} = 0 , \\ \Phi_{aR}^g\big|_{\Gamma_H} = \hat{R}_H \Phi_{aR}^g + [\hat{R}_H \Phi_a^+] . \end{cases} \tag{7}$$

EQUATIONS FOR THE SFC IN THE CASE OF NONORTHOTROPIC BOUNDARY

In the case of an arbitrary reflecting surface the contribution into the irradiance due to reflection from a surface is described by the boundary problem with a source $E = \hat{R}_H \Phi^0$, which is the irradiance of the surface produced by the radiation of intensity $\Phi^{(0)} = \Phi^0 + \Phi_a$ from an isolated layer.

Consider now the problem on calculating the irradiance Φ_q in a horizontally homogeneous system

$$\{\hat{K}\Phi_q = 0\ , \quad \Phi_q\big|_{\Gamma_0} = 0\ , \quad \Phi_q\big|_{\Gamma_H} = q\hat{R}_H\Phi_q + qE\ .$$

By introducing a parametric series $\Phi_q = \sum_{k=1}^{\infty} \varepsilon^k \Phi_k$ we arrive at a system of recurrent problems, corresponding to the kth order of reflection from the underlying surface ($k \geq 1$, $\Phi_0 = E$),

$$\{\hat{K}\Phi_k = 0\ , \quad \Phi_k\big|_{\Gamma_0} = 0\ , \quad \Phi_k\big|_{\Gamma_H} = q\hat{R}_H\Phi_{k-1}\ . \tag{8}$$

Let $E = E(s)$ be independent of the coordinate $r_\perp$. Then the linear approximation, relative to the albedo variations, $\check{\Phi}_1(z, p, s) = \check{q}(p)W_1(z, p, s)$ can be written in terms of the SFS. Here W_1 is the solution of the problem

$$\{\hat{L}(p)W_1 = 0\ , \quad W_1\big|_{\Gamma_0} = 0\ , \quad W_1\big|_{\Gamma_H} = E(s)\ , \tag{9}$$

and the nonlinear terms of the series are found using the nonlinear SFC's $W_k(z, p_k, \ldots, p_1, s)$ which are the solutions of the system of complex radiation transfer equations

$$\begin{cases} \hat{L}(p_k)W_k = 0\ , \\ W_k\big|_{\Gamma_0} = 0\ , \\ W_k\big|_{\Gamma_H} = \hat{R}_H W_{k-1}(p_{k-1}, \ldots, p_1)\ . \end{cases} \tag{10}$$

The irradiance due to radiation reflected from the surface is calculated using the functional

$$\Phi_q = \sum_{k=1}^{\infty} \frac{1}{(2\pi)^{2k}} \int_{-\infty}^{\infty} \ldots \int \check{q}(p_1)\check{q}(p_2 - p_1)\ldots\check{q}(p_k - p_{k-1}) \times$$

$$\times\ W_k(z, p_k, \ldots, p_1, s) \exp[-i(p_k, r_\perp)]\mathrm{d}p_k \ldots \mathrm{d}p_1\ . \tag{11}$$

If the albedo has the component $\bar{q} = \mathrm{const}$ then the irradiance can be divided into two components

$$\Phi_q = \Phi^{(\bar{q})} + \Phi^{(\bar{q}\,\tilde{q})}\ ,$$

which are the solutions of the following problems:

$$\begin{cases} \hat{K}_z\Phi^{(\bar{q})} = 0\ , \quad \Phi^{(\bar{q})}\big|_{\Gamma_0} = 0\ , \\ \Phi^{(\bar{q})}\big|_{\Gamma_H} = \bar{q}\hat{R}_H\Phi^{(\bar{q})} + \bar{q}E\ ; \end{cases} \tag{12}$$

$$\begin{cases} \hat{K}\Phi^{(\bar{q}\,\tilde{q})} = 0\ , \quad \Phi^{(\bar{q}\,\tilde{q})}\big|_{\Gamma_0} = 0\ , \\ \Phi^{(\bar{q}\,\tilde{q})}\big|_{\Gamma_H} = \bar{q}\hat{R}_H\Phi^{(\bar{q}\,\tilde{q})} + \tilde{q}[\hat{R}_H\Phi^{(\bar{q}\,\tilde{q})}\hat{R}_H\Phi^{(\bar{q})} + E]\ . \end{cases} \tag{13}$$

The first problem will be analyzed in the next section, while the solution of the second one may be presented in the form of a parametric series. The problem

$$\begin{cases} \hat{K}\Phi_1 = 0\ , \quad \Phi_1\big|_{\Gamma_0} = 0, \\ \Phi_1\big|_{\Gamma_H} = \bar{q}\hat{R}_H\Phi_1 + \tilde{q}E_q \end{cases} \tag{14}$$

is being solved within a linear approximation, with the source function $E_q(s) \equiv \hat{R}_H \mathrm{F}^{(\bar{q})} + E$ while nonlinear approximations ($k \geq 2$) are related to each other recurrently

$$\begin{cases} \hat{K}\Phi_k = 0\ , \quad \Phi_k\big|_{\Gamma_0} = 0, \\ \Phi_k\big|_{\Gamma_H} = \bar{q}\hat{R}_H\Phi_k + \hat{R}_H\Phi_{k-1}\ . \end{cases} \tag{15}$$

The linear SFC $W_1(z, p, s)$ is being found from the problem

$$\begin{cases} \hat{L}W_1 = 0\ , \quad W_1\big|_{\Gamma_0} = 0, \\ W_1\big|_{\Gamma_H} = \bar{q}\hat{R}_H W_1 + E_q\ . \end{cases} \tag{16}$$

Nonlinear approximations describing the irradiance due to the reflections from the underlying surface are expressed in terms of the nonlinear SFC's, which are the solutions of the recurrent system ($k \geq 2$)

$$\begin{cases} \hat{L}W_k(p_k, \ldots, p_1) = 0\ , \quad W_k\big|_{\Gamma_0} = 0, \\ W_k\big|_{\Gamma_H} = \bar{q}\hat{R}_H W_k + \hat{R}_H W_{k-1}(p_{k-1}, \ldots, p_1)\ . \end{cases} \tag{17}$$

The functional representation of the component $\mathrm{F}^{(\bar{q}\tilde{q})}$ has the same form as Eq. (11) for Φ_q. A principle difference between the cases of the Lambertian and non–Lambertian underlying surfaces, when accounting for the irradiance due to the reflections from the underlying surface, is that the nonlinear SFC's of Lambertian surfaces can be factorized over spatial frequencies and, as a result, can be expressed in terms of a linear SFC, while for the nonorthotropic reflection the factorisation is impossible and the linear SFC depends on the character of the surface irradiation. In the latter case the nonlinear SFC's are essentially influenced by the reflection law.

