ANALYTICAL METHODS IN THE THEORY OF OPTICAL RADIATION PROPAGATION THROUGH MACROINHOMOGENEOUS STOCHASTIC SCATTERING MEDIA

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Basic ideas of two new methods for solving the stochastic radiative transfer equation (local homogeneity approximation and linear (quadratic) approximation) are described. The expressions for the moments of stochastic medium transmittance in the small–angle approximation are given. Optical image transfer through a stochastic scattering medium is discussed.

It is generally recognized that full description of radiation propagation through the Earth's atmosphere calls for the statistical methods in the study of physical processes. These methods provide the construction of the most adequate mathematical models of the Earth's atmosphere, which account for the stochastic variations of the parameters in space and time, as well as the statistical description of optical radiation propagation through this medium.[1] The basic problem of such an approach is to determine the relation between the statistical characteristics of the scattering parameters of a medium (e.g., probability densities, mean values, and variances) and the corresponding characteristics of radiation fields.

Two classes of statistical models of the atmosphere (one– and three–dimensional models) are of particular interest for practical implementation. Within the scope of one–dimensional stochastic model the parameters of the atmosphere are modelled by one–dimensional random functions of altitude, and within the scope of a three–dimensional model these parameters are random functions of three spatial coordinates (random fields). In this case the simplest one–dimensional model is a good approximation for modelling of the light propagation through cloudless atmosphere and continuous cloudiness. The most complicated three–dimensional model makes it possible to describe the effect of spatial fluctuations of the parameters of a medium in cloudless atmosphere and fog.[2]

The most glowing example of a medium with three–dimensional random fluctuations of scattering parameters is the atmosphere with cumulus clouds.

The level of spatial fluctuations of scattering parameters of the atmosphere can be conveniently characterized by the dimensionless parameter $S = M\{\varepsilon\}\, l_\varepsilon$. Hereinafter, $M\{\varepsilon\}$ defines the mean value of the random variable ε, and l_ε is the spatial scale of fluctuations in the extinction coefficient ε. For $S < 1$ the extinction coefficient fluctuations can be considered to be weak, and for $S > 1$ they are assumed to be strong. For real natural conditions the following values of the parameter S are representative: for cloudless atmosphere $S \simeq 10^{-2}$–10^{-1}, for fogs $S \simeq 10^{-1}$–10^{1}, and for cumulus clouds $S \simeq 10^{1}$–10^{2}.

At present the theory of wave propagation through macroinhomogeneous stochastic media is being increasingly developed. It is based on numerical and analytical methods for solving the stochastic transfer equation in which the scattering parameters are considered to be three–dimensional inhomogeneous random functions. The Monte Carlo method is the most advanced numerical method for solving this equation in the case of solar radiation propagation through cumulus clouds.[3,4] The first analytic solutions of the stochastic transfer equation were obtained in Refs. 5 and 6 for a limiting case $S \ll 1$.

During the past several years the more effective methods of analytic solution of the stochastic transfer equation have been developed. They are applicable to arbitrary values of the parameter S and based on the small–angle approximation of the transfer equation (approximation of local homogeneity and linear (quadratic) approximation).

The present paper gives an overview of basic ideas of these methods and describes future trends in the development of the analytical theory.

APPROXIMATION OF LOCAL HOMOGENEITY

In the context of the transfer theory light fields from different sources can be conveniently found from the Green's function $G_0(\boldsymbol{\rho}_0;\, \mathbf{r};\, \boldsymbol{\Omega}_0;\, \boldsymbol{\Omega})$ which describes a light field produced at a point of $\mathbf{r}$ in the direction $\boldsymbol{\Omega}$ with a point–size unidirectional source located in the plane $z = 0$ at a point specified by a radius vector $\boldsymbol{\rho}_0$, which radiates in the direction $\boldsymbol{\Omega}_0$. For a horizontally homogeneous medium the Green's function is spatially invariant and has the form

$$G_0(\boldsymbol{\rho}_0;\, \mathbf{r};\, \boldsymbol{\Omega}_0;\, \boldsymbol{\Omega}) = G_0(\boldsymbol{\rho} - \boldsymbol{\rho}_0 - z\, \boldsymbol{\Omega}_\perp / \Omega_z;\, z;\, \boldsymbol{\Omega}_0;\, \boldsymbol{\Omega}) , \qquad (1)$$

