THE POYNTING THEOREM AND RADIATION DAMPING FOR WEAKLY NONLINEAR MEDIA

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The poynting theorem for weakly nonlinear absorbing media under conditions of self-action of monochromatic waves is formulated. An analog of the Bouguer law for wings of spectral lines of media with cubic nonlinearity is considered. It is shown that at large optical depths (of the order of several units) the difference from a classical form is not so important.

A problem of determining experimentally some constants of nonlinearity is raised.

1. Equations for a field in an isotropic nonmagnetic medium with time dependence separated out in the form of the functional factor $\exp(-i\omega t)$ satisfy the equations written in the time–independent form[1]

$$\mathrm{rot}\,\mathbf{E} - ik_0\mathbf{H} = 0\ ,\quad \mathrm{rot}\,\mathbf{H} + ik_0\varepsilon_L\mathbf{E} = 0\ , \tag{1}$$

where $k = \omega/c$ and ε_L is the dielectric constant.

In the case of sufficiently strong fields linear equations (1) can be given in the form

$$\mathrm{rot}\,\mathbf{E} - ik_0\mathbf{H} = 0\ ,\quad \mathrm{rot}\,\mathbf{H} + ik_0(\varepsilon_L + \varepsilon_{NL}(|\mathbf{E}|^2))\,\mathbf{E} = 0\ . \tag{2}$$

Equations (2) describe self–action of plane monochromatic waves. If we consider the first nonzero correction to ε_L which depends on the field strength ("cubic" media), then the latter equation takes the form

$$\mathrm{rot}\,\mathbf{H} + ik_0(\varepsilon_L + \varepsilon_2|\mathbf{E}|^2)\,\mathbf{E} = 0\ . \tag{3}$$

It is assumed that

$$\mathbf{E} = \rho(\mathrm{S})\,\mathbf{E}\ ,\quad \mathrm{S} = |\mathbf{E}|^2\ , \tag{4}$$

where ρ is the complex function, in general ε_2 is also a complex quantity. It is also assumed that the fields are locally transverse. In the geometric optics the approximation $\mathrm{S} = |\mathbf{E}^*\times\mathbf{H} + \mathbf{E}\times\mathbf{H}^*| = |\mathbf{S}|$, where $\mathbf{S}$ is the Poynting vector (necessary factors, in units of CGSE are omitted); $\mathrm{grad}\ \mathrm{S} = -2\,k_0\kappa\mathbf{S}$ (this relation follows from the Poynting theorem for the vector $\mathbf{S}\mathbf{n}$, $\mathbf{n}$ is the unit vector along the Poynting vector, $\mathrm{div}\mathbf{n} = 0$); $\varepsilon_L = m^2$, $m = n + i\kappa$ is the complex index of refraction.

Along with definition (4) of a nonlinear electric field we define with the help of Eq. (2) the nonlinear magnetic vector

$$\mathbf{H} = \rho\,\mathbf{H} + \frac{1}{ik_0}[\nabla\rho\times\mathbf{E}]\ . \tag{5}$$

From Eqs. (1) – (3) we obtain an equation for the function ρ. Solutions of this equation for the spectral line wings are given in Ref. 2. In this paper they are used to study the monochromatic radiation damping in weakly nonlinear media.

2. Let us formulate the Poynting theorem for the case of self–action effects in a monochromatic field. Using the definitions of the electric (4) and magnetic (5) nonlinear fields one can construct for the Poynting vector the combination

$$\mathbf{E}^*\times\mathbf{H} = |\rho|^2\,\mathbf{E}^*\times\mathbf{H} + \frac{1}{ik_0}\rho^*\,\mathbf{E}^*\times[\nabla\rho\times\mathbf{E}]\ ,$$

and then, based on formulas of vector analysis, we obtain

$$\mathrm{div}[\mathbf{E}^*\times\mathbf{H}] = \mathrm{div}(|\rho|^2[\mathbf{E}^*\times\mathbf{H}] + \rho^*\,\{[\nabla\rho\times\mathbf{H}]\,\mathbf{E}^* -$$

$$- [\nabla\rho\times\mathbf{E}]\,\mathbf{H}^*\} - \frac{k_0}{i}\varepsilon_{NL}\,|\rho|^2\,|\mathbf{E}|^2\ .$$

Combination of this expression with its complex conjugate gives

$$\mathrm{div}\,\Pi = |\rho|^2\,\mathrm{div}\,\mathbf{S} - 2k_0\,(\mathrm{Im}\,\varepsilon_{NL})\,|\rho|^2\,\mathrm{S}\ ;$$

$$\Pi = \mathbf{E}^*\times\mathbf{H} + \mathbf{E}\times\mathbf{H}^*\ ,\quad \Pi = |\Pi|\ . \tag{6}$$