In case the irradiance $E(r_\perp, s)$ is a function of the horizontal coordinate $r_\perp$ (as in the case of the problem (2)) the linear approximation of the solution of system (8) will be given by the convolution

$$\check{\Phi}_1(z, p, s) = \Big(\check{q}(p_0) * W_1(z, p, p_0, s)\Big) \tag{18}$$

with the SFC $W_1(z, p, p_0, s)$ which is a function of two parameters

$$\begin{cases} \hat{L}(p_1)W_1(p,\ p_0) = 0\ , \\ W_1\big|_{\Gamma_0} = 0\ , \\ W_1\big|_{\Gamma_H} = \check{E}(p_0,\ s)\ . \end{cases} \tag{19}$$

One more parameter will also appear in the nonlinear SFC's:

$$\begin{cases} \hat{L}W_k(p_k,\dots,\ p_0) = 0\ , \\ W_k\big|_{\Gamma_0} = 0\ , \\ W_k\big|_{\Gamma_H} = \hat{R}_H W_{k-1}(p_{k-1},\dots,\ p_0)\ . \end{cases} \tag{20}$$

and Eq. (11) will be slightly modified as follows:

$$\Phi_q = \sum_{k=1}^{\infty} \frac{1}{(2\pi)^{2k+1}} \int_{-\infty}^{\infty} \dots \int \check{q}(p_0)\check{q}(p_1 - p_0)\dots\check{q}(p_k - p_{k-1}) \times$$

$$\times\ W_k(z,\ p_k,\dots,\ p_0,\ s)\exp[-i((p_k,\ r_\perp))]dp_k\dots\ dp_0\ . \tag{21}$$

THE NEUMANN SERIES IN THE CASE OF A HOMOGENEOUS BOUNDARY

A system of equations (12) describes a one–dimensional problem with a nonorthotropic boundary, homogeneously and anisotropically illuminated $E = E(s)$. The analytical dependence on the parameter $\overline{q}$ may be found in two different ways.

First, taking into account that at $q = \overline{q} = \text{const}$ the Fourier transform of the albedo is $\check{q} = (2\pi)^2\,\overline{q}\delta(p)$, we obtain a particular expression from functional (11)

$$\Phi^{(\overline{q})}(z,\ s) = \sum_{k=1}^{\infty} \overline{q}^k\ W_k^0(z,\ s)\ , \tag{22}$$

where the functions $W_k^0(z,\ s) \equiv W_k(z,\ p_k = 0,\dots,\ p_1 = 0,\ s)$ are the solutions of the recurrent system of problems (k ≥ 1):

$$\{\hat{K}_z W_k^0 = 0,\ \ W_k^0\big|_{\Gamma_0} = 0,\ \ W_k^0\big|_{\Gamma_H} = \hat{R}_H W_{k-1}^0 \tag{23}$$

with its initial approximation $\hat{R}_H W_0^0 = E(s)$. Equation (22) presents the sum of the Neumann series over the orders of reflection from the underlying surface with the succeeding multiple scattering in the atmospheric layer. Equations (23) are a particular case of the system (10), all of the spatial frequencies $p_k = 0$, $k = 1,\ 2,\ \dots$.

Second, a series over the orders of re–reflections from the surface can be introduced from the very begining with its terms satisfying the system of equations

$$\{\hat{K}_z\Phi_k = 0,\ \ \Phi_k\big|_{\Gamma_0} = 0,\ \ \Phi_k\big|_{\Gamma_H} = \overline{q}\hat{R}_H\Phi_{k-1} \tag{24}$$

with the initial approximation $\hat{R}\Phi_0 = E$. It is obvious that $\Phi_k = \overline{q}^k W_k^0$, and, hence, we arrive at Eq. (22), which describes the iterative process over the orders of the reflection from the surface and allows one to make a complete account for multiple scattering in the atmospheric layer.

It should be noted that in contrast to the case of the Lambertian surface, in the case of the reflection from the nonorthotropic surface the irradiance of the atmospheric layer due to reflections from the underlying surface depends on the azimuth, and if one uses the expansion of the solution over azimuthal harmonics, a large number of equations for different harmonics have to be solved. The boundary problems, like those presented by Eqs. (23), are the ordinary one–dimensional problems, and their solution is not very difficult. The authors of other approaches[7-9] construct the solution of the same problem using integral equations for layer reflectivities and transmissivities, and propose to use the method of successive approximations. Iterations in the form of Eq. (22) are simpler and do not need for any additional restrictions on the parameters of the layer. In addition, this approach allows the spatial and angular distributions of intensity to be directly obtained, i.e., we obtain a complete solution of the problem.

THE INFLUENCE AND TRANSMISSION FUNCTIONS OF A LAYER WITH A UNIFORMLY NONORTHOTROPIC BOUNDARY

If the ocean surface could be considered as a uniform ($q = \overline{q} = \text{const}$) nonorthotropically reflecting boundary the field of radiation in the "atmosphere–ocean" (ANOSS) system is being sought–after as a solution of the problem

$$\begin{cases} \hat{K}_z\Phi = 0\ , \\ \Phi\big|_{\Gamma_0} = \pi S_\lambda\delta(s - s_0)\ , \\ \Phi\big|_{\Gamma_H} = q\hat{R}_H\Phi \end{cases} \tag{25}$$

in the form of a superposition $\Phi = \Phi^0 + \Phi_a + \Phi_q$, where $\Phi_q = \Phi_{aR}$ and $\Phi^{(0)} = \Phi^0 + \Phi_a$. The components Φ^0 and Φ_a are solutions of problems (3) and (4), respectively.

The boundary problem for the irradiance due to reflections in a horizontally homogeneous system of a layer and underlying surface with nonorthotropic reflection, described by the operator $\hat{R}_H$,

$$\begin{cases} \hat{K}_z\Phi_q = 0\ , \\ \Phi_q\big|_{\Gamma_0} = 0\ , \\ \Phi_q\big|_{\Gamma_H} = q\hat{R}_H\Phi_q + qE \end{cases} \tag{26}$$

involves the radiation source described by the irradiance $E(s) = \hat{R}_H\Phi^{(0)}$, which is produced by an isolated layer that, in turn, is a function of the direction $s \in \Omega^{-1}$.