where $\boldsymbol{\rho}$ and $\boldsymbol{\Omega}_\perp$ are the projections of the vectors $\mathbf{r}$ and $\boldsymbol{\Omega}$ onto the plane $z = 0$, Ω_z is the direction cosine of the angle between the unit vector $\boldsymbol{\Omega}$ and the coordinate axis z. A three–dimensional inhomogeneous stochastic medium can be approximately considered to be horizontally homogeneous only within a small region whose dimensions are smaller than a horizontal scale of fluctuations of the scattering parameters of a medium $l_\perp$. Therefore when the transverse width of the Green's function $R_\perp < l_\perp$, the Green's function within this region can also be assumed to be spatially invariant, and its form is determined solely by the law of variation of the scattering parameters along the beam axis. From the above reasoning we have for the Green's function G of a three–dimensional stochastic layer

$$G_0(\boldsymbol{\rho}_0;\, \mathbf{r};\, \boldsymbol{\Omega}_0;\, \boldsymbol{\Omega}) \approx G_0(\boldsymbol{\rho}_0;\, \boldsymbol{\rho} - \boldsymbol{\rho}_0;\, z;\, \boldsymbol{\Omega}_0;\, \boldsymbol{\Omega}) , \qquad (2)$$

where G_0 is the Green's function of a horizontally homogeneous medium whose scattering characteristics in the direction $\mathbf{\Omega}_0$ are identical to those of a three–dimensional layer along the axis of the Green's function. Moreover, since we believe that in the horizontal direction within the width of the Green's function R the scattering parameters do not change, the Green's function G_0 is approximately equal to the Green's function G'_0 of a horizontally homogeneous medium whose scattering characteristics are identical to those of a three–dimensional inhomogeneous layer in the direction of the unit vector $\mathbf{\Omega}$ whose origin is placed at a point $\mathbf{r}$:

$$G_0(\rho_0; \mathbf{r}; \mathbf{\Omega}_0; \mathbf{\Omega}) \approx G'_0(\rho; \rho - \rho_0; z; \mathbf{\Omega}_0; \mathbf{\Omega}) . \quad (3)$$

Relations (2) and (3) form the basis for the approximation of local homogeneity. In this approximation the problem of determining the Green's function of a three–dimensional stochastic medium is reduced to a simpler problem of finding the Green's function of a one–dimensional stochastic medium. The stochastic realizations of the light field brightness $I(\mathbf{r}; \mathbf{\Omega})$ produced by arbitrary sources with spatial–angular distribution of brightness $I_0(\mathbf{r}; \mathbf{\Omega})$ can be found from the formulas

$$I_0(\mathbf{r}; \mathbf{\Omega}) = \int\int d\rho_0\, d\Omega_0\, I_0(\mathbf{r}_0; \mathbf{\Omega}_0)\, G_0(\rho_0; \rho - \rho_0; z; \mathbf{\Omega}_0; \mathbf{\Omega}) \approx$$

$$\approx \int\int d\rho_0\, d\Omega_0\, I_0(\mathbf{r}_0; \mathbf{\Omega}_0)\, G'_0(\rho; \rho - \rho_0; z; \mathbf{\Omega}_0; \mathbf{\Omega}) , \quad (4)$$

and the statistical parameters of light fields can be determined by statistical averaging of these expressions.

In practical applications of expressions (4) the Green's functions G_0 and G'_0 can be found by any convenient method. However by now the methods of modelling of the light fields have been developed only in the small–angle approximation of the transfer equation. This is accounted for by the fact that in this approximation the problem is reduced to well–developed divisions of the theory of random fields including the theory of characteristic functions and functionals.[7,8] Using the results of these divisions the authors of Refs. 1, 10, and 11 put forward a number of models of stochastic media which sufficiently simply describe the optical radiative transfer. Within the frameworks of these models it is possible to find not only the mean values of propagating radiation fields but also, in some cases, the moments of arbitrary order and probability densities. This calculational procedure may be illustrated by considering, by way of example, the statistical characteristics of the transmittance of a stochastic layer, illuminated by solar radiation, based on the Gaussian model of this layer. In this case $I_0(\rho; \mathbf{\Omega}) = I_0\delta(\mathbf{\Omega} - \mathbf{\Omega}_0)$, where δ is the delta function and I_0 is the incident beam intensity, and in the small–angle approximation[12] the transmittance is

$$T_0(\mathbf{r}; \mathbf{\Omega}_0) = \exp\left\{-\int_0^{z_0} k^*(\mathbf{r}'_u)\, d\, u/\mu_0\right\} , \quad (5)$$

where z_0 is the layer thickness, k^* is the effective absorptance by the medium,[12] $\{\mathbf{r}'_u\} = \{\rho - \mathbf{b}(z_0 - u); u\}$, $\mathbf{b} = \mathbf{\Omega}_{0\perp}/\mathbf{\Omega}_{0z}$, $\mathbf{r}'_u$ is the radius vector specifying the straight line through the point ρ in the direction $\mathbf{\Omega}_0$, and $\mu_0 = \mathbf{\Omega}_{0z}$. Then the moments of the order n of the transmittance are

$$M_{nt} = M\left\{ T_0^n(\mathbf{r}; \mathbf{\Omega}_0) \right\} , \quad (6)$$

or

$$M_{nt} = M\left\{ \exp\left[-n\int_0^{z_0} k^*(\mathbf{r}'_u)\, d\, u/\mu_0 \right]\right\}. \quad (7)$$

