ON THE THEORY OF OPTICAL TRANSFER OPERATOR

T.A. Sushkevich, A.K. Kulikov, and S.V. Maksakova

M.V. Keldysh Institute of Applied Mathematics, the Russian Academy of Sciences, Moscow

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In this paper we present optical transfer operator (OTO) constructed based on the consideration of the general boundary–value problem of the radiation transfer theory for the case of a plane layer with a reflecting bottom and finite sources of radiation. The kernel of the OTO is constructed of the influence functions (IF) identical to the point spread function or of the spatial frequency characteristics (SFCH), which coincide with the optical transfer function (OTF). The IF and SFCH are versatile linear transfer functions of the system "atmosphere (ocean, cloudiness, hydrom) – underlying surface" that are determined from a solution of the boundary–value problem of the theory of radiation transfer for a plane layer with nonreflecting boundaries irradiated with a source of like a cw laser beam. We have constructed an OTO for the case of horizontally inhomogeneous boundary with an anisotropic reflection when no splitting of spatial and angular variables is used in the scattering coefficient. Such an OTO has the most general form and is expressed in terms of IF and SFCH. We show in this paper that all other expressions for OTO are particular cases of the derived formula.

INTRODUCTION

The model (approximate or empirical) linear optical transfer function (OTF) and point spread functions (PSF), formulated at a physically rigorous level[1], are normally used in multidimensional problems on radiation correction of remote sensing data when studying various targets and environment, in processing of optical information, in the theories of vision and image transfer in turbid media, and in theoretical foundations for computational techniques for opto-electronic observational systems. The problems of radiation transfer through 3D plane-parallel layers with horizontally inhomogeneous reflecting boundaries are more complicated, since several theoretical principles put into the basis of the theory of linear systems are not satisfied then. These are the invariance principle, the theorem of optical reciprocity and isoplanarity.[1,2] Development of nonlinear approximations of the techniques of spatial frequency characteristics (SFC) and functions of influence (FI)[2] methods is of principal importance. First, one needs to estimate the role of nonlinear approximations in solution of specific applied problems. Second, it is important to reveal in explicit form the relations of either recorded or computed radiation to the characteristics of reflecting boundary. Third, it is necessary to formulate an efficient mathematical approach to construction of the optical transfer operator (OTO), that could be used as either exact or approximate solution to the general boundary–value problem of radiation transfer through a plane-parallel layer with finite sources of radiation and a horizontally inhomogeneous Lambertian or anisotropically reflecting underlying surface.

The approach proposed uses the series of perturbation theory[3-5] and generalized solution to the general boundary–value problem of radiation transfer through scattering and absorbing media (such as atmosphere, ocean, clouds and hydrometeors) above a reflecting bottom.[2,6-14] It is based on the physical characteristics of the transfer system that accounts for the fact that, in agreement with the physics of the phenomenon, the norm of the reflection operator does not exceed unity, so the constructed series are convergent. Analitical isolation of the "average" horizontally homogeneous component enables one to lower the norm of the horizontally variable component of the reflection operator so that the series converge faster.

Below we present some new results to demonstrate how, using the universal linear transfer characteristics (OTF identical to SFC and PSF identical to FI) one can obtain a solution to an approximation accounting for an arbitrary order interaction of radiation with the layer boundary and construct an OTO for the task of remote sensing of the underlying surface. The optical transfer operator formulated in this paper for the case of a horizontally inhomogeneous, anisotropically reflecting underlying surface, when no splitting of spatial and angular coordinates is feasible, is the most general form of OTO, from which one may derive all the particular presentations of OTO for any linear and nonlinear approximations available from literature.

FORMULATION OF THE PROBLEM

Consider a plane-parallel layer, infinite horizontally ($-\infty < x, y < \infty$) and having a finite height ($0 \le z \le H$), illuminated from either the top, bottom, or inside. The "layer - underlying surface" system is considered non-multiplicating at the level $z = H$. The direction in which the radiation propagates, $s = \{\vartheta, \varphi\}$ ($\mu = \cos\vartheta$) is described using spherical coordinates; $\vartheta = \arccos\mu$, $\vartheta \in [0, \pi]$ is the zenith angle, i.e. the angle between the propagation direction and the direction of the internal normal to the top boundary of the layer $z = 0$ which coincides with the z axis, and $\varphi \in [0, 2\pi]$ is the azimuth angle taken from positive direction of the x axis. The whole set of all directions s makes up a unit sphere $\Omega \equiv [-1, 1] \times [0, 2\pi]$ in which $\mu \in [-1, 1]$ and $\varphi \in [0, 2\pi]$; $\Omega^+ \equiv [0, 1] \times [0, 2\pi]$ and $\Omega^- \equiv [-1, 0] \times [0, 2\pi]$ are hemispheres for the downward (transmitted) and the upward (reflected) going radiation, respectively. For a convenience in formulation of the boundary conditions let us introduce two sets