Representing the illumination in terms of the integral with a δ–function as

$$E(s) = \frac{1}{2\pi}\int_{\Omega^-} E(s_0)\,\delta(s - s_0)ds_0\ , \tag{27}$$

one can seek after the irradiance due to the reflections from the underlying surface in the form of the functional

$$\Phi_q(z,\ s) = \frac{1}{2\pi} \int\limits_{\Omega^-} E(s_0)\ \theta_q^0(z,\ s,\ s_0)\mathrm{d}s_0\ , \tag{28}$$

its kernel being the IF $\theta_q^0(z,\ s,\ s_0)$, which is the solution of the boundary problem with its parameter $s_0 \in \Omega^{-1}$

$$\begin{cases} \hat{K}_z\theta_q^0 = 0\ , \\ \theta_q^0\Big|_{\Gamma_0} = 0\ , \\ \theta_q^0\Big|_{\Gamma_H} = q\hat{R}_H\theta_q^0 + q\delta(s - s_0). \end{cases} \tag{29}$$

Let us introduce a parametric series

$$\theta_q^0(z,\ s,\ s_0) = \sum_{k=1}^{\infty} q^k\, \theta_k^0(z,\ s,\ s_0) \tag{30}$$

and proceed to the system of recurrent problems

$$k = 1: \begin{cases} \hat{K}_z\theta_1^0 = 0\ , \\ \theta_1^0\Big|_{\Gamma_0} = 0\ , \\ \theta_1^0\Big|_{\Gamma_H} = \delta(s - s_0), \end{cases}$$

$$k \ge 2: \begin{cases} \hat{K}_z\theta_k^0 = 0\ , \\ \theta_k^0\Big|_{\Gamma_0} = 0\ , \\ \theta_k^0\Big|_{\Gamma_H} = \hat{R}_H\theta_{k-1}^0\ . \end{cases}$$

It can be shown by the method of induction that the kth approximation can be presented by the operator, whose kernel is the linear IF $\theta_1^0(z,\ s,\ s_0)$ $(k \ge 2)$.

As a result[2] the IF $\theta_q^0(z,\ s,\ s_0)$ can be expressed in terms of the linear parametric IF $\theta_1^0(z,\ s,\ s_0)$ for any law of reflection

$$\theta_q^0(z, s, s_0) = q\theta_1^0(z, s, s_0) + \frac{q^2}{2\pi}\int\limits_{\Omega^-} \theta_1^0(z, s, s_1)\,\big[\hat{R}_H\,\theta_1^0\big](s_1, s_0)\mathrm{d}s_1 +$$

$$+ \sum_{k=3}^{\infty} \frac{qk}{(2\pi)^{k-1}} \int\limits_{\Omega^-} \theta_1^0(z,\ s,\ s_{k\text{-}1})\mathrm{d}s_{k-1} \times$$

$$\times \int\limits_{\Omega^-} \big[\hat{R}_H\ \theta_1^0\big](s_{k-1},\ s_{k-2})\mathrm{d}s_{k\text{-}2}\cdots \int\limits_{\Omega^-} \big[\hat{R}_H\ \theta_1^0\big](s_3,\ s_2)\mathrm{d}s_2 \times$$

$$\times \int\limits_{\Omega^-} \big[\hat{R}_H\ \theta_1^0\big](s_2,\ s_1)\ \big[\hat{R}_H\ \theta_1^0\big](s_1,\ s_0)\mathrm{d}s_1\ . \tag{31}$$

Using presentation (31) for q_q^0 we obtain an explicit expression for the irradiance Φ_q due to reflections from the underlying surface in terms of the surface illumination, the operator of reflection, and the linear q_1^0 of an isolated layer

$$\Phi_q(z,\ s) = \frac{q}{2\pi}\int\limits_{\Omega^-} E(s_0)\ \theta_1^0(z,\ s,\ s_0)\mathrm{d}s_0 +$$

$$+ \frac{q^2}{(2\pi)^2}\int\limits_{\Omega^-} E(s_0)\mathrm{d}s_0 \int\limits_{\Omega^-} \theta_1^0(z,\ s,\ s_1)\ \big[\hat{R}_H\,\theta_1^0\big](s_1,\ s_0)\mathrm{d}s_1 +$$

$$+ \sum_{k=3}^{\infty}\frac{q^k}{(2\pi)^k}) \int\limits_{\Omega^-} E(s_0)\mathrm{d}s_0 \int\limits_{\Omega^-} \theta_1^0(z,\ s,\ s_{k-1})\mathrm{d}s_{k-1} \times$$

$$\times \int\limits_{\Omega^-} \big[\hat{R}_H\ \theta_1^0\big](s_{k-1},\ s_{k-2})\mathrm{d}s_{k-2}\cdots \int\limits_{\Omega^-} \big[\hat{R}_H\ \theta_1^0\big](s_3,\ s_2)\mathrm{d}s_2 \times$$

$$\times \int\limits_{\Omega^-} \big[\hat{R}_H\ \theta_1^0\big](s_2,\ s_1)\big[\hat{R}_H\,\theta_1^0\big](s_1,\ s_0)\mathrm{d}s_1\ . \tag{32}$$

Introducing new notation for the variables: $s_{k-1} = s_0^*$, $s_{k-2} = s_1^*, \ldots,\ \ s_1 = s_{k-2}^*,\ \ s_0 = s_{k-1}^*$ we obtain an operator expression in which the kernel (i.e., the linear IF θ_1^0) as well as the linear and nonlinear corrections of the illumination $E(s_0)$ for the re–reflections of photons from the surface and for their multiple scatterings in the atmosphere are separated out

$$\Phi_q(z,\ s) = \frac{q}{2\pi}\int\limits_{\Omega^-} \theta_1^0(z,\ s,\ s_0^*)\ \mathrm{d}s_0^*\ \Bigg\{E(s_0^*) +$$

$$+ \frac{q}{2\pi}\int\limits_{\Omega^-} [\hat{R}_H\ \theta_1^0](s_0^*,\ s_1^*)\ E(s_1^*)\ \mathrm{d}s_1^* +$$

$$+ \sum_{k=3}^{\infty}\frac{q^{k-1}}{(2\pi)^{k-1}}\int\limits_{\Omega^-} \big[\hat{R}_H\ \theta_1^0\big](s_0^*,\ s_1^*)\ \mathrm{d}s_1^*\ldots$$

$$\ldots\int\limits_{\Omega^-}\big[\hat{R}_H\,\theta_1^0\big](s_{k-3}^*, s_{k-2}^*)\ \mathrm{d}s_{k-2}^*\int\limits_{\Omega^-}\big[\hat{R}_H\ \theta_1^0\big](s_{k-2}^*,\ s_{k-1}^*)\,\mathrm{d}s_{k-1}^*\Bigg\}. \tag{33}$$

This expression we call the optical transfer operator (OTO).