By the definition,[7] the characteristic functional of random process $k^*(\mathbf{r}'_u)$ is

$$\Phi_k(\nu) = M\left\{ \exp\left[i\int_0^{z_0} k^*(\mathbf{r}'_u)\, \nu(u)\, d\, u \right]\right\} , \quad (8)$$

where $\nu(u)$ is the arbitrary function and i is the unit imaginary number. Then it is easy to see that

$$M_{nt} = \Phi_k\,[-\,i\, n/\mu_0] . \quad (9)$$

For the Gaussian model of a stochastic medium we assume that the quantity

$$\tau_s = \int_0^{z_0} k^*(\overline{\mathbf{r}}'_u)\, d\, u/\mu_0$$

is the Gaussian random variable. Then the moments M_{nt} are

$$M_{nt} = \exp\left[-\,n\, M\{\tau_s\} + n^2 D\{\tau_s\}/2\right] , \quad (10)$$

where

$$M\{\tau_s\} = \int_0^{z_0} M\,\{k^*(\mathbf{r}'_u)\}\, d\, u/\mu_0 , \quad (11)$$

the variance of τ_s

$$D\{\tau_s\} = \int_0^{z}\int_0^{z} R\,(\mathbf{r}_{u1}; \mathbf{r}_{u2})\, d\, u_1\, d\, u_2\,/\mu_0^2 , \quad (12)$$

$$R(\mathbf{r}'_u; \mathbf{r}''_u) = M\left\{\left[k^*(\mathbf{r}'_u) - M\,\{k^*(\mathbf{r}'_u)\}\right]\left[k^*(\mathbf{r}''_u) - M\,\{k^*(\mathbf{r}''_u)\}\right]\right\}$$

represents the correlation function for the three–dimensional random process $k^*(\mathbf{r})$, and

$$\{\mathbf{r}_{u1}\} = \{\rho - \mathbf{b}(z_0 - u_1); u_1\},\ \ \{\mathbf{r}_{u2}\} = \{\rho - \mathbf{b}(z_0 - u_2); u_2\} .$$

The probability density of transmittance for the Gaussian model of a stochastic layer obeys a log–normal distribution. The Gaussian model is applicable for $M\{\tau_s\} > nD\{\tau_s\}/2$.

Some other models of a stochastic medium and light fields in them were described in Refs. 1, 9, and 13. A comparison of the results of calculations with the numerical data obtained by the Monte Carlo method and with the results of field measurements performed in Refs. 1 and 13 showed good agreement of the results for an optically not very thick cloud. For a sufficiently large optical thickness of clouds the agreement deteriorates and the results of calculations made in the small–angle approximation become inapplicable. Therefore it is of interest to obtain sufficiently convenient mathematical expressions for the average Green's functions G_0 and G_0' applicable at large optical thicknesses as well. The experience in developing the classical (deterministic) transfer theory showed that it can be done using the small–angle diffusion or diffusion approximations of the transfer equation.

OPTICAL IMAGE TRANSFER THROUGH A STOCHASTIC SCATTERING LAYER

Let us consider the case of observation through a stochastic scattering layer of a harmonic plane test object with brightness distribution

$$B(\rho) = B_0[1 + k_0 \exp(i\, \omega_0\, \rho)] , \qquad (13)$$

where B_0 is the mean brightness of the object, k_0 is its contrast, and ω_0 is the spatial frequency. Substituting Eq. (13) into the lower equality of Eq. (4), we obtain

$$I(\mathbf{r};\, \Omega) = B_0\, T(\mathbf{r};\, \Omega)\Big[\, 1 + \tau(\omega_0;\, \rho;\, \Omega)\, k_0 \exp(i\, \omega_0\, \rho)\Big] , \qquad (14)$$

where

$$\tau(\omega_0;\, \rho;\, \Omega) = T(\omega_0;\, \mathbf{r};\, \Omega)/\, T(\mathbf{r};\, \Omega),\;\; T(\mathbf{r};\, \Omega) = T(\omega = 0;\, \mathbf{r};\, \Omega),$$

$$T(\omega;\, \mathbf{r};\, \Omega) = \int\!\!\int d\xi\; d\,\Omega_0\, G_0'(\rho;\, \xi;\, z;\, \Omega_0;\, \Omega)\, \exp(\, i\, \omega\, \xi)$$

is the Fourier transform of the Green's stochastic function of the diffuse source and $\xi = \rho - \rho_0$.

The expression (14) is analogous to that describing an image of a harmonic plane object transmitted through a deterministic scattering layer. The only difference is that in this case the functions $\tau(\omega_0;\, \rho;\, \Omega)$ and $T(\omega_0;\, \rho;\, \Omega)$ are random and depend on the spatial coordinate $\mathbf{r}$. As for the deterministic layer, the function $\tau(\omega_0;\, \rho;\, \Omega)$ determines the image contrast which, however, is random for a stochastic layer and depends on the spatial coordinate $\mathbf{r}$. Starting from these facts it was suggested in Ref. 1 to call the function $T(\omega_0;\, \rho;\, \Omega)$ the unnormalized local optical transfer function (OTF) of a stochastic layer, and the function $\tau(\omega_0;\, \rho;\, \Omega)$ – the normalized local OTF.

The statistical characteristics of the image contrast of self–illuminating and reflecting objects illuminated by the solar radiation were studied in Refs. 1, 10, 13, and 14.

LINEAR AND QUADRATIC APPROXIMATIONS

One more method for solving the stochastic equation was proposed in Ref. 15. The essence of this method can be explained in a rather simple way using the operator methods. In operator form the transfer equation is $LG = 0$, where L is the stochastic operator of the transfer equation and $G(0)$ is its stochastic Green's function. This function is represented by a sum of the Green's functions of unscattered G_u and multiply scattered G^* light. The equation for the function G has the form $LG^* = Q$, where Q is the functional describing the singly scattered radiation. The stochastic operator L and the functional Q are represented by a superposition of deterministic L_0 and Q_0 and random V and F components, and instead of the equation $LG^* = Q$ we consider an equivalent system of equations

$$L_0\, G_0^* = Q_0 - M\{V\, \tilde{G}^*\}\ ,$$

$$L_0\, \tilde{G}^* = F + M\, \{V\, \tilde{G}^*\}\, -\, V\, G_0^* - V\, \tilde{G}^*\ , \qquad (15)$$

where G_0^* and $\tilde{G}^*$ are the deterministic and random components of the function G^*. Instead of the system of equation (15) it is more convenient to consider another system which differs only by a dimensionless parameter s

$$L_0\, G_0 = Q_0 - s\, M\, \{V\, \tilde{G}^*\}\ ,$$

$$L_0\, \tilde{G}^* = F + s\, M\, \{V\, \tilde{G}^*\} - s\, V\, G_0^* - s\, V\, \tilde{G}^*\ . \qquad (16)$$

The mathematical analysis of the problem revealed that in this system the action of the operator V can be considered to be weak as compared to that of the operator F. Physically this corresponds to the fact that at an arbitrary point of a medium the radiation fluctuations due to singly scattered light are considered to be much weaker than the fluctuations caused by multiply scattered light. As a result, the solution of Eq. (16) can be found as a series expansion in the parameter s. In this case if in the expansion of deterministic components we take into consideration only the terms linear with respect to s, we will have linear approximation. If we take into consideration quadratic terms, we can obtain quadratic approximation.[15] In linear approximation for regular and variable components of the Green's function G_1^* and $\tilde{G}_1^*$ we have

$$L_0\, G_1^* = Q_0\, ,\; L_0\, \tilde{G}_1^* = F\ . \qquad (17)$$

In the quadratic approximation for regular and variable components of the Green's function G_2^* and $\tilde{G}_2^*$ we obtain the following equations:

$$\{L_0 - M\, \{V\, L_0^{-1}\, V\}\}\;\, G_2^* =$$

$$= Q_0 - M\, \{V\, L_0^{-1}\, F\} - M\, \{V\, L_0^{-1}\, V\, L^{-1}\, F\,\}\ ,$$

$$L_0\, \tilde{G}_2^* = F + M\, \{V\, L_0^{-1}\, F\} - V\, G_2^* - V\, L_0^{-1}\, F\ . \qquad (18)$$

where L_0^{-1} is the operator inverse to L_0.

The solution of the systems of equations (17)–(18) for mean fields were found in Ref. 15 in the small–angle approximation of the radiative transfer equation. These solutions are in good agreement with the numerical results obtained by the Monte Carlo method.

By now the linear and quadratic approximations have been developed much more poorly than the local homogeneity approximation. Since the physical and mathematical interpretations of these approximations are not so clear as of the local homogeneity approximations, we need some additional examinations of convergence conditions used in solving the series. It is also of interest to derive the solutions of Eqs. (17)–(18) at large optical thicknesses.

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