By substituting the relation $\mathrm{div}\,\mathbf{S} = -\sigma\mathrm{S}$ from the linear theory (σ is the absorption coefficient, $\sigma = 2\,k_0\kappa$) into Eq. (6) we finally have

$$\mathrm{div}\,\Pi = -\,(\sigma + 2k_0\,\mathrm{Im}\,\varepsilon_{NL})\,|\rho|^2\,\mathrm{S} \tag{7}$$

or for cubic media

$$\mathrm{div}\,\Pi = -\,(\sigma \pm 2k_0\,|\mathrm{Im}\,\varepsilon_2|\,|\rho|^2\,\mathrm{S})\,|\rho|^2\,\mathrm{S}\ . \tag{8}$$

In Eq. (8) the lower sign stands for the decrease in the effective absorption coefficient and, therefore, it is related to a partial clearing–up of the medium. According to Ref. 2 this case is associated with the solution of the equation for function ρ in the form of a hyperbolic secant.

3. Similarly, for the nonlinear Poynting vector one can write

$$\Pi = \rho^*\,\mathbf{E}^*\times\Big(\rho\,\mathbf{H} + \frac{1}{ik_0}[\nabla\rho\times\mathbf{E}]\Big) + \mathrm{c.\ c.} =$$

$$= |\rho|^2\,\mathbf{S} - \frac{\mathrm{S}}{ik_0}\{\rho\,\nabla\rho^* - \rho^*\,\nabla\rho\}\ .$$

The expression in braces gives

$$\Pi = |\mathbf{E}|^2\,\{1 + P\}\ ; \tag{9}$$

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$$P = \frac{2kS}{i}\left[\left(\frac{\dot{\rho}}{\rho}\right)^* - \frac{\dot{\rho}}{\rho}\right]. \qquad (10)$$

Dotted ρ denotes here the first derivative of the function ρ with respect to its argument. An explicit form of the expression (10) is given below. Thus the field intensity should be determined not by the value $|\mathbf{E}|^2$ (as in Ref. 2) but by the Poynting vector (9). Let us then clarify the effect of the correction P on the squared amplitude of the field and show that this correction cannot be neglected.

Using the approaches described in Ref. 2 when calculating Eq. (10) we obtain

$$\Pi = |\mathbf{E}|^2\left\{1 - \frac{2k}{i}\left(2i\Phi'_\xi + i\frac{n}{k}\right)\right\}.$$

According to Ref. 2, for $\mathrm{Im}\,\varepsilon_2 = \delta < 0$,

$\Phi'_\xi = \frac{\beta}{2}\,\mathrm{th}\,\vartheta$, $\vartheta = \frac{1}{2}\xi - C$, $\beta = n/\kappa$, $\xi = -\tau = -\sigma z$ (optical depth)

$E = \frac{A}{x^{1/2}}\,\mathrm{sech}\,(\vartheta)\,\exp\,(i\Phi(\vartheta))\,\mathrm{E}(z)$, $x = \exp\,(-\tau)$,

and $A^2 = \frac{3}{4}\beta/|\delta|\ S_0$,

where S_0 is the intensity of the field incident on the medium, A is the amplitude factor, C is the additive constant, and z is the distance.

It should be noted that the radiation damping is described by a downward branch of the function sech ϑ. For the Poynting vector we have

$$\Pi = |E|^2\left\{1 - 2n\left[1 - \mathrm{th}\left(\frac{\tau}{2} + C\right)\right]\right\} \qquad (11)$$

when $\tau = 0$ (upon entrance into the medium)

$$\Pi = A^2\,\mathrm{sech}^2(C)\,\{1 - 2n\,(1 - \mathrm{th}\,C)\}\,|\mathbf{E}_0|^2. \qquad (12)$$

When $C = 0$ we have th 0 = 0, and, therefore, the value Π becomes negative. The condition $\Pi > 0$ can hold only when $C \neq 0$. As a result it is necessary that

$$A^2\,\mathrm{sech}^2(C)\,\{1 - 2\,n\,(1 - \mathrm{th}\,C)\} = 1. \qquad (13)$$

According to relation (13) one can see that the dependence on the field E_0 incident on the medium is extended to the additive constant C. Table I shows in what manner the choice of this additive constant is related to the normalized constant A for $n = 1$ (and in terms of this constant to the value of the initial field E_0).

TABLE I.