Let us now consider problem (12) reduced to the system of recurrent problems (23). Using the IF $\theta_1^0(z,\ s,\ s')$ the linear approximation may be defined as

$$\Phi_1(z, s) = \frac{q}{2\pi} \int\limits_{\Omega^-} \theta_1^0(z, s, s_0)\, E(s_0) \mathrm{d}s_0$$

and a recurrent relation between two successive approximations is then as follows:

$$\Phi_k(z, s) = \frac{q}{2\pi} \int\limits_{\Omega^-} \theta_1^0(z, s, s_0)\, [\hat{R}_H \Phi_{k-1}]\,(s_0)\, \mathrm{d}s_0\ .$$

Let us now introduce the operator

$$[\hat{\mathcal{R}}_0 f](s) = \frac{q}{2\pi} \int\limits_{\Omega^-} [\hat{R}_H \theta_1^0]\,(s, s')\, f(s')\mathrm{d}s' \qquad (34)$$

which acts at the boundary $z = H$.

For $k \geq 1$ the representation

$$\Phi_k(z, s) = \frac{q}{2\pi} \int\limits_{\Omega^-} \theta_1^0(z, s, s_0)\, [\hat{\mathcal{R}}_0^{k-1} E](s_0)\mathrm{d}s_0$$

is valid.

As a consequence the additional irradiance due to the re–reflections from the surface can be written in terms of the functional with the kernel being the linear IF $\theta_1^0(z, s, s')$:

$$\Phi_q = \sum_{k=1}^{\infty} \Phi_k = \frac{q}{2\pi} \int\limits_{\Omega^-} \theta_1^0(z, s, s_0) \left[\sum_{k=1}^{\infty} \hat{\mathcal{R}}_0^{k-1} E \right](s_0)\mathrm{d}s_0 =$$

$$= \frac{q}{2\pi} \int\limits_{\Omega^-} \theta_1^0(z, s, s_0)\, E_q(s_0)\, \mathrm{d}s_0\ , \qquad (35)$$

where

$$E_q(s_0) \equiv [\hat{V}_0 E](s_0) \equiv \sum_{k=0}^{\infty} [\hat{\mathcal{R}}_0^k E](s_0) = [(\hat{E} - \hat{\mathcal{R}}_0)^{-1} E](s_0).$$

Expression (35) is an optical transfer operator of a plane layer with a uniformly nonorthotropic boundary. The terms constituting the operator $\hat{V}_0$ are related to the corresponding terms of the Neumann series over the orders of the reflection from the boundary with the full account for multiple scattering in the layer. Representations (33) and (35) of the OTO are completely equivalent to each other. Thus, in particular case of the Lambertian reflection (E = const) we have

$$[\hat{R}_H \theta_1^0]\,(s, s_0) = [\hat{R}\theta_1^0]\,(s_0)\ , \qquad (36)$$

the transmission function is

$$W_0(z, s) = \frac{1}{2\pi} \int\limits_{\Omega^-} \theta_1^0(z, s, s_0)\, \mathrm{d}s_0\ , \qquad (37)$$

and the spherical albedo is

$$c_0 \equiv \hat{R} W_0 = \frac{1}{2\pi} \int\limits_{\Omega^-} [\hat{R}\, \theta_1^0](s_0)\, \mathrm{d}s_0\ . \qquad (38)$$

The equation for the transmission function $W_0(z, s)$ is obtained by integration of the equation for $\theta_1^0(z, s, s_0)$ over the parameter $s_0 \in \Omega-$ under the boundary conditions

$$\{\hat{K}_z W_0 = 0, \quad W_0\big|_{\Gamma_0} = 0, \quad W_0\big|_{\Gamma_H} = 1\ . \qquad (39)$$

Using the expression for the IF:

$$\theta_q^0(z, s, s_0) = q\theta_1^0(z, s, s_0) + q^2[\hat{R}\, \theta_1^0](s_0)\, W_0(z, s) +$$

$$+ \sum_{k=3}^{\infty} q^k [\hat{R}_H\, \theta_1^0](s_0)\, W_0(z, s)\, c_0^{k-2} \qquad (40)$$

one can calculate the additional irradiance due to the reflections from the surface as follows:

$$\Phi_q(z, s) = \frac{1}{2\pi} \int\limits_{\Omega^-} E\, \theta_q^0(z, s, s_0)\mathrm{d}s_0 =$$

$$= Eq\left\{ \frac{1}{2\pi} \int\limits_{\Omega^-} \theta_1^0(z, s, s_0)\mathrm{d}s_0 + qW_0(z, s) \frac{1}{2\pi} \int\limits_{\Omega^-} [\hat{R}\, \theta_1^0](s_0)\mathrm{d}s_0 + \right.$$

$$\left. + \sum_{k=3}^{\infty} q^{k-1} c_0^{k-2}\, W_0(z, s) \frac{1}{2\pi} \int\limits_{\Omega^-} [\hat{R}\, \theta_1^0](s_0)\mathrm{d}s_0 \right\} =$$

$$= EqW_0(z, s) \left\{ 1 + qc_0 + \sum_{k=2}^{\infty} q^k c_0^k \right\} =$$

$$= EqW_0(z, s)/(1 - qc_0)\ . \qquad (41)$$

As a result we arrive at a generalization of the Sobolev formula[7] allowing for the additional irradiance of the layer due to the reflections from the uniformly Lambertian surface provided that its illumination is homogeneous:

$$\Phi_q(z, s) = EqW_0(z, s)/(1 - qc_0)\ . \qquad (42)$$

THE FUNCTION OF INFLUENCE AND THE SFC OF A LAYER WITH NONUNIFORMLY NONORTHOTROPIC BOUNDARY

Let us construct a functional expression to describe the contribution of the additional irradiance $\Phi_q(z, r_\perp, s)$ due to reflections from the nonuniform nonorthotropic boundary of the layer into its total irradiance. The function $\Phi_q(z, r_\perp, s)$ is, in this case, a solution of the boundary problem with inhomogeneous albedo and nonorthotropic law of reflection:

$$\begin{cases} \hat{K}\Phi_q = 0\,, \quad \Phi_q\big|_{\Gamma_0} = 0\,, \\ \Phi_q\big|_{\Gamma_H} = q(r_\perp)\,\hat{R}_H\Phi_q + q(r_\perp)(s)\,E(s)\,, \end{cases} \tag{43}$$

where $E(s) = [\hat{R}_H\Phi^{(0)}](s)$ and $q(r_\perp) = \varepsilon\tilde{q}(r_\perp)$. Using the parametric series we can proceed to the system of recurrent problems

$$k = 1\colon \{\hat{K}\Phi_1 = 0, \quad \Phi_1\big|_{\Gamma_0} = 0, \quad \Phi_1\big|_{\Gamma_H} = \tilde{q}(r_\perp)\,E(s)\,,$$

$$k \geq 2\colon \{\hat{K}\Phi_k = 0, \quad \Phi_k\big|_{\Gamma_0} = 0, \quad \Phi_k\big|_{\Gamma_H} = \tilde{q}(r_\perp)\,\hat{R}_H\Phi_{k-1}\,.$$

