PROBABILITY OF MULTIRAY PROPAGATION OF OPTICAL WAVES BEHIND A RANDOM PHASE SCREEN

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Dependence of the mean–square number of rays on a distance in the region behind a phase screen is obtained. The probabilities of single–, three–, and five–ray propagation are found using the average number of rays and their mean square. It is shown that the multiray propagation is manifested at distances much longer than the distance at which the strong intensity fluctuations of a wave start to manifest themselves.

As optical waves propagate through a randomly inhomogeneous medium or behind a random phase screen, the large–scale (compared to the wavelength) inhomogeneities of the refractive index act as lenses resulting in formation of caustics. A number of interesting effects accompany the formation of caustic singularities. For example, it was shown in Ref. 1 that the region in which the probability of formation of even a single caustic differed noticeably from zero could be considered as the start of the region in which the strong fluctuations of the wave intensity occur. At the same time, it is well known[2] that the multiray propagation may accompany the formation of caustic. In this case not one but already several rays with different initial coordinates arrive at a fixed point in space.

The statistical properties of multiray propagation must be known for its description. Dependence of the average number of rays $\langle N(t)\rangle$ in the region behind the screen on the coordinate t of wave propagation was found in Ref. 3. In addition to the fact that knowledge of the number of rays arriving at the given point (x, t) is important (here for simplicity we will consider only the transverse coordinate x, the quantity $\langle N\rangle$ bears supplemental information about the multiray propagation. First, it allows us to estimate the applicability limits of the single–ray approximation and second, it can be used to obtain the probability of the three–ray propagation $P(3; t)$.

Analogously to Ref. 3, i.e., using the normalization condition for the probability density and definition of the average, we can write down

$$\begin{aligned}
&P(1; t) + P(3; t) + \ldots + P(N; t) + \ldots = 1 ,\\
&1\,P(1; t) + 3\,P(3; t) + \ldots + N\,P(N; t) + \ldots = \langle N(t)\rangle ,\\
&1^2P(1; t) + 3^2P(3; t) + \ldots + N^2P(N; t) + \ldots = \langle N^2(t)\rangle ,\\
&\ldots\ldots\ldots\ldots\ldots\\
&1^kP(1; t) + 3^kP(3; t) + \ldots + N^kP(N; t) + \ldots = \langle N^k(t)\rangle .\\
&\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{1}$$

Here the probability $P(N, t)$ is equal to the normalized length of the intervals along the transverse axis at the distance t from the screen within which N rays fall. As far as the number of rays N infinitely increases[3] with increase of t, then it follows from the system of equations (1) that all the moments $\langle N^{\kappa}\rangle$, where $\kappa = 1, 2, \ldots$, must be known to obtain the probabilities $P(N; t)$. If we ignore the probability of occurence of seven and more rays, then the first three equations of the system of equations (1) will form the closed system from which the probabilities $P(1; t)$, $P(3; t)$, and $P(5; t)$ can be found providing $\langle N(t)\rangle$ and $\langle N^2(t)\rangle$ are known. The relationship between $\langle N(t)\rangle$ and the Lagrangian average of the modulus of the beam divergence has been found in Ref. 3

$$\langle N(x, t)\rangle = \langle |J(x, t)|\rangle_{\mathrm{L}} .$$

Here we can calculate $\langle N^2(x, t)\rangle$. To do this for the coordinate X of the ray coming from the point (y, t), the angle of ray arrival V, and the ray divergence J, we transform from the Lagrangian two–point probability density to the Eulerian one

$$\begin{aligned}
&W^{\mathrm{L}}_{X,\,V,\,J}(x_1, x_2, \upsilon_1, \upsilon_2, j_1, j_2; y_1, y_2, t) =\\
&= \langle \delta(X(y_1, t) - x_1)\,\delta(X(y_2, t) - x_2)\,\delta(V(y_1, t) - \upsilon_1) \times\\
&\times \delta(V(y_2, t) - \upsilon_2)\,\delta(J(y_1, t) - j_1)\,\delta(J(y_2, t) - j_2)\rangle_{\mathrm{L}} =\\
&= \frac{1}{|j_1 j_2|} \Big\langle \sum_{n=1}^{N(x_1,\,t)} \sum_{m=1}^{N(x_2,\,t)} \delta(y_n(x_1, t) - y_1) \times\\
&\times \delta(y_m(x_2, t) - y_2)\,\delta(\upsilon_n(x_1, t) - \upsilon_1)\,\delta(\upsilon_m(x_2, t) - \upsilon_2) \times\\
&\times \delta(j_n(x_1, t) - j_1)\,\delta(j_m(x_2, t) - j_2)\Big\rangle_{\mathrm{E}} .
\end{aligned} \tag{2}$$

