APPLICABILITY LIMITS OF THE METHOD OF PHYSICAL OPTICS IN THE PROBLEMS ON LIGHT SCATTERING BY LARGE CRYSTALS. PART. I. SCATTERING BY A ROUND PLATE

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Analytical relations for cross sections and efficiency factors of the extinction, scattering, and absorption are obtained within the framework of the method of physical optics for a round plate when the normal to its base coincides with the direction of incident wave propagation. It is shown based on the energy relationships relating the sought characteristics that the error in the method of physical optics is estimated by a linear combination of two integrals whose values depend only on the diffraction parameter $p = ka$ (a is the radius of a plate). Different numerical estimates of the relationship of the method uncertainty with the diffraction parameter are presented.

The method of physical optics is shown to be the most optimal one in describing the scattering of optical radiation which interacts with spatially oriented atmospheric crystals.[1,2] Actually, an electromagnetic field near the scatterer surface is formed as beams of parallel rays due to its polyhedral shape. It should be noted that linear dimensions of each beam forming the field for any atmospheric crystal many times exceed the wavelength not only in the visible but also in the near and middle IR ranges. If an electromagnetic field in each beam cross section and contours of this section are known, then the method of physical optics allows one to simply recalculate electromagnetic field of refracted beams from the near to the far zone. This method does not take into account possible edge perturbation of the electromagnetic field of refracted beams. However the uncertainty of the method is negligible provided that the boundary region of the beam cross section where the distortions occur is far less than the entire cross section of a beam. Finally, this requirement is reduced to the statement that the larger are the linear dimensions of a polyhedron compared to the wavelength, the more accurate is the physical optics description of the electromagnetic field scattering by a polyhedron. In this paper the aforementioned statement is embodied in quantitative estimates obtained based on the law of energy conservation.

As a crystal model, the simplest geometric shape of a round plate with a complex refractive index $\tilde{n} = n + i\kappa$, radius a and thickness d was chosen. The normal to the plate base is assumed to be oriented along the direction of propagation of a plane wave incident on it. Such a problem formulation makes it possible to obtain all energy characteristics of scattering in the simplest and most suitable for analysis form.

The efficiency factors of extinction (Q_{ext}), scattering (Q_{sca}), and absorption (Q_{abs}) for any scatterer, including those for a round plate, are related by the following expressions[3]:

$$Q_{ext} = Q_{sca} + Q_{abs}\,; \qquad \kappa \neq 0\,, \tag{1}$$

$$Q_{ext} = Q_{sca}\,; \qquad \kappa = 0\,. \tag{2}$$

Based on relation (2) we find the applicability limits of the method of physical optics for the particular case ($\kappa = 0$) of the aforementioned problem of scattering. Then we make necessary generalization when proceeding to the general case of the problem for $\kappa \neq 0$ that makes relation (1) valid.

At normal incidence of the wave onto the plate base the solution of the scattering problem, by virtue of symmetry, is reduced to a scalar case. In addition, in the formula for a scattered field written in a spherical system of coordinates (r, ϑ, φ) there should be no dependence on the azimuthal angle φ. With these simplifications taken into account the scattering on the plate of any component of the electromagnetic field with unit amplitude is described by the relation

$$\Psi(r,\,\vartheta) = S(\vartheta)\,\exp(i\,\kappa\,r)/\,i\,\kappa\,r\,, \tag{3}$$

where

$$S(\vartheta) = (1 - T)\,F_1(\vartheta) + R\,F_2(\vartheta)\,. \tag{4}$$

The first term in relation (4), with the account of mutual phase shifts, couples a diffracted field with the scattered fields of all beams emerging from the plate along the direction of incident wave propagation. The second term involves the scattered fields of the reflected beam and of all beams emerging from the plate along the backward direction. Within the framework of the problem formulated T and R are determined as the Fresnel coefficients for a plane wave interacting with a semitransparent layer and have the form

$$T = \frac{t\,\exp[i\,\kappa\,d\,(n-1)]}{1 - r\,\exp(2\,i\,\kappa\,d\,n\,)}\,, \tag{5}$$

$$R = \frac{n-1}{n+1}\left(1 - \frac{t\,\exp(2\,i\,\kappa\,d\,n)}{1 - r\,\exp(2\,i\,\kappa\,dn\,)}\right), \tag{6}$$

where

$$t = \frac{4\,n}{(n+1)^2}\,; \quad r = \left(\frac{n-1}{n+1}\right)^2.$$

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The amplitude functions $F_1(\vartheta)$ and $F_2(\vartheta)$ characterizing the scattering of the propagated and reflected beams are determined as Fraunhofer integrals over the plate base. Finally, these integrals are reduced to the analytical expressions typical for scattering theory[3,4]

$$F_1(\vartheta) = p^2 \frac{1 + \cos\vartheta}{2} \frac{J_1(p \sin\vartheta)}{p \sin\vartheta} , \tag{7}$$

$$F_2(\vartheta) = p^2 \frac{1 + \cos(\pi - \vartheta)}{2} \frac{J_1(p \sin(\pi - \vartheta))}{p \sin(\pi - \vartheta)} , \tag{8}$$

where $p = \kappa a$ is the diffraction parameter; κ is the wave number, and $J_1(z)$ is the first–order Bessel function.