Solutions of these problems are expressed in terms of the SFS's $\psi_k(z, p_k, \ldots, p_1, s)$

$$\check{\Phi}_1(z, p_1, s) = \check{\tilde{q}}\,\Psi_1(z, p_1, s)\,,$$

$$\check{\Phi}_k(z, p_1, s) = \frac{1}{(2\pi)^{2(k-1)}} \int_{-\infty}^{\infty}\!\!\ldots\!\int \check{\tilde{q}}(p_1)\check{\tilde{q}}(p_2 - p_1)\ldots\check{\tilde{q}}(p_k - p_{k-1}) \times$$

$$\times\, \Psi_k(z, p_k, \ldots, p_1, s)\,\mathrm{d}p_{k-1}\ldots\mathrm{d}p_1\,, \tag{44}$$

which satisfy the complex equations of radiation transfer

$$\{\hat{L}(p_1)\,\Psi_1(z, p_1, s) = 0\,, \quad \Psi_1\big|_{\Gamma_0} = 0\,, \quad \Psi_1\big|_{\Gamma_H} = E(s)\,, \tag{45}$$

$$\begin{cases} \hat{L}(p_k)\Psi_k = 0\,, \\ \Psi_k\big|_{\Gamma_0} = 0\,, \\ \Psi_k\big|_{\Gamma_H} = \hat{\mathrm{R}}_H\,\Psi_{k-1}(H, p_{k-1}, \ldots, p_1, s)\,. \end{cases} \tag{46}$$

Let now the source function be introduced as

$$E(s) = \frac{1}{2\pi}\int_{\Omega-} E(s_0)\,\delta(s - s_0)\mathrm{d}s_0\,. \tag{47}$$

If the linear SFC is sought—after as a functional

$$\Psi_1(z, p_1, s) = \frac{1}{2\pi}\int_{\Omega^-} E(s_0)\,\theta_1(z, p_1, s, s_0)\mathrm{d}s_0\,, \tag{48}$$

its kernel will satisfy the complex equation of radiation transfer with the parameters p_1 and s_0

$$\{\hat{L}(p_1)\,\theta_1 = 0\,, \quad \theta_1\big|_{\Gamma_0} = 0\,, \qquad \theta_1\big|_{\Gamma_H} = \delta(s - s_0)\,. \tag{49}$$

It can be shown by the induction method that an arbitrary approximation of the SFC of the order $k \geq 2$ is expressed in terms of the linear IF θ_1

$$\Psi_k(z, p_k, \ldots, p_1, s) = \frac{1}{2\pi}\int_{\Omega^-} \theta_1(z, p_k, s, s_{k-1})\mathrm{d}s_{k-1}\,\times$$

$$\times \frac{1}{2\pi}\int_{\Omega^-}[\hat{R}_H\theta_1](p_{k-1}, s_{k-1}, s_{k-2})\mathrm{d}s_{k-2}\cdots\frac{1}{2\pi}\int_{\Omega^-}[\hat{R}_H\theta_1](p_2, s_2, s_1)\mathrm{d}s_1 \times$$

$$\times\,\frac{1}{2\pi}\int_{\Omega^-} E(s_0)[\hat{R}_H\,\theta_1](p_1, s_1, s_0)\mathrm{d}s_0\,. \tag{50}$$

As a result one obtains a functional expression for the additional irradiance of the layer due to the reflections from the underlying surface

$$\Phi_q(z, r_\perp, s) = \frac{1}{2\pi}\int_{\Omega^-} E(s_0)\mathrm{d}s_0 =$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\check{\tilde{q}}(p_1)\theta_1(z, p_1, s, s_0)\exp[-i(p_1, r_\perp)]\mathrm{d}p_1 +$$

$$+\sum_{k=2}^{\infty}\frac{1}{2\pi}\int_{\Omega^-}\mathrm{d}s_{k-1}\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\theta_1(z, p_k, s, s_{k-1})\exp[-i(p_k, r_\perp)]\mathrm{d}p_k \times$$

$$\times\frac{1}{2\pi}\int_{\Omega^-}\mathrm{d}s_{k-2}\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\check{\tilde{q}}(p_k - p_{k-1})[\hat{R}_H\,\theta_1](p_{k-1}, s_{k-1}, s_{k-2})\mathrm{d}p_{k-1}\times$$

$$\times\frac{1}{2\pi}\int_{\Omega^-}E(s_0)\mathrm{d}s_0\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\check{\tilde{q}}(p_1)\,\check{\tilde{q}}(p_2 - p_1)[\hat{R}_H\,\theta_1](p_1, s_1, s_0)\mathrm{d}p_1 \tag{51}$$

that is the optical transfer operator.

Let us construct the optical transfer operator by making use of the operation at the boundary $z = H$

$$[\hat{\mathfrak{R}}\check{f}] \equiv \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\check{\tilde{q}}(p - p')\mathrm{d}p' \times$$

$$\times\,\frac{1}{2\pi}\int_{\Omega^-}[\hat{R}_H\,\theta_1](p', s, s')\,\check{f}(p', s')\mathrm{d}s'\,. \tag{52}$$

Within the linear approximation we have

$$\check{\Phi}_1((z, p, s) = \frac{\check{\tilde{q}}(p)}{2\pi}\int_{\Omega-}\theta_1(z, p, s, s_0)\,E(s_0)\mathrm{d}s_0\,.$$

It can be shown[2] that for $k \geq 1$ the representation

$$\check{\Phi}_k(z, p, s) = \frac{1}{2\pi}\int_{\Omega-}\theta_1(z, p, s, s_0)\,[\hat{\mathfrak{R}}^{k-1}\,\check{\tilde{q}}\,E](p, s_0)\,\mathrm{d}s_0 \tag{53}$$

is valid and, as a consequence, the additional irradiance is determined by the functional

$$\Phi_q(z, r_\perp, s) = \frac{1}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \exp[-i(p, r_\perp)]\mathrm{d}p \times$$

$$\times \frac{1}{2\pi} \int\limits_{\Omega-} \theta_1(z, p, s, s_0) [\hat{V} \check{\tilde{q}} E](p, s_0)\mathrm{d}s_0 . \quad (54)$$

The sum of the Neumann series is denoted here as:

$$[\hat{V}\check{q}E](p, s_0) = \sum_{k=0}^{\infty} [\hat{\mathfrak{R}}^k \check{q} E](p, s_0) =$$

$$= [(\hat{E} - \hat{\mathfrak{R}})^{-1} \check{q} E](p, s_0) . \quad (55)$$

Expression (54) is the optical transfer operator of a plane layer with nonuniformly, nonorthotropic boundary.