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$\Gamma_0 = \{z, r_\perp, s: z=0, s \in \Omega^+\}$, $\Gamma_H = \{z, r_\perp, s: z=H, s \in \Omega^-\}$.

Following T.A. Germogenova,[13] we use term general boundary–value problem of the theory of radiation transfer

$$\{\hat{K}\Phi = 0, \quad \Phi|_{\Gamma_0} = 0, \quad \Phi|_{\Gamma_H} = \varepsilon \hat{R}\Phi + \varepsilon E(r_\perp, s), \quad (1)$$

provided that the source E and the reflection operator R do not turn to zero simultaneously, and $0 < \varepsilon \le 1$.

The first boundary–value problem for a 3D equation of radiation transfer

$$\{\hat{K}\Phi = 0, \qquad \Phi|_{\Gamma_0} = 0, \qquad \Phi|_{\Gamma_H} = E(r_\perp, s) \quad (2)$$

with its linear transfer operator

$$\hat{D} \equiv (s, \text{grad}) + \sigma(z) = \hat{D}_z + \left(s_\perp, \frac{\partial}{\partial r_\perp}\right), \ \hat{D}_z \equiv \mu\frac{\partial}{\partial z} + \sigma(z),$$

the integral of collisions

$$\hat{S}\Phi \equiv \sigma_s(z)\int_\Omega \gamma(z, s, s')\Phi(z, r_\perp, s')\, d s',$$

and the integro-differential operator $\hat{K} \equiv \hat{D} - \hat{S}$, is reduced, through a Fourier transform over the coordinate $r_\perp = \{x,y\}$

$$\check{f}(p) \equiv F[\, f(r_\perp)](\, p) = \int_{-\infty}^{\infty} f\,(r_\perp)\exp[\, i\,(\, p, r_\perp)]\, d\, r_\perp,$$

(the spatial frequency $p = \{p_x, p_y\}$ here takes only real values $-\infty < p_x, p_y < \infty$) to the boundary–value problem for a parametric complex one-dimensional equation of radiation transfer [2]

$$\left\{\hat{L}(\, p)\,\check{\Phi} = 0, \qquad \check{\Phi}|_{\Gamma_0} = 0, \qquad \check{\Phi}|_{\Gamma_H} = \check{E}(\, p, s) \quad (3)\right.$$

with the operator $\hat{L}(\, p) \equiv \hat{D}_z - i\,(\, p, s_\perp) - \hat{S}$,
$(p, s_\perp) = p_x \sin\vartheta\cos\varphi + p_y \sin\vartheta\sin\varphi$.

Fourier images are marked with hacek. Optical properties of the medium are described by vertical distributions of the extinction coefficient $\sigma(z) = \sigma_s(z) + \sigma_{abs}(z)$, absorption coeffecient $\sigma_{abs}(z)$, total scattering coefficient $\sigma_s(z)$, and the total scattering phase function $\gamma(z, s, s')$ normalized according to the expression $\int_\Omega \gamma(z, s, s')\, ds' = 1$.

By analogy with the theory of constant coefficient differential equations in partial derivatives[8-12], the solution to the problem (2) is presented as a linear functional [2,15]

$$\Phi(z, r_\perp, s) = (\Theta, E) = \frac{1}{2\pi}\int_{\Omega^-} d\, s^- \int_{-\infty}^{\infty} \Theta(s^-; z, r_\perp - r'_\perp, s) E(r'_\perp, s^-) dr'_\perp$$

with the kernel FI $\Theta\,(s^-;\ z, r_\perp, s)$ being the solution to a boundary–value problem

$$\left\{\hat{K}\, Q = 0, \ \ \Theta|_{\Gamma_0} = 0, \ \ \Theta|_{\Gamma_H} = f_\delta(s^-; r_\perp, s);\right.$$