Let us consider $x_2 = x_1 + s$ and introduce the notation for the numbers of rays arriving at the points (x_1, t) and (x_2, t)

$$N(x_1, t) = N ,\ \ N(x_2, t) = M .$$

Integrating equality (2) over y_1 and y_2 and taking the formula for the total probability into account, we derive

$$\begin{aligned}
&\int\limits_{-\infty}^{+\infty} W^{\mathrm{L}}(x_1, x_2, \upsilon_1, \upsilon_2, j_1, j_2; y_1, y_2, t)\,\mathrm{d}y_1\mathrm{d}y_2 =\\
&= \frac{1}{|j_1 j_2|} \sum_{N=1}^{\infty} \sum_{M=1}^{\infty} P(N, M; s, t) \times\\
&\times \sum_{n=1}^{N} \sum_{m=1}^{M} W^{\mathrm{E}}_{nm}(\upsilon_1, j_1; x_1, t\,|N; \upsilon_2, j_2; x_2, t\,|\,M) ,
\end{aligned}$$

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where $P(N, M; s, t)$ is the probability that N rays will arrive at the point (x_1, t) and M rays – at the point $(x_1 + s, t)$; W^E_{nm} is the joint probability density of the Eulerian fields t and j in the nth and mth rays under the same conditions. We multiply the last equality by $|j_1 j_2|$ and integrate it over j_1, j_2, υ_1, and υ_2

$$\int\limits_{-\infty}^{+\infty} |j_1 j_2|\; W^{\mathrm{L}}(x_1, x_2, j_1, j_2; y_1, y_2, t)\, \mathrm{d}y_1 \mathrm{d}y_2\, \mathrm{d}j_1 \mathrm{d}j_2 =$$

$$= \sum_{N,\,M}^{\infty} NMP(N, M; s, t)\ . \tag{3}$$

It can be seen that the right–side expression represents the correlation function of the number of rays

$$K_N(s, t) = \langle N(x, t)\, N(x + s, t)\rangle\ .$$

Let us simplify the left side of Eq. (3) accounting for the statistical homogeneity of the medium. To do this, we go over to the coordinates

$$s = x_1 - x_2\ ,\quad s_0 = y_1 - y_2\ ,\quad q_0 = \frac{y_1 + y_2}{2}$$

and integrate over q_0

$$K_N(s, t) = \int\limits_{-\infty}^{+\infty} |j_1 j_2|\; W^{\mathrm{L}}(s, j_1, j_2; s_0, t)\, \mathrm{d}s_0 \mathrm{d}j_1 \mathrm{d}j_2\ . \tag{4}$$

By definition,

$$W^{\mathrm{L}}(s, j_1, j_2; s_0, t) =$$

$$= \langle\delta(X(s_0, t) - X(0, t) - s)\, \delta(J(s_0, t) - j_1)\, \delta(J(0, t) - j_2)\rangle\ .$$

Substituting W^{L} into Eq. (4) on account of the properties of the δ–function, we obtain

$$K_N(s, t) = \int\limits_{-\infty}^{+\infty} \langle|J(0, t)\, J(s_0, t)|\, \delta(X(s_0, t) - X(0, t) - s)\rangle \mathrm{d}s_0. \tag{5}$$

We employ the relations for the coordinate and divergence of the geometric optical ray behind the phase screen[3]

$$X\,(y, t) = y + \upsilon_0(y)\, t\ ,$$

$$J\,(y, t) = 1 + u\,(y)\, t\ ,\quad u = \upsilon_0'(y)\ .$$

Then, taking these equations into account, we can perform the averaging in Eq. (5) using the joint probability density of the quantities $(s - s_0)/t$, $p_1 = u(0)\, t$, and $p_2 = u\,(s_0)\, t$

$$K_N(s, t) = \frac{1}{t} \int\limits_{-\infty}^{+\infty} |(1 + p_1)\,(1 + p_2)| \times$$

$$\times\, W\left(\frac{s - s_0}{t}, p_1, p_2; s, t\right) \mathrm{d}s_0\, \mathrm{d}p_1\, \mathrm{d}p_2\ . \tag{6}$$

The sought–after quantity $\langle N^2(t)\rangle$ can be obtained from Eq. (6) for $s = 0$.

The field $\upsilon_0(y)$ is taken to be Gaussian with zero average and covariance function

$$B_{\upsilon_0}(s_0) = \sigma_0^2 \exp(-\, s_0^2/d^2)\ ,$$

where d is the characteristic scale of the inhomogeneities of the screen. The probability density of the three–dimensional normal distribution[4] is

$$W(\xi_1, \xi_2, \xi_3) = \frac{1}{(2\pi)^{3/2}\, \sigma_1\sigma_2\sigma_3 \sqrt{\Delta}} \times$$

$$\times \exp\left\{ -\frac{1}{2\Delta}\left[\frac{\xi_1^2}{\sigma_1^2}(1 - R_{23}^2) + \frac{\xi_2^2}{\sigma_2^2}(1 - R_{13}^2) + \frac{\xi_3^2}{\sigma_3^2}(1 - R_{12}^2) + \right.\right.$$

$$+ \frac{2\xi_1\xi_2}{\sigma_1\sigma_2}(R_{12} - R_{13}R_{23}) + \frac{2\xi_2\xi_3}{\sigma_2\sigma_3}(R_{23} - R_{12}R_{13}) +$$

$$\left.\left. + \frac{2\xi_1\xi_3}{\sigma_1\sigma_3}(R_{13} - R_{12}R_{23})\right]\right\}\ ,$$