Thus the solution of the problem on additional irradiance of the layer due to the reflections from its boundary Φ_q is reduced to the solution of the one–dimensional complex problem with the parameters p and s_0 for the linear IF $\theta_1(z, p, s, s_0)$ and to calculation of the OTO (51). In the case of a uniformly underlying surface, i.e., at q = const, we have

$$\Phi_q(z, s) = \sum_{k=1}^{\infty} q^k \Psi_k^0(z, s) , \theta_1^0(z, s, s_0) = \theta_1(z, p = 0, s, s_0) , \quad (56)$$

$$k = 1: \Psi_1^0(z, s) = \Psi_1(z, p_1 = 0, s) = \frac{1}{2\pi} \int\limits_{\Omega^-} E(s_0)\ \theta_1^0(z, s, s_0)\mathrm{d}s_0,$$

$$k \geq 2: \Psi_k^0(z, s) = \Psi_k(z, p_k = 0, \ldots, p_1 = 0, s) =$$

$$= \frac{1}{2\pi} \int\limits_{\Omega^-} \theta_1^0(z, s, s_{k-1})\mathrm{d}s_{k-1} \frac{1}{2\pi} \int\limits_{\Omega^-} [\hat{R}_H \theta_1^0](s_{k-1}, s_{k-2})\mathrm{d}s_{k-2} \ldots$$

$$\ldots \frac{1}{2\pi} \int\limits_{\Omega^-} E(s_0) [\hat{R}_H \theta_1^0](s_1, s_0)\mathrm{d}s_0 .$$

If the underlying surface is uniform and orthotropic, and $E = E(s)$ expression (51) is transformed into the generalized Sobolev formula (42). When the albedo has a constant component, the problem on the additional irradiance of the layer due to the reflections from its boundary

$$\begin{cases} \hat{K}\Phi_q = 0 , \\ \Phi_q|_{\Gamma_0} = 0 , \\ \Phi_q|_{\Gamma_H} = (\overline{q} + \varepsilon\tilde{q})\ \hat{R}_H\Phi_q + (\overline{q} + \varepsilon\tilde{q})\ E_0(s) \end{cases} \quad (57)$$

can be reduced to the system of recurrent problems

$$k = 0: \{\hat{K}_z\Phi_0 = 0 , \quad \Phi_0|_{\Gamma_0} = 0 , \quad \Phi_0|_{\Gamma_H} = \overline{q}\hat{R}_H\Phi_0 + \overline{q}E_0(s)\} ,$$

$$k = 1: \{\hat{K}\Phi_1 = 0 , \quad \Phi_1|_{\Gamma_0} = 0 , \quad \Phi_1|_{\Gamma_H} = \overline{q}\hat{R}_H\Phi_1 + \tilde{q}E(s)\} ,$$

$$k \geq 2: \{\hat{K}\Phi_k = 0 , \quad \Phi_k|_{\Gamma_0} = 0 , \quad \Phi_k|_{\Gamma_H} = \overline{q}\hat{R}_H\Phi_k + \tilde{q}\hat{R}_H\Phi_{k-1}\}$$

using the parametric series, and assuming that $E(s) = E_0(s) + \hat{R}_H\Phi_0$.

Using the IF $\theta_q^0(z, s, s_0) \equiv \theta_q(z, p = 0, s, s_0)$ we obtain

$$\Phi_0(z, s) = \frac{1}{2\pi} \int\limits_{\Omega^-} E_0(s_0)\ \theta_q^0(z, s, s_0)\mathrm{d}s_0 \quad (58)$$

or

$$\Phi_0(z, s) = \frac{\overline{q}}{2\pi} \int\limits_{\Omega^-} \theta_1^0(z, s, s_0) [\hat{V}_0 E_0](s_0)\mathrm{d}s_0 . \quad (59)$$

In the linear approximation we have

$$\check{\Phi}_1(z, p_1, s) = \check{\tilde{q}}(p_1)\ \Psi_{1q}(z, p_1, s) ,$$

where the SFC Ψ_{1q} is found from the problem

$$\{\hat{L}(p_1)\ \Psi_{1q} = 0 , \quad \Psi_{1q}|_{\Gamma_0} = 0 , \quad \Psi_{1q}|_{\Gamma_H} = \overline{q}\hat{R}_H\Psi_{1q} + E(s)$$

in terms of the IF $\theta_q(z, p, s, s_0)$ as

$$\Psi_{1q}(z, p_1, s) = \frac{1}{2\pi} \int\limits_{\Omega^-} E(s_1)\ \theta_q(z, p_1, s, s_1)\mathrm{d}s_1 ,$$

which satisfies the complex equation

$$\begin{cases} \hat{L}(p)\ \theta_q(z, p, s, s_0) = 0 , \\ \theta_q|_{\Gamma_0} = 0 , \\ \theta_q|_{\Gamma_H} = \overline{q}\hat{R}_H\theta_q + \delta(s - s_0) \end{cases} \quad (60)$$

with the parameters p and s_0.

Introducing the parametric series

$$\theta_q(z, p, s, s_0) = \sum_{k=1}^{\infty} \overline{q}^k\ \theta_{qk}(z, p, s, s_0) , \quad (61)$$

we arrive at the recurrent system of equations

$$k = 0: \{\hat{L}(p)\ \theta_{q0} = 0 , \quad \theta_{q0}|_{\Gamma_0} = 0 , \quad \theta_{q0}|_{\Gamma_H} = \delta(s - s_0) ,$$

$$k \geq 1: \{\hat{L}(p)\ \theta_{qk} = 0 , \quad \theta_{qk}|_{\Gamma_0} = 0 , \quad \theta_{qk}|_{\Gamma_H} = \hat{R}_H\theta_{q,\, k-1} .$$

It is obvious that $\theta_{q0}(z, p, s, s_0) = \theta_1(z, p, s, s_0)$. All other approaches are expressed by means of the operators, their kernels being the influence function $\theta_1(z, p, s, s_0)$.