The extinction cross section σ_{ext} is found from the formula[3,4]

$$\sigma_{\text{ext}} = \frac{4\pi}{\kappa^2} \operatorname{Re}(S(0)) . \tag{9}$$

Substituting Eq. (4) into Eq. (9) and taking into account that $F_2(0) = 0$ and $F_1(0) = (1/2)\, p^2 = (1/2)(\kappa\, a)^2$ we obtain for the extinction cross section

$$\sigma_{\text{ext}} = 2\pi\, a^2 (1 - \operatorname{Re}(T)) , \tag{10}$$

and the extinction efficiency factor sought is determined by the relation

$$Q_{\text{ext}} = \sigma_{\text{ext}}/\pi a^2 = 2(1 - \operatorname{Re}(T)) . \tag{11}$$

The scattering cross section σ_{sca} and the scattering efficiency factor Q_{sca} are calculated by the formulas[4]

$$\sigma_{\text{sca}} = \int_0^{2\pi} \int_0^{\pi} \frac{|S(\vartheta)|^2}{\kappa^2} \sin\vartheta \, d\vartheta \, d\varphi = \frac{2\pi}{\kappa^2} \int_0^{\pi} |S(\vartheta)|^2 \sin\vartheta \, d\vartheta, \tag{12}$$

$$Q_{\text{sca}} = \sigma_{\text{sca}}/\pi\, a^2 . \tag{13}$$

The relation for $|S(\vartheta)|^2$ is found using Eq. (4). Following some simple transformations we obtain

$$|S(\vartheta)|^2 = [F_1(\vartheta)]^2\, |1 - T|^2 + [F_2(\vartheta)]^2\, |R|^2 +$$

$$+ 2\, F_1(\vartheta)\, F_2(\vartheta) \operatorname{Re}[(1 - T)\, R^*] . \tag{14}$$

For the amplitude functions $F_1(\vartheta)$ and $F_2(\vartheta)$ it is easy to prove the identities

$$\int_0^{\pi} [F_1(\vartheta)]^2 \sin\vartheta \, d\vartheta = \int_0^{\pi} [F_2(\vartheta)]^2 \sin\vartheta \, d\vartheta , \tag{15}$$

$$\int_0^{\pi} F_1(\vartheta)\, F_2(\vartheta) \sin\vartheta \, d\vartheta = 2 \int_0^{\pi/2} F_1(\vartheta) F_2(\vartheta) \sin\vartheta \, d\vartheta \tag{16}$$

With relations (14)–(16) taken into account the expression for the scattering efficiency factor takes the form

$$Q_{\text{sca}} = \left(|1 - T|^2 + |R|^2 \right) A(p) + 2\operatorname{Re}[(1 - T)\, R^*]\, B(p), \tag{17}$$

where

$$A(p) = 2 \int_0^{\pi} \left(\frac{1 + \cos\vartheta}{2} \right)^2 \frac{J_1^2(p \sin\vartheta)}{\sin\vartheta} \, d\vartheta , \tag{18}$$

$$B(p) = \int_0^{\pi/2} \sin\vartheta \, J_1^2(p \sin\vartheta) \, d\vartheta . \tag{19}$$

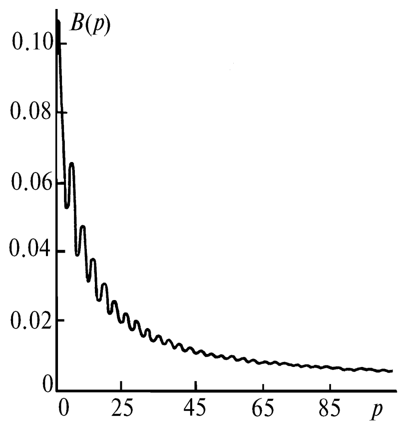

FIG. 1. A plot of values of the integrals A(p) vs the diffraction parameter p.

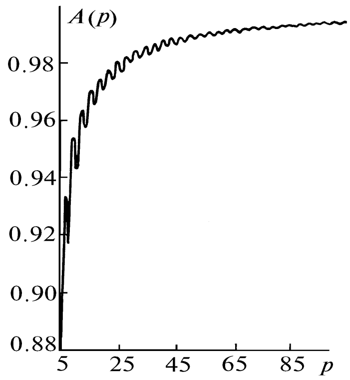

FIG. 2. A plot of values of the integrals B(p) vs the diffraction parameter p.