where

$$\Delta = 1 - R_{12}^2 - R_{13}^2 + R_{23}^2 + 2R_{12}R_{13}R_{23}\ .$$

In our case

$$\xi_1 = \frac{s - s_0}{t}\ ,\ \xi_2 = p_1\ ,\ \xi_3 = p_2\ ,\ \sigma_1 = \sigma_s\ ,\ \sigma_2 = \sigma_3 = \sigma_p\ ,$$

while the cross–correlation coefficients have the form

$$R_{12} = R(\,p_1, s_0) = \frac{t}{\sigma_p\sigma_s}\frac{\mathrm{d}B_{\upsilon_0}(s_0)}{\mathrm{d}s_0}\ ,$$

$$R_{13} = R(\,p_2, s_0) = R_{12}\ ,$$

$$R_{23} = R(\,p_1, p_2) = -\frac{t^2}{\sigma_p^2}\frac{\mathrm{d}^2B_{\upsilon_0}(s_0)}{\mathrm{d}s_0^2}\ .$$

In addition, the variances σ_s and σ_p of the fields s and $p_{1,2}$ are also expressed in terms of the function $B_{\upsilon_0}(s_0)$

$$\sigma_s^2 = 2\big(\sigma_0^2 - B_{\upsilon_0}(s_0)\big)\ ,\ \sigma_p^2 = -\, t^2 \left.\frac{\mathrm{d}^2B_{\upsilon_0}(s_0)}{\mathrm{d}s_0^2}\right|_{s_0=0}.$$

Further we can conveniently use the new dimensionless coordinates

$$\eta_0 = \frac{s_0}{d}\ ,\ z = \frac{t}{t_0} \equiv t\,\frac{\sigma_0}{d}\ .$$

Here d is the characteristic focal length,[5] i.e., the distance from the screen at which the strong fluctuations of the wave intensity occur.

Substituting now the function $W(\eta_0, p_1, p_2; z)$ into Eq. (6), we finally derive

$$\langle N^2(z)\rangle = \frac{1}{8\pi^{3/2} z^3} \int\limits_{-\infty}^{+\infty} |(1 + p_1)\,(1 + p_2)|\, \frac{1}{\sqrt{A_1 A_2}} \times$$

$$\times \exp\left\{ -\frac{1}{4\, z^2 A_4}\left[\eta_0^2 A_0 + \frac{(p_1^2 + p_2^2)\, A_2}{A_1} + \right.\right.$$

$$+ 2\eta_0^2(p_1 + p_2) - \frac{2p_1 p_2 A_3}{A_1} \Bigg]\Bigg\} d\eta_0 dp_1 dp_2 \,. \qquad (7)$$

Here we introduce the notation

$A_0 = 1 - (2\eta_0^2 - 1)\, e^{-\eta_0^2}$, $A_1 = 1 + (2\eta_0^2 - 1)\, e^{-\eta_0^2}$,

$A_2 = 1 - e^{-\eta_0^2} - \eta_0^2\, e^{-2\eta_0^2}$,

$A_3 = (1 + \eta_0^2\, e^{-\eta_0^2} - 2\,\eta_0^2 - e^{-\eta_0^2})\, e^{-\eta_0^2}$, $A_4 = A_2 + A_3$.

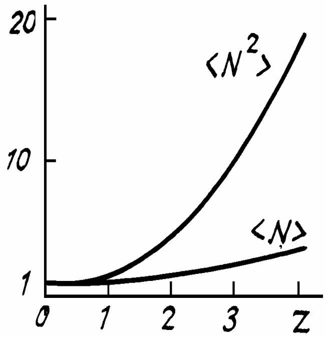

FIG. 1. Dependence of the mean square $\langle N^2\rangle$ and the average number $\langle N\rangle$ of rays on the distance in the region behind the phase screen.

Figure 1 shows the dependence $\langle N^2(z)\rangle$ obtained by numerical integration of Eq. (7). It also shows the dependence $\langle N(z)\rangle$ calculated by the formula given in Ref. 3

$$\langle N(z)\rangle = \Phi\left(\frac{1}{z}\right) + \sqrt{\frac{2}{\pi}}\, z \exp\left(-\frac{1}{2z^2}\right),$$

where $\Phi(\tau)$ is the error integral.

Now from the system of equations (1) we can derive the probabilities of the single–, three–, and five–ray propagation

$$P(1;\, z) = \frac{15 - 8\langle N\rangle + \langle N^2\rangle}{8},$$

$$P(3;\, z) = \frac{-5 + 6\langle N\rangle - \langle N^2\rangle}{4},$$

$$P(5;\, z) = \frac{3 - 4\langle N\rangle + \langle N^2\rangle}{8}.$$

For example, at $z = 1$, i.e., in the region of strong focusing, by substituting $\langle N\rangle = 1.167$ and $\langle N^2\rangle = 1.675$ into the above relations, we obtain $P(3;\, z = 1) = 0.08175$ and $P(5;\, z = 1) = 0.000875$.

This means that the multiray propagation is manifested at distances much longer than the distances at which the strong overshoots of the wave intensity occur in the vicinities of the originating caustics.

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