The nonlinear approaches of the additional irradiance Φ_k of the layer are written, similarly to expression (44), using the nonlinear SFC's $\Psi_{kq}(z, p_k, \ldots, p_1, s)$ which bring the solution to the recurrent system of complex equations

$$\begin{cases} \hat{L}(p_k)\Psi_{kq} = 0 , \\ \Psi_{kq}\big|_{X_0} = 0 , \\ \Psi_{kq}\big|_{X_H} = \bar{q}\hat{R}_H\Psi_{kq} + [\hat{R}_H\Psi_{k-1,\, q}]\ (p_{k-1}, \ldots , p_1, s) . \end{cases}$$

Two successive approaches of the SFC are related by the recurrent expression

$$\Psi_{kq}(z, p_k, \ldots, p_1, s) = \frac{1}{2\pi}\int\limits_{\Omega^-}\theta_q(z, p_k, s, s_0)\times$$

$$\times [\hat{R}_H\Psi_{k-1,\, q}](p_{k-1}, \ldots , p_1, s_0)\mathrm{d}s_0 .$$

And for $k = 2$ we have

$$\Psi_{2q}(z, p_2, p_1, s) = \frac{1}{2\pi}\int\limits_{\Omega^-}\theta_q(z, p_2, s, s_1)\mathrm{d}s_1\times$$

$$\times\frac{1}{2\pi}\int\limits_{\Omega^-}[\hat{R}_H\theta_q](p_1, s_1, s_0)\, E(s_0)\mathrm{d}s_0 .$$

It can also be shown by the induction method that the operator expression (50) is valid for the SFC Ψ_{kq} if the IF θ_1 is substituted in it for the IF θ_q. The parameters $p_k, \ldots, p_1$ are split in this case, but it is then impossible to sum the Neumann series. In this case the representation of Φ_q by expression (51) is valid olny if the IF θ_1 is replaced for the IF θ_q, which, in its turn, is completely defined in terms of the IF $\theta_1(z, p, s, s_0)$

$$\theta_q(z, p, s, s_0) = \theta_1(z, p, s, s_0) +$$

$$+\frac{\bar{q}}{2\pi}\int\limits_{\Omega^-}\theta_1(z, p, s, s_1)\,[\hat{R}_H\theta_1](p, s_1, s_0)\mathrm{d}s_1 +$$

$$+\sum_{k=2}^{\infty}\frac{\bar{q}^k}{2\pi}\int\limits_{\Omega^-}\theta_1(z, p, s, s_k)\mathrm{d}s_k\frac{1}{2\pi}\int\limits_{\Omega^-}[\hat{R}_H\theta_1](p, s_k, s_{k-1})\mathrm{d}s_{k-1}\ldots$$

$$\ldots\frac{1}{2\pi}\int\limits_{\Omega^-}[\hat{R}_H\theta_1](p, s_2, s_1)[\hat{R}_H\theta_1](p, s_1, s_0)\mathrm{d}s_1 . \tag{62}$$

By making use of the recurrent relation

$$\theta_{qk}(z, p, s, s_0) = \frac{1}{2\pi}\int\limits_{\Omega^-}\theta_1(z, p, s, s_1)\,[\hat{R}_H\theta_{q,\, k-1}](p, s_1, s_0)\mathrm{d}s_1 \tag{63}$$

and of the operation at the boundary $z = H$

$$[\hat{\mathcal{R}}_c\check{f}](p, s, s_0) \equiv \frac{1}{2\pi}\int\limits_{\Omega^-}[\hat{R}_H\theta_1](p, s, s_1)\,\check{f}(p, s_1, s_0)\mathrm{d}s_1 \tag{64}$$

it is quite easy to find the functionals for the approaches at $k \ge 1$

$$\theta_{qk}(z, p, s, s_0) = \frac{1}{2\pi}\int\limits_{\Omega^-}\theta_1(z, p, s, s_1)[\hat{\mathcal{R}}_c^{k-1}\hat{R}_H\theta_1](p, s_1, s_0)\mathrm{d}s_1 .$$

Therefore, the IF θ_q can be represented as a sum of the Neumann series

$$\theta_q(z, p, s, s_0) = \theta_1(z, p, s, s_0) +$$

$$+\frac{\bar{q}}{2\pi}\int\limits_{\Omega^-}\theta_1(z, p, s, s_1)\, E_0(p, s_1, s_0)\mathrm{d}s_1 , \tag{65}$$

where

$$E_0(p, s_1, s_0) \equiv \sum_{k=0}^{\infty}\bar{q}^k\,[\hat{\mathcal{R}}_c^k\hat{R}_H\theta_1](p, s_1, s_0) =$$

$$= [\hat{V}_c\hat{R}_H\theta_1](p, s_1, s_0) , \tag{66}$$

$$V_c\check{f} \equiv \sum_{k=0}^{\infty}\bar{q}^k\,\hat{\mathcal{R}}_c^k\check{f} = [\hat{E} - \bar{q}\,\hat{\mathcal{R}}_c]^{-1}\check{f} . \tag{67}$$

It is obvious that $\bar{q}\,\hat{\mathcal{R}}_c$ coincides with $\hat{\mathcal{R}}$ at $\tilde{q} = \bar{q} = \mathrm{const}$. The successive approaches are related to each other by the reccurent relation

$$\check{\Phi}_k(z, p, s) = \frac{1}{2\pi}\int\limits_{\Omega^-}\theta_q(z, p, s, s_0)\mathrm{d}s_0\times$$

$$\times\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\check{\tilde{q}}(p - p')[\hat{R}_H\check{\Phi}_{k-1}](p', s_0)\mathrm{d}p' .$$

Let us introduce the operation

$$[\hat{\mathcal{R}}_q\check{f}](p, s) \equiv \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\check{\tilde{q}}(p - p')\mathrm{d}p'\times$$

$$\times\frac{1}{2\pi}\int\limits_{\Omega^-}[\hat{R}_H\theta_q](p', s_1, s')\check{f}(p', s')\mathrm{d}s' \tag{68}$$

at the boundary $z = H$.