Depicted in Figs. 1 and 2 are the plots of integrals $A(p)$ and $B(p)$. Analysis of expressions (18) and (19) reveals that

$$\lim_{p\to\infty} A(p) = 1, \quad \lim_{p\to\infty} B(p) = 0. \tag{20}$$

As a result, for $p \to \infty$ the proof of relation (2) is reduced to the proof of the identify

$$2(1 - \operatorname{Re}(T)) = |1 - T|^2 + |R|^2 , \tag{21}$$

which can readily be transformed to a simpler form

$$|T|^2 + |R|^2 = 1. \tag{22}$$

Substitution of expressions (5) and (6) for T and R into identity (22) allows it to be easily proved. It should be noted that in identity (22) the law of electromagnetic energy conservation is represented in a more explicit form than that in relation (2). Actually, it follows from Eq. (22) that in the absence of absorption in the plate ($\kappa = 0$) the sum of intensities of the propagated and reflected electromagnetic fields is equal to the incident field intensity.

As it should be expected, the method of physical optics accurately describes the scattering only as $p \to$ \$. Let us estimate the uncertainty of this method for an arbitrary value of the diffraction parameter p. Let us use the designation

$$\frac{Q_{\text{ext}} - Q_{\text{sca}}}{Q_{\text{ext}}} = \Delta(p). \tag{23}$$

Expression (2) interrelates the local and integral values of the amplitude function $S(\vartheta)$. Then the difference between the values Q_{ext} and Q_{sca} is taken as an estimate of the uncertainty of the method used for obtaining the amplitude function $S(\vartheta)$. To determine the upper limit for $\Delta(p)$ let us use the inequality

$$-(|1-T|^2 + |R|^2) \le 2\text{Re}[(1-\text{T})R^*] \le |1-T|^2 + |R|^2 . \tag{24}$$

Inequality (24) can readily be proved in the general form, i.e., it is not related to the magnitudes of the complex values T and R. Let us transform inequality (24) with the account of identity (21) and expressions (11), (17) describing Q_{ext} and Q_{sca}. As a result, we have

$$Q_{\text{ext}}(A(p) - B(p)) \le Q_{\text{sca}} \le Q_{\text{ext}}(A(p) + B(p)). \tag{25}$$

By combining relation (23) and the left–hand side of inequality (25) we obtain the estimate for relative characteristic (23)

$$\Delta(p) \le 1 - A(p) + B(p) . \tag{26}$$

The latter inequality yields the following numerical estimates: $\Delta(p) < 10\%$ when $p > 10$ ($a > 1.6\ \lambda$); $\Delta(p) < 5\%$ when $p > 20$ ($a > 3.2\ \lambda$); $\Delta(p) < 2\%$ when $p > 55$ ($a > 8.8\ \lambda$); and, $\Delta(p) < 1\%$ when $p > 120$ ($a > 19.1\ \lambda$). It follows from inequality (26) that only the highest degree of localization of amplitude functions around the directions of output refracted beams ($A(p) \approx 1$, $B(p) \approx 0$) provides for small values of the relative uncertainty.

Let the absorption coefficient of the plate κ differ from zero. This complication of the problem results only in formal modifications of the relations obtained. In particular, the real refractive index n in formulas for T and R must be replaced by the complex index $\tilde{n} = n + i\kappa$. After such a substitution the aforementioned Fresnel coefficients are denoted as $\tilde{T}$ and $\tilde{R}$. The procedures for solving the formulated problem of scattering for the general ($\kappa \ne 0$) and particular ($\kappa = 0$) cases coincide. Therefore the values Q_{ext} and Q_{sca} for $\kappa \ne 0$ are determined from the same relations (11) and (17) in which new designations $\tilde{T}$ and $\tilde{R}$ must be used instead of T and R.

Let us now determine the absorption efficiency factor Q_{abs} from Eq. (1) assuming that $A(p) = 1$ and $B(p) = 0$.

The result is

$$Q_{\text{abs}} = 1 - |\tilde{T}|^2 - |\tilde{R}|^2 . \tag{27}$$

Referring to identify (22) proved for $\kappa = 0$ it can readily be seen that the right–hand side of relation (27) involves the only possible combination of Fresnel coefficients which describes the absorption of electromagnetic energy in the plate. Actually, if the intensities of the propagated and reflected electromagnetic fields are subtracted from the intensity of the field incident onto the plate, the difference obtained determines the intensity losses due to absorption. Thus the only possible form of the right–hand side of relation (27) proves the validity of identity (1). As a result, obtained estimate (26) for the relative uncertainty of the method of physical optics remains valid in the case of absorbing scatterers, as well.

Using Eq. (27) we write the formula for the absorption cross section σ_{abs} which can be useful for estimating the absorbed energy of optical radiation interacting with a system of oriented plate crystals. It has the form

$$\sigma_{\text{abs}} = \pi a^2 (1 - |\tilde{T}|^2 - |\tilde{R}|^2). \tag{28}$$

Remind that $\tilde{T}$ and $\tilde{R}$ are determined from relations (5) and (6) where n must be replaced by $\tilde{n}$.

Linear dimensions of spatially oriented atmospheric crystals are, as a rule, hundreds and even thousands of micrometers. Therefore the obtained numerical estimates determining the applicability limits of the method of physical optics ensure its correct use for the aforementioned crystals with high enough accuracy not only in the visible but also in the IR ranges.

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