$$f_\delta(s^-; r_\perp, s) = \delta(\, r_\perp)\, \delta(s - s^-), \quad (4)$$

or in terms of Fourier transforms it may be treated as a solution to the problem (3) in the form of a linear functional

$$\check{\Phi}(z, p, s) = (\Psi, \check{E}) = \frac{1}{2\pi}\int_{\Omega^-} \Psi(s^-; z, p, s)\, \check{E}(\, p, s^-)\, d\, s^-$$

with the kernel in the form of SFC $\Psi\,(s^-, z, p, s)$ which is a solution of the boundary–value problem for the parametric complex equation of radiation transfer

$$\left\{\hat{L}(\, p)\,\Psi = 0, \qquad \Psi|_{\Gamma_0} = 0, \qquad \Psi|_{\Gamma_H} = \check{f}_\delta(s^-; p, s), \quad (5)\right.$$

where $\check{f}_\delta(s^-; p, s) = F\big[f_\delta(s^-; r_\perp, s)\big] = \delta(\, s - s^-)$, since $F\,[\delta(r_\perp)] = 1$.

The function of influence and the spatial frequency characteristic are related through the Fourier transform

$$\Theta(s^-; z, r_\perp, s) = F^{-1}\big[\Psi(s^-; z, p, s)\big];$$
$$\Psi(s^-; z, p, s) = F\big[\Theta(s^-; z, r_\perp, s)\big].$$

If a linear functional

$$(\Theta, f\,)(\, s^-; z, r_\perp, s) = \frac{1}{2\pi}\int_{\Omega^-} d\, s^{-'} \int_{-\infty}^{\infty} \Theta(\, s^{-'}; z, r_\perp - r'_\perp, s) \times$$
$$\times f(s^-; H, r'_\perp, s^{-'})\, d\, r'_\perp \quad (6)$$

is defined for the function $f(s^-; H, r_\perp, s)$ with its parameter $s^- \in \Omega^-$, then its linear Fourier transfrom is

$$F[(\Theta, f\,)] = (\Psi, \check{f})\,(s^-; z, p, s) = \frac{1}{2\pi}\int_{\Omega^-} \Psi(s^{-'}; z, p, s) \times$$
$$\times \check{f}(s^-; H, p, s^{-'})\, d\, s^{-'}. \quad (7)$$

Problems (4) and (5) correspond to the simplest linear systems of radiation transfer, their parameters independent of the horizontal coordinates and of the properties of the reflecting boundary.

Mathematical models of SFC, FI, and OTO of the system of the radiation transfer we obtain both phenomenologically and rigorously, from the general boundary–value problem of the theory of radiation transfer (1). The idea of such an approach is as follows. The initial 3D equation of radiation transfer is replaced by a system of recursive equations for approximations of a series for perturbations over the parameter, ε, which presents the process of interaction of radiation with the boundary. We construct a fundamental solution using the Fourier transform over x and y. As a result boundary–value problem (1) in a five-dimensional phase space $R^2 \times [0, H] \times \Omega = \{x, y, z, \vartheta, \varphi\}$, in which the sources and boundary conditions have complicated dependences on the spatial, x, y, and angular, ϑ,φ coordinates (including those discontinuous in x, y, which result in singular solutions and discontinuities of the first kind) is reduced to a parametric set of boundary–value problems (5), which are one-dimensional in space and have three variables z, ϑ,φ and regular coefficients. Universal functions are then selected, invariant with respect to horizontal variations and to the angular dependences of both the boundary conditions and sources of the initial problem. Having a parametric set of such invariant functions, which are called SFC's one can obtain a solution of the problem (1) for various spatial and angular structures of the reflection coefficient and for sources at the boundary $z = H$ via the functionals and perturbation series. Thus constructed series are Neumann series over the order of interaction of radiation with the reflecting boundary.

A single act of interaction of radiation with the reflecting boundary may be described by the operators

$$[\hat{R}_\nu \Phi](H, r_\perp, s) \equiv \int_{\Omega^+} \Phi(H, r_\perp, s^+)\, P_\nu(r_\perp, s, s^+)\, d s^+;$$

$$[\hat{R}_c \Phi](H, r_\perp, s) \equiv \int_{\Omega^+} \Phi(H, r_\perp, s^+)\, P_c(s, s^+)\, d s^+;$$

$$\hat{R} \Phi = \hat{R}_\nu \Phi + \hat{R}_c \Phi,$$

or by their Fourier transforms

$$\left[\check{R}_\nu \check{\Phi}\right](H, p, s) \equiv F[\hat{R}_\nu \Phi] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d p' \times$$

$$\times \int_{\Omega^+} \check{\Phi}(H, p', s^+)\, \check{P}_\nu(p - p', s, s^+)\, d s^+;$$

$$\left[\hat{R}_c \check{\Phi}\right](H, p, s) \equiv F[\hat{R}_c \Phi] = \int_{\Omega^+} \check{\Phi}(H, p, s)\, P_c(s, s^+)\, d s^+;$$

$$\left[\check{R} \check{\Phi}\right](H, p, s) \equiv F[\hat{R} \Phi] = \check{R}_\nu \check{\Phi} + \hat{R}_c \check{\Phi}.$$

With the account of contribution from multiple scattering in a medium and depending on the structure of characteristics of reflection, the process of formation of illumination due to repeated reflections of radiation from the boundary is described by the general boundary–value problem (1) and by the general boundary–value problems:

$$\left\{\hat{K} \Phi_c = 0,\ \Phi_c\,|_{\Gamma_0} = 0\ \Phi_c\,|_{\Gamma_H} = \varepsilon \hat{R}_c \Phi_c + \varepsilon E_c(r_\perp, s);\right. \quad (8)$$

$$\left\{\hat{K} \Phi_\nu = 0,\ \Phi_\nu\,|_{\Gamma_0} = 0,\quad \Phi_\nu\,|_{\Gamma_H} = \varepsilon \hat{R}_\nu \Phi_\nu + \varepsilon E_\nu(r_\perp, s).\right. \quad (9)$$

Sources $E_c = \left\{\hat{R}_c \Phi^0, \hat{R}_c \Phi^H\right\}$, $E_v = \left\{\hat{R}_\nu \Phi^0, \hat{R}_\nu \Phi^H\right\}$, and $E = \left\{\hat{R} \Phi^0, \hat{R}_\nu \Phi^0\right\}$ in these problems are defined in terms of the background radiation Φ^0 or Φ^H, which is the solution to the problem with sources $E^0(r_\perp, s)$ or $E^H(r_\perp, s)$

$$\left\{\hat{K} \Phi^0 = 0,\ \Phi^0\,|_{\Gamma_0} = E^0(r_\perp, s),\ \Phi^0\,|_{\Gamma_H} = 0;\right.$$

$$\left\{\hat{K} \Phi^H = 0,\ \Phi^H\,|_{\Gamma_0} = 0,\ \Phi^H\,|_{\Gamma_H} = E^H(r_\perp, s).\right.$$

One may seek the solution to each of the problems (1), (8), and (9) in two forms, either as a series over the order of reflection, or as linear functionals, with their kernels being FI or SFC, overburdened with contributions from multiple scattering and reflection.

OPTICAL TRANSFER OPERATOR FOR A HORIZONTALLY HOMOGENEOUS OPERATOR OF REFLECTION

A generalized solution to the general boundary–value problem (8) may be presented in the form of linear functionals

$$\Phi_c(z, r_\perp, s) = (\Theta_c, E_c), \qquad \check{\Phi}_c(z, p, s) = \left(\Psi_c, \check{E}_c\right), \quad (10)$$

their kernels $\Theta_c(s^-; z, r_\perp, s)$ and $\Psi_c(s^-; z, p, s) = F[\Theta_c]$ satisfying the boundary–value problems

$$\left\{\hat{K} Q_c = 0,\ \Theta_c\,|_{\Gamma_0} = 0,\ \Theta_c\,|_{\Gamma_H} = \varepsilon \hat{R}_c Q_c + f_\delta(s^-; r_\perp, s);\right. \quad (11)$$

$$\left\{\hat{L}(p)\, \Psi_c = 0,\ \Psi_c\,|_{\Gamma_0} = 0,\ \Psi_c\,|_{\Gamma_H} = \varepsilon \hat{R}_c \Psi_C + \check{f}_\delta(s^-; s).\right. \quad (12)$$

Let us define operations of interaction between radiation and horizontally homogeneous boundary at $z = H$ in terms of FI $\Theta(s^-; z, r_\perp, s)$

$$\left[\hat{G}_c f\right](s^-; H, r_\perp, s) = \hat{R}_c(\Theta, f) = \left(\left[\hat{R}_c \Theta\right], f\right) = \frac{1}{2\pi} \int_{\Omega^-} d s^{-\prime} \int_{\Omega^+} P_c(s, s^+)\, d s^+ \int_{-\infty}^{\infty} f(s^-; r'_\perp, s^{-\prime}) \times \Theta(s^{-\prime}; H, r_\perp - r'_\perp, s^+)\, d r'_\perp \quad (13)$$

or using SFC $\Psi(s^-, z, p, s)$

$$\left[\hat{Q}_C \check{f}\right](s^-; H, p, s) = F\left[\hat{G}_C f\right] = \hat{R}_C(\Psi, \check{f}) = \left(\hat{R}_C \Psi\right], \check{f}\Big) = \frac{1}{2\pi} \int_{\Omega^-} \check{f}(s^-; p, s^{-\prime})\, d s^{-\prime} \int_{\Omega^+} P_c(s, s^+) \Psi(s^{-\prime}; H, p, s^+)\, d s^+. \quad (14)$$

The components of perturbation series

$$\Theta_c(s^-; z, r_\perp, s) = \sum_{n=0}^{\infty} \varepsilon^n\, \Theta_{cn}(s^-; z, r_\perp, s) \quad (15)$$

are solutions of the system of recursive problems

$$n = 0:\ \{\hat{K} Q_{c0} = 0,\ \Theta_{c0}\,|_{\Gamma_0} = 0, \quad \Theta_{c0}\,|_{\Gamma_H} = f_\delta(s^-; r_\perp, s);$$

$$n \ge 1:\ \left\{\hat{K} Q_{cn} = 0,\ \Theta_{cn}\,|_{\Gamma_0} = 0 \qquad \Theta_{cn}\,|_{\Gamma_H} = \hat{R}_c\, \Theta_{cn-}\right](s^-; H, r_\perp, s),$$

and may be presented as nonlinear functionals ($\Theta_{c0} = (\Theta, f_\delta) = \Theta$)

$$\Theta_{c1}(s^-; z, r_\perp, s) = (\Theta, \hat{R}_c\, \Theta) = \left(\Theta, \left[\hat{R}_c(\Theta, f_\delta)\right]\right) = \left(\Theta, \hat{G}_c\, f_\delta\right) = \frac{1}{2\pi}\int_{\Omega^-} d\, s_1^- \int_{-\infty}^{\infty} Q\, (s_1^-; z, r_\perp - r_{\perp 1}, s) \left[\hat{R}_c\Theta\right] \times$$

$$\times (s^-; H, r_{\perp 1}, s_1^-)\, d\, r_{\perp 1} = \frac{1}{2\pi}\int_{\Omega^-} d\, s_1^- \int_{-\infty}^{\infty} Q\; s_1^-; z, r_\perp - r_{\perp 1}, s)\, d\, r_{\perp 1} \int_{\Omega^+} \Theta(s^-; H, r_{\perp 1}, s_0^+)\, P_c(s_1^-, s_0^+)\, d\, s_0^+;$$

$$\Theta_{cn}(s^-; z, r_\perp, s) = (\Theta, \hat{R}_c, \Theta_{cn-1}) = \left(\Theta, \hat{G}_c^n\, f_\delta\right) = \left(\Theta, \hat{G}_c^{n-1}\left[\hat{R}_c\Theta\right]\right) = \frac{1}{2\pi}\int_{\Omega^-} d\, s_n^- \int_{-\infty}^{\infty} Q(s_n^-; z, r_\perp - r_{\perp n}, s)\, d\, r_{\perp n} \times$$

$$\times \frac{1}{2\pi}\int_{\Omega^-} d\, s_{n-1}^- \int_{-\infty}^{\infty} d\, r_{\perp n-1} \int_{\Omega^+} P_c(s_n^-, s_{n-1}^+)\, Q(s_{n-1}^-; H, r_{\perp n} - r_{\perp n-1}, s_{n-1}^+)\, d\, s_{n-1}^+ \; \frac{1}{2\pi}\int_{\Omega^-} d\, s_{n-2}^- \int_{-\infty}^{\infty} d\, r_{\perp n-2} \times$$

$$\times \int_{\Omega^+} P_c(s_{n-1}^-, s_{n-2}^+)\, Q(s_{n-2}^-; H, r_{\perp n-1} - r_{\perp n-2}, s_{n-2}^+)\, d\, s_{n-2}^+ \ldots \frac{1}{2\pi}\int_{\Omega^-} d\, s_2^- \int_{-\infty}^{\infty} d\, r_{\perp 2} \times$$

$$\times \int_{\Omega^+} P_c(s_3^-, s_2^+)\, Q(s_2^-; H, r_{\perp 3} - r_{\perp 2}, s_2^+)\, d\, s_2^+ \; \frac{1}{2\pi}\int_{\Omega^-} d\, s_1^- \int_{-\infty}^{\infty} d\, r_{\perp 2} \int_{\Omega^+} P_c(s_2^-, s_1^+)\, Q(s_1^-; H, r_{\perp 2} - r_{\perp 1}, s_1^+)\, d\, s_1^+ \times$$

$$\times \int_{\Omega^+} P_c(s_1^-, s_0^+)\, Q(s^-; H, r_{\perp 1}, s_0^+)\, d\, s_0^+ = \frac{1}{2\pi}\int_{\Omega^-} d\, s_n^- \int_{-\infty}^{\infty} Q(s_n^-; z, r_\perp - r_{\perp n}, s)\, d\, r_{\perp n} \times \frac{1}{2\pi}\int_{\Omega^-} d\, s_{n-1}^- \times$$

$$\times \int_{-\infty}^{\infty} \left[\hat{R}_c\Theta\right] s_{n-1}^-; H, r_{\perp n} - r_{\perp n-1}, s_n^-\big)\, d\, r_{\perp n-1} \; \frac{1}{2\pi}\int_{\Omega^-} d\, s_{n-2}^- \; \frac{1}{2\pi}\int_{\Omega^-} d\, s_{n-2}^-; H, r_{\perp n-1} - r_{\perp n-2}, s_{n-1}^-\big)\, d\, r_{\perp n-2} \ldots \times$$

$$\times \frac{1}{2\pi}\int_{\Omega^-} d\, s_2^- \times \int_{-\infty}^{\infty} \left[\hat{R}_c\Theta\right](s_2^-; H, r_{\perp 3} - r_{\perp 2}, s_3^-\big)\, d\, r_{\perp 2} \; \frac{1}{2\pi}\int_{\Omega^-} d\, s_1^- \int_{-\infty}^{\infty} \left[\hat{R}_c\Theta\right](s_1^-; H, r_{\perp 2} - r_{\perp 1}, s_2^-\big) \left[\hat{R}_c\Theta\right]\big(s^-; H, r_{\perp 1}, s_1^-\big)\, d\, r_{\perp 1}.$$

The sum of series (15) is an exact solution of the general boundary–value problem (11)

$$\Theta_c\big(s^-; z, r_\perp, s\big) = \sum_{n=0}^{\infty} \left(\Theta, \hat{G}_c^n\, f_\delta\right) = \left(\Theta, \hat{Y}_c\, f_\delta\right), \tag{16}$$

where

$$\hat{Y}_c\, f_\delta \equiv \sum_{n=0}^{\infty} \hat{G}_c^n\, f_\delta = \left[\hat{E} - \hat{G}_c\right]^{-1} f_\delta \tag{17}$$

is the sum of the Neumann series over the order of interaction of radiation with a horizontally homogeneous anisotropically reflecting boundary.

Terms of the parametric series

$$\Psi_c(s^-; z, p, s) = \sum_{n=0}^{\infty} \varepsilon^n\, \Psi_{cn}(s^-; z, p, s) \tag{18}$$

satisfy the system of recursive problems

$$n = 0:\; \left\{\hat{L}(\,p)\, \Psi_{c0} = 0,\;\; \Psi_{c0}\,\big|_{\Gamma_0} = 0,\;\; \Psi_{c0}\,\big|_{\Gamma_H} = \check{f}_\delta(s^-; s);\right.$$

$$n \ge 1:\; \left\{\hat{L}(\,p)\, \Psi_{cn} = 0,\;\; \Psi_{cn}\,\big|_{\Gamma_0} = 0,\; \Psi_{cn}\,\big|_{\Gamma_H} = [\hat{R}_c\, \Psi_{cn-1}](s^-; H, p, s),\right.$$

and for $n \ge 1$ they are defined as nonlinear functionals (apparently, $\Psi_{c0} = (\Psi, \check{f}_\delta) = \Psi$)

$$\Psi_{c1}(s^-; z, p, s) = (\Psi, \hat{R}_c\, \Psi) = (\Psi, [\hat{R}_c\, (\Psi, \check{f}_\delta)]) = (\Psi, \hat{Q}_c\, \check{f}_\delta) = \frac{1}{2\pi}\int_{\Omega^-} \Psi(s_1^-; z, p, s)\, [\hat{R}_c\, \Psi](s^-; H, p, s_1^-)\, d\, s_1^- =$$

$$= \frac{1}{2\pi} \int\limits_{\Omega^-} \Psi(s_1^-; z, p, s)\, d s_1^- \int\limits_{\Omega^+} P_c(s_1^-, s_0^+)\, \Psi(s^-; H, p, s_0^+)\, d s_0^+ ;$$

$$\Psi_{cn}(s^-; z, p, s) = (\Psi, \hat{R}_c \Psi_{cn-1}) = (\Psi, \hat{Q}_c^n \check{f}_\delta) = (\Psi, \hat{Q}^{n-1}_c[\hat{R}_c \Psi]) = \frac{1}{2\pi} \int\limits_{\Omega^-} \Psi(s_n^-; z, p, s)[\hat{Q}_c^n \check{f}_\delta]\,(s^-; H, p, s_n^-)\, d s_n^- =$$

$$= \frac{1}{2\pi} \int\limits_{\Omega^-} \Psi(s_n^-; z, p, s)\, d s_n^- \frac{1}{2\pi} \int\limits_{\Omega^-} [\hat{R}_c \Psi](s_{n-1}^-; H, p, s_n^-)\, d s_{n-1}^- \times$$

$$\times \frac{1}{2\pi} \int\limits_{\Omega^-} [\hat{R}_c \Psi](s_{n-2}^-; H, p, s_{n-1}^-)\, d s_{n-2}^- \ldots \frac{1}{2\pi} \int\limits_{\Omega^-} [\hat{R}_c \Psi](s_2^-; H, p, s_3^-)\, d s_2^- \frac{1}{2\pi} \int\limits_{\Omega^-} [\hat{R}_c \Psi](s_1^-; H, p, s_2^-) \times$$

$$\times [\hat{R}_c \Psi](s^-; H, p, s_1^-)\, d s_1^- = \frac{1}{2\pi} \int\limits_{\Omega^-} \Psi(s_n^-; z, p, s)\, d s_n^- \frac{1}{2\pi} \int\limits_{\Omega^-} d s_{n-1}^- \int\limits_{\Omega^+} P_c(s_n^-, s_{n-1}^+)\, \Psi(s_{n-1}^-; H, p, s_{n-1}^+)\, d s_{n-1}^+ \times$$

$$\times \frac{1}{2\pi} \int\limits_{\Omega^-} d s_{n-2}^- \int\limits_{\Omega^+} P_c(s_{n-1}^-, s_{n-2}^+)\, \Psi(s_{n-2}^-; H, p, s_{n-2}^+)\, d s_{n-2}^+ \ldots \frac{1}{2\pi} \int\limits_{\Omega^-} d s_2^- \int\limits_{\Omega^+} P_c(s_3^-, s_2^+)\, \Psi(s_2^-; H, p, s_2^+)\, d s_2^+ \times$$

$$\times \frac{1}{2\pi} \int\limits_{\Omega^-} d s_1^- \int\limits_{\Omega^+} P_c(s_2^-, s_1^+)\, \Psi(s_1^-; H, p, s_1^+)\, d s_1^+ \int\limits_{\Omega^+} P_c(s_1^-, s_0^+)\, \Psi(s^-; H, p, s_0^+)\, d s_0^+ .$$

The sum of series (18) is the exact solution to the problem (12)

$$\Psi_c = \sum_{n=0}^{\infty} (\Psi, \hat{Q}_c^n \check{f}_\delta) = (\Psi, \hat{Z}_c \check{f}_\delta), \tag{19}$$

$$\hat{Z}_c \check{f}_\delta \equiv \sum_{n=0}^{\infty} \hat{Q}_c^n \check{f}_\delta = [\hat{E} - \hat{Q}_c]^{-1} \check{f}_\delta \tag{20}$$

is the sum of the Neumann series over the order of interaction of radiation with the reflecting boundary in terms of the Fourier transforms.

If we introduce a series over the order of reflection from the boundary

$$\Phi_c(z, r_\perp, s) = \sum_{k=1}^{\infty} \varepsilon^k\, \Phi_{ck}(z, r_\perp, s) \tag{21}$$

with its terms being solutions to the system of recursive problems

$$k = 1: \left\