It can be shown by the induction method that for $k \ge 1$

$$\check{\Phi}_k(z, p, s) = \frac{1}{2\pi}\int\limits_{\Omega^-}\theta_q(z, p, s, s_0)\,[\hat{\mathcal{R}}_q^{k-1}\check{\tilde{q}}E](p, s_0)\mathrm{d}s_0 .$$

The additional irradiance of the layer due to the reflections from its boundary is calculated using the optical transfer operator, its kernel being the IF θ_q

$$\Phi_q(z, r_\perp, s) = \frac{1}{2\pi}\int\limits_{\Omega^-}\theta_q^0(z, s, s_0)\, E_0(s_0)\mathrm{d}s_0 +$$

$$+\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\exp[-i(p,\ r_\perp)]\mathrm{d}p\ \times$$

$$\times\frac{1}{2\pi}\int_{\Omega^-}\theta_q(z,\ p,\ s,\ s_0)\,[\hat{V}_q\,\check{\tilde{q}}E](p,\ s_0)\mathrm{d}s_0\ , \qquad (69)$$

and then the sum of the Neumann series is

$$\hat{V}_q\check{f}\equiv\sum_{k=0}^{\infty}\hat{\mathcal{R}}_q^k\check{f}=[\hat{E}-\hat{\mathcal{R}}_q]^{-1}\check{f}\ . \qquad (70)$$

FUNDAMENTAL SOLUTION OF THE BOUNDARY PROBLEM FOR THE ADDITIONAL IRRADIANCE AT AN ARBITRARY LAW OF REFLECTION

When the albedo is perturbed at a point ($q(r_\perp)=\tilde{q}\delta(r_\perp)$) the fundamental solution of the boundary problem for additional irradiance of the layer can be found using expression (44) and the definition of the IF as $\theta_k(z,\ r_{\perp k},\ldots,\ r_{\perp 1},s)=F^{-1}[\Psi_k]$ as follows:

$$\Phi_\delta(\mathbf{z},\ r_\perp,\ s)=\sum_{k=1}^{\infty}\frac{\tilde{q}^k}{(2\pi)^k}\int\ldots\int_{-\infty}^{\infty}\Psi_k(z,\ p_k,\ldots,\ p_1,s)\ \times$$

$$\times\exp[-i(p_k,\ r_\perp)]\mathrm{d}p_k\ldots\mathrm{d}p_1=\sum_{k=1}^{\infty}\tilde{q}^k\theta_k(z,\ r_{\perp k},\ 0,\ldots,\ 0,\ s)\ . \qquad (71)$$

Using expression (50) for the SFC Ψ_k we can find its more detailed representation based on the use of the linear IF $\theta_1(z,\ p,\ s,\ s')$, i.e.,

$$\Phi_\delta(\mathbf{z},\ r_\perp,\ s)=\sum_{k=1}^{\infty}\tilde{q}^k\,\frac{1}{2\pi}\int_{\Omega^-}\mathrm{d}s_{k-1}\,\frac{1}{(2\pi)^2}\ \times$$

$$\times\int_{-\infty}^{\infty}\theta_1(z,\ p_k,\ s,\ s_{k-1})\,\exp[(-i(p_k,\ r_\perp)]\mathrm{d}p_k\ \times$$

$$\times\frac{1}{2\pi}\int_{\Omega^-}\mathrm{d}s_{k-2}\,\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}[\hat{R}_{\mathrm{H}}\theta_1](p_{k-1},\ s_{k-1},\ s_{k-2})\mathrm{d}p_{k-1}\ldots$$

$$\ldots\frac{1}{2\pi}\int_{\Omega^-}\mathrm{d}s_1\,\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}[\hat{R}_{\mathrm{H}}\theta_1](p_2,\ s_2,\ s_1)\mathrm{d}p_2\ \times$$

$$\times\frac{1}{2\pi}\int_{\Omega^-}E(s_0)\mathrm{d}s_0\,\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}[\hat{R}_{\mathrm{H}}\theta_1](p_1,\ s_1,\ s_0))\mathrm{d}p_1=$$

$$=\sum_{k=1}^{\infty}\tilde{q}^k\,\frac{1}{2\pi}\int_{\Omega^-}\theta(z,\ r_\perp,\ s,\ s_{k-1})\mathrm{d}s_{k-1}\,\frac{1}{2\pi}\int_{\Omega^-}c_1(s_{k-1},\ s_{k-2})\,\mathrm{d}s_{k-2}\ldots$$

$$\ldots\frac{1}{2\pi}\int_{\Omega^-}c_1(s_2,\ s_1)\mathrm{d}s_1\,\frac{1}{2\pi}\int_{\Omega^-}E(s_0)\,c_1(s_1,\ s_0)\mathrm{d}s_0\ , \qquad (72)$$

where

$$\theta(z,\ r_\perp,\ s,\ s')\equiv\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\theta_1(z,\ p,\ s,\ s')\,\exp[(-i(p,\ r_\perp)]\mathrm{d}p\ , \qquad (73)$$

and

$$c_1(s,\ s')\equiv\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}[\hat{R}_H\,\theta_1](p,\ s,\ s')\mathrm{d}p\ . \qquad (74)$$

The formulated mathematical models of the OTO, SFC, and IF make it possible to construct new techniques of remote sensing of nonorthotropic surfaces (such as land and ocean), based on the fundamental solutions of the boundary problems of the theory of radiation transfer. As was shown above, in the case of uniform nonorthotropic surfaces the IF of the atmosphere is a response of the medium on the propagation of a wide monodirected beam, and in the case of nonuniform surface it corresponds to the response on the propagation of a laser beam. In the case of the Lambertian surface the IF is determined by isotropic sources.

REFERENCES

1. T.A. Sushkevich, S.A. Strelkov, and A.A. Ioltukhovskii, *The Method of Characteristics in Problems of Atmospheric Optics* (Nauka, Moscow, 1990).
2. T.A. Sushkevich, *"Solving the boundary problems of theory of transfer for nonorthotropic boundaries using the SFC and IF techniques,"* Preprint, No. 107, M.V. Keldysh Institute of Appllied Mathematics, Academy Sciences of the USSR, Moscow (1990) 32 pp.
3. A.A. Ioltukhovskii and T.A. Sushkevich, *"Numerical solution of the equation of transfer with analitical account of albedo,"* Preprint, No. 87, M.V. Keldysh Institute of Appllied Mathematics, Academy Sciences of the USSR, Moscow (1983), 28 pp.
4. V.S. Vladimirov, *Equations of Mathematical Physics* (Nauka, Moscow, 1967), 436 pp.
5. V.S. Vladimirov, *Generalized Functions in Mathematical Physics* (Nauka, Moscow, 1976), 280 pp.
6. T.A. Germogenova, *Local Aspectss of Solution of the Transfer Equation* (Nauka, Moscow, 1986), 272 pp.
7. V.V. Sobolev, *Transfer of Radiant Energy in the Atmospheres of Stars and Planets,* (Gos. Izdat. Tekn. Teor. Lit., Moscow, 1956) 391 pp.
8. S.D. Gutshabash and V.M. Kochetkov, Izv. Akad. Nauk SSSR, FAO **11**, No. 12, 1272–1283 (1975).
9. E.G. Yanovitskii, *Radiation Fields in Inhomogeneous Planetary Atmospheres,* Doc. Phys.–Math. Sci. Dissert., Leningrad State Univ., Leningrad (1984).
10. E.P. Zege, A.P. Ivanov, and I.L. Katsev, *Image Transfer in Scattering Media* (Nauka i tekhnika, Minsk, 1985), 327 pp.