C	th C	2th C − 1	sech C	$A^2 = \frac{1}{\mathrm{sech}^2 C}\,\frac{1}{2\,\mathrm{th}\,C - 1}$
0.55	0.501	0.001	0.866	1335
0.95	0.739	0.480	0.673	4.60
1	0.762	0.523	0.648	4.55
2	0.964	0.928	0.266	15.23
3	0.995	0.990	0.099	103
5	0.9999	0.9998	0.013	5918

As follows from the table, the curve described by relation (13) has a minimum. We interpret the quantity $C \sim 1$ related to this minimum to be connected with the maximum field E_0 which is admissible for the model of a cubic medium[2] to work.

When $C > 1$ the quantity E_0 decreases and the value Π for C 5 completely corresponds to the ordinary Bouguer law for attenuation of radiation

For example, for $C = 1$, we have

$$\Pi = 4.55\,\mathrm{sech}\left(-\frac{\tau}{2} - 1\right)\left\{2\,\mathrm{th}\left(\frac{\tau}{2} + 1\right) - 1\right\}|\mathbf{E}_0|^2. \qquad (14)$$

At large optical depths (ϑ is of the order of several units)

$\mathrm{th}\,\vartheta \sim 1$, $\mathrm{sech}\,\vartheta \sim 2\exp\,(-\vartheta)$,

and, therefore,

$$\Pi(C = 1) \approx 2.46\,\exp\,(-\tau)\,|\mathrm{E}_0|. \qquad (15)$$

For large amplitudes (less than S_0) the numerical factor in Eq. (15) decreases, and for a sufficiently small S_0 the minimum value of this factor is close to unity. For example,

$$\Pi(C = 5) \approx 1.07\,\exp\,(-\tau)\,|\mathrm{E}_0|^2. \qquad (16)$$

Thus, it follows from the expression for Π that the energy flux in the medium (n, κ, ε_2) depends not only on the field amplitude but also on the phase characteristics of the field. When $\mathrm{Im}\,\varepsilon_2 < 0$ there occurs a partial clearing–up of the medium.

4. Analogous treatment of the opposite sign in Eq. (8) forces us to consider the solution for the function ρ in the form of cosech ϑ ($\mathrm{Im}\,\varepsilon_2 > 0$). Here

$$\Phi'_\xi = \frac{\beta}{2}\,\mathrm{cth}\vartheta,\quad E = \frac{A}{x^{1/2}}\,\mathrm{cosech}\,(\vartheta)\,\exp\,(i\Phi)\,\mathrm{E}_L$$

and we can write

$$\Pi = A^2\,\mathrm{cosech}^2(\vartheta)\,\{1 - 2n\,(1 - \mathrm{cth}\,\vartheta)\}\,|\mathrm{E}_0|^2.$$

Upon entrance into the medium ($\tau = 0$)

$$\Pi = A^2\,\mathrm{cosech}^2(C)\,\{1 - 2n\,(1 - \mathrm{cth}\,C)\}\,|\mathrm{E}_0|^2.$$

Corresponding estimates of the A dependence of C for $n = 1$ and $\Pi(\tau = 0) = S_0$ are given in Table II.

TABLE II.

C	cth C	2 cth C − 1	cosech C	A^2
0.01	100.0	199.	100	50.25
1	1.3130	1.6260	0.8509	0.8493
2	1.0373	1.0746	0.2757	12.2428
5	1.0001	1.0002	0.0135	5486

Here one can also note that the value of the constant $C \sim 1$ is related to the maximum value of the field E_0 which is admissible for the model of a cubic medium to be applicable. When the field E_0 decreases (i.e., with increasing A) the law of radiation attenuation follows the standard form of the exponential decay.

Following the above procedure let us consider the case of $C = 1$. For larger ϑ we have

$\mathrm{cth}\,\vartheta \sim 1 + 2\exp\,(-2\vartheta)$, $\mathrm{cosech}\,\vartheta \sim 2\exp\,(-\vartheta)$,

$$\Pi(C = 1) \approx 0.46\,\exp\,(-\tau)\,|\mathbf{E}_0|^2, \qquad (17)$$

here the preexponential factor is less than unity as it follows from Eq. (8) with the plus sign. For $C = 5$ we have

$$\Pi(C = 5) \approx 0.996 \exp(-\tau) S_0 . \tag{18}$$

In general, the sign of the quantity Im ε_2 and its value must be determined from specific conditions of a problem under study.[3]

It should be noted that in addition to the discussed task of the radiation damping another aspect of this problem is quite interesting. It is related to the observation of spectral lines and possible experimental determination of the constant ε_2.

REFERENCES

1. M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, N.G., 1980) [Russian translation] (Nauka, Moscow, 1973), 856 pp.
2. E.V. Lugin, Opt. Atm. **3**, No. 10, 977–983 (1990).
3. L.D. Ivleva and M.A. Kovner, Opt. Spektrosk. **26**, No. 4, 601–606 (1969); **29**, No. 5, 1002–1004 (1970).