THE INTEGRAL MATRIX NONSTATIONARY RADIATIVE TRANSFER EQUATION

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Received May 21, 1992

The integral matrix nonstationary radiative transfer equation (NRTE) has been derived to study a diffuse light field. The properties of its solution are discussed. The angular distribution of scattered radiation brightness is analyzed. Recommendations for the use of the NRTE in different approximations are presented.

The need to study a diffuse light field (DLF) based on a nonstationary radiative transfer equation (NRTE) arises in the determination of the spatial and temporal shape distortion and dissipation of the signal of a pulsed emitter in the process of light beam propagation through the thickness of a turbid medium (atmospheric aerosol, fog, clouds, natural water, optical glasses, vegetation, etc).

In the aforementioned natural turbid media the time of scattering by individual particles τ (scattering is treated as absorption with subsequent re–emission) is much smaller than the transit time of light between two successive scattering events $t_\sigma = (\varepsilon\nu)^{-1}$ (the lifetime of the photon in the medium), i.e., $t_\sigma \gg \tau$ so that scattering can be considered to be instantaneous.

TABLE I.

Medium	σ, km^{-1}	L_σ, km	t_r, s	r, μm	τ, s	t_0, s
Haze	0.1	10	$3\cdot10^{-5}$	< 1	$< 10^{-13}$	10^{-7}
Light fog	1.0	1.0	10^{-6}	10	10^{-12}	10^{-8}
Dense cloud	50	0.02	10^{-7}	10	10^{-12}	10^{-9}
Sea water	40	0.025	10^{-7}	10	10^{-12}	10^{-9}

Note: L_σ is the mean free path of photon (for scattering), r is the radius of scattering particles, t_0 is the pulse duration in the medium treated as the instantaneous pulse.

By way of example Table I is borrowed from Ref. 1 in which the relation between τ and t_σ for some typical turbid media is given.

It is interesting to note that formulation of the nonstationary problem adds the new variable t which, in its turn, imposes an additional constraint: the time of averaging must be substantially greater than the coherence time for an interference pattern formed due to multiple scattering of radiation in the thickness of the medium. For most representative turbid media this time is greater than 10^{-9} s (see Ref. 1). For this reason the propagation of pulses shorter than 1 ns cannot be described by the NRTE.

The matrix radiative transfer equation expresses the local energy conservation law for the parameter of the Stokes vector S_i and in a nonstationary case has the form[2]

$$(\hat{l}, \nabla)S_i(\hat{q}; \mathbf{r}; \hat{l}; t) + \frac{1}{\nu}\frac{\partial S_i(\cdot)}{\partial t} + Q_{ik}(\mathbf{r})[S_k(\cdot) - S_i(\cdot)] =$$

$$= \int_0^t \oint M_i(\varphi_1) D_{ik}(\hat{l}\cdot\hat{l}'; t - t') M_k(\varphi_2) S_k(\hat{q}; \mathbf{r}; \hat{l}'; t') d\hat{l}' dt, \quad (1)$$

where $M(\varphi)$ is the clockwise rotational matrix of a reference plane, and the corresponding angles of rotation of the scattering plane $\hat{l}\times\hat{l}'$ counted off from the planes of incidence $\hat{r}\times\hat{l}'$ and reception $\hat{r}\times\hat{l}$ of radiation are

$$M(\varphi) = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\varphi & \sin 2\varphi & 0 \\ 0 & -\sin 2\varphi & \cos 2\varphi & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix},$$

$$\left\{ \begin{aligned} \varphi_1 &= \arccos \frac{((\hat{l}' \times \hat{l})(\hat{r} \times \hat{l}'))\hat{l}'}{|\hat{l}' \times \hat{l}|\;|\hat{r} \times \hat{l}'|} \\ \varphi_2 &= \arccos \frac{((\hat{l}' \times \hat{l})(\hat{r} \times \hat{l}))\hat{l}}{|\hat{l}' \times \hat{l}|\;|\hat{r} \times \hat{l}|} \end{aligned} \right\},$$

and the source function is

$$S_k^0(\hat{q}_i; \mathbf{r}; \hat{l}; t) = \Phi_k \nu^{-1} Q_{ik}^{-2}(\mathbf{r}) \delta(\hat{q} - \hat{l}) \delta(\mathbf{r}) \delta(t)$$

(below, for simplicity, we set $\Phi_k \overset{\mathrm{df}}{=} 1$).

In the case of isotropy of local inhomogeneities (this is exactly the case of most natural media) the matrix $Q_{ik}(r)$ degenerates into the scalar quantity $Q_{ik} = \varepsilon\delta_{ik}$ (here ε is the coefficient of radiation extinction in the medium and δ_{ik} is Kronecker's delta symbol). In this case Bouguer's term and the source function of the NRTE are simplified, i.e.,

$$Q_{ik}(r)[S_k(\hat{q}; r; \hat{l}; t) - S_k^0(\cdot)] = \varepsilon[S_i(\cdot) - S_i^0(\cdot)] . \quad (2)$$

In an elementary scattering event a photon is for some time in the absorbed state. The law of decay (or re–emission) of this radiation depends on specific physical conditions. In a number of the most important situations (in such media as atmospheric aerosol and natural water) the probability of re–emission is[3]

$$\exp\left\{ -\frac{t - t'}{\tau} \right\} \frac{dt}{\tau},$$

hence, the matrix of the local transformation of a wave train in the medium is reduced to the form

$$D_{ik}(\hat{l}\cdot\hat{l}', t - t') = \frac{1}{\tau} \exp\left\{ -\frac{t - t'}{\tau} \right\} D_{ik}(\hat{l}\cdot\hat{l}') . \quad (3)$$

0235-6880/92/08 539-03 $02.00

Let us use the following designations for the dimensionless time:

$$u = \frac{t}{t_\sigma + \tau}, \ \beta_1 = \frac{\varepsilon}{t_\sigma + \tau}, \ \beta_2 = \frac{t_\sigma}{t_\sigma + \tau},$$

which are substantially simplified for $t_\sigma \gg \tau$

$$u = \varepsilon v t, \ \beta_1 \to 0, \ \beta_2 \to 1 . \quad (4)$$

After introduction of the variables u and $\mathbf{r} = \mathbf{R} - l\hat{l}$ and simple transformations on account of Eqs. (2)–(4), the NRTF changes in the following way:

$$(\hat{l}, \nabla)S_i(\hat{q};\ \mathbf{R} - l\hat{l};\ \hat{l};\ u) + \varepsilon \frac{\partial S_i(\cdot)}{\partial u} + \varepsilon[S_i(\cdot) - S_i^0(\cdot)] =$$

$$= \oint M_i(\varphi_1)D_{ik}(\hat{l}\times\hat{l}')M_k(\varphi_2)S_k(\hat{q};\ R - l\hat{l};\ \hat{l}';\ u)\mathrm{d}\hat{l}' . \quad (5)$$

Here the function of the sources, when using the properties of the δ–function, has the from

$$S_i^0(\hat{q}; R - l\hat{l};\ \hat{l};\ u) = (\varepsilon l^2)^{-1}\delta(\hat{q} - \hat{l})\delta(\hat{R} - \hat{l})\delta(R - l)\delta(u). \quad (6)$$

The Fourier transform of a kinetic term assumes the form

$$\int \frac{\partial S_i(\cdot)}{\partial u} \mathrm{e}^{-i|wu|}\mathrm{d}u = iwS_i(\hat{q};\ \mathbf{r};\ \hat{l};\ w) .$$

Let us make use of this Fourier transform

$$S_i(\hat{q};\ \mathbf{r};\ \hat{l};\ w) = \int S_i(\hat{q};\ \mathbf{r};\ \hat{l};\ u) \exp(-\ iwu)\mathrm{d}u$$

and write down the NRTE for the Fourier transforms of the goal function

$$(\hat{l}, \nabla)S_i(\hat{q};\ \mathbf{R} - l\hat{l};\ \hat{l};\ w) + \bar{\varepsilon}S_i(\cdot) =$$

$$= F_{sk}(\cdot) + \frac{1}{l^2}\,\delta(\mathbf{R} - l)\,\delta(\hat{R} - \hat{l})\delta(\hat{q} - \hat{l}) , \quad (7)$$

where $\bar{\varepsilon} = \varepsilon(1 + iw)$ and

$$F_{sk}(\hat{q};\ \mathbf{r};\ \hat{l};\ w) = \int \mathrm{e}^{-iwu}\oint M_i(\varphi_1)\ D_{ik}(\hat{l}\cdot\hat{l}') \times$$

$$\times\ M_k(\varphi_2)S_k(\hat{q};\ \mathbf{r};\ \hat{l}';\ u)\mathrm{d}\hat{l}'\mathrm{d}u$$

with corresponding boundary conditions

$$\begin{cases} S_i(\hat{q};\ \mathbf{R} - l\hat{l};\ \hat{l};\ w) \to 0,\ l \to \infty, \\ S_i(\hat{q};\ \mathbf{R} - l\hat{l};\ \hat{l};\ w) \to S_i^0(\hat{q};\ R;\ \hat{l};\ w),\ l \to 0 . \end{cases} \quad (8)$$

Let a natural system of coordinates centered at the point R and a basis unit vector oriented in the direction opposite to the unit vector $\hat{l}$ be taken as a basis. By multiplying Eq. (7) by $\exp(\bar{\varepsilon}l)$ we obtain

$$-\nabla_l[S_i(\cdot)\mathrm{e}^{-\bar{\varepsilon}l}] = F_{sk}(\cdot)\mathrm{e}^{-\bar{\varepsilon}l} + l^2\delta(R - l)\delta(\hat{R} - \hat{l})\delta(\hat{q} - \hat{l})\mathrm{e}^{-\bar{\varepsilon}l}. \quad (9)$$

By solving Eq. (9) as an ordinary linear differential equation on account of boundary conditions (8) and properties of the δ–function, we derive the equality

$$S_i(\hat{q};\ \mathbf{R};\ \hat{l};\ w) =$$

$$= \int \mathrm{e}^{-\bar{\varepsilon}l}F_{sk}(\hat{q};\ \mathbf{R} - l\hat{l};\ \hat{l};\ w)\mathrm{d}l + \frac{\mathrm{e}^{-\bar{\varepsilon}R}}{R^2}\,\delta(\hat{R} - \hat{l})\varepsilon(\hat{q} - \hat{l}). \quad (10)$$

Recall the properties of the δ–function in the Fourier transform

$$\int \delta(x)\mathrm{e}^{-i|wx|}\mathrm{d}x \overset{df}{=} 1 \text{ and } (2\pi)^{-1}\int \mathrm{e}^{-i|wk|}\mathrm{d}w = \delta(x) .$$

Let us take the inverse Fourier transform of Eq. (10)

$$S_i(\hat{q};\ \mathbf{R};\ \hat{l};\ u) = \frac{1}{2\pi}\int S_i(\hat{q};\ \mathbf{R};\ \hat{l};\ w)\mathrm{e}^{iwu}\mathrm{d}w .$$

As a result, we obtain the sought–after representation of the matrix NRTE in the integral form

$$S_i(\hat{q};\ \mathbf{R};\ \hat{l};\ u) = \frac{\mathrm{e}^{-\bar{\varepsilon}R}}{R^2}\,\delta(\hat{R} - \hat{l})\delta(\hat{q} - \hat{l})\delta(u - \varepsilon R) + \int_0^\infty \mathrm{e}^{-\varepsilon R} \times$$

$$\times \oint M_i(\varphi_1)\ D_{ik}(\hat{l}\cdot\hat{l}')\ M_k(\varphi_2)S_k(\hat{q};\ \mathbf{R} - l\hat{l};\ \hat{l}';\ u - \varepsilon l))\mathrm{d}\hat{l}'\,\mathrm{d}l \quad (11)$$

or in the operator form

$$S_i = S_i^0 + \bar{K}S_i , \quad (12)$$

which is the Fredholm equation of the second kind with the domain of integration being the n–dimensional Euclidean space, where $\{S_i,\ S_i^0\} \in L$, $\bar{K} \in [L - L_1]$, and L and L_1 are the Banach spaces of the integrated functions.[4] In this case $\|\|S_i\| = \oint |S_i(\hat{l})|\,\mathrm{d}\hat{l}$ and

$$|\bar{K}| \le \sup \int_0^\infty \mathrm{e}^{-\varepsilon R}\oint\ |M_i(\varphi_1)D_{ik}(\hat{l}\cdot\hat{l}')M_k(\varphi_2)\mathrm{d}\hat{l}'\ \mathrm{d}\hat{l} .$$

The physical meaning of Eq. (12) is apparent: it describes the resultant effect of multiple scattering of radiation in the form of the local radiant energy conservation law.

Let us now discuss the properties of its solution based on the most general considerations. The time u in dimensionless units (of an optical path) is employed as one of the variables of the goal function. It is natural that for $u < \varepsilon R$ the Stokes parameter $S_i(u < \varepsilon R) = 0$. It follows from random trajectories of photons (particles of photon gas) that the photon displacement is proportional to the square root of the travelled distance.[5] Hence, to reach the point R the photon must travel the distance $u = (1 - g)(\varepsilon R)^2$, where $g = \overline{\cos\gamma}$ is the mean cosine of the scattering phase function. Thus, when $u = \varepsilon R$ we record only direct and singly scattered radiations at the calculated point of the field R while at the point $u = (1 - g)(\varepsilon R)^2$ the goal function attains its maximum value. It is natural that for $u \gg \varepsilon R$ the Stokes parameter $S_i(u \gg \varepsilon R) \to 0$.

We dwell on the analysis of the angular distribution of the brightness of scattered light. The brightness field (the first Stokes parameter $L \equiv S_i$) has two asymptotes. At the initial instants of time u the singly scattered radiation arrives predominantly at the point R and, hence, the brightness field is determined by the scattering phase function of an elementary radiation scattering event $x(\gamma)$. After a long time u the structure of the brightness field is no longer dependent on $x(\gamma)$ and boundary and initial conditions, i.e., at large optical depth the brightness field becomes quasi–isotropic (this is the specific feature of the depth behavior of light).

As is well known, the solution of the equation in the form of formula (12) is the Neumann series

$$S_i = \sum_{n=0}^{\infty} \bar{K}^n S_i^0 , \qquad (13)$$

which, in many applications of radiative transfer theory, can be represented in the approximate form using the small–angle modification of the iteration method

$$S_{i(n)} = \sum_{m=0}^{n-1} \bar{K}^m S_i^0 + \bar{K}^n S_i^{\text{sas}} , \qquad (14)$$

where S_i^{sas} is the Stokes parameter determined in the approximation of small–angle scattering of radiation.

The solution of Eq. (14) has a clear physical meaning: the resultant sum correctly takes into account the terms associated with the source radiation which undergoes from 0 to m scattering events while the remaining terms are taken into account approximately through a single effective scattering event with a brightness field being quasi–similar to the brightness field of the mth multiplicity of scattering.

This model is the direct generalization of the principles of depth behavior of light and remains independent of the boundary conditions. However, the correct account of the first nth multiplicities of scattering enables one to employ it from the initial instants of time and from the distance $R = 0$.

It is interesting to note that the speed of convergence of Eq. (14) is higher than that of Eq. (13), since the total contribution of multiple anisotropic scattering is in fact taken into consideration by the single term of series (14) rather than by an infinite sum of series (13) in which every subsequent term increases the multiplicity of the integral by three units. In this connection it may be possible to estimate the accuracy of $S_{i(n)}$ from the difference between two subsequent iterations, i.e., as $S_{i(n+1)} - S_{i(n)}$.

The efficiency of application of Eq. (14) obviously depends on the geometry of the problem and successful, from the physical point of view, choice of S_i^{sas}. The most natural media are characterized by true absorption and highly anisotropic scattering of radiation in their thickness. For this reason the solution of the NRTE in the transport approximation must be used for S_i^{sas} since it describes fairly well the energy redistribution in the process of multiple scattering of radiation. In this case the structure (angular distribution) of the brightness field is described by the exactly calculated terms of series (14).

In conclusion we note that to solve a wide range of applied problems in photometric theory of the DLF we may use the solutions of the NRTE derived in the approximations of quasi–single $S_{i(1)}$ (for $n = 1$ in Eq. (14)) and quasi–double $S_{i(2)}$ (for $n = 2$ in Eq. (14)) scattering of radiation. It should be noted that $S_{i(1)}$ is applicable only in the case in which $x(\gamma)$ is more forward–peaked function in comparison with the source scattering phase function (the Green's function of the matrix NRTE for an elementary isotropic emitter), otherwise $S_{i(2)}$ must be used (the Green's function of the matrix NRTE for an elementary collimated emitter). As a result, the method of the Green's functions provides highly accurate calculations of the structure and integral energetic parameters of the DLF of arbitrary form in turbid atmospheric aerosol and natural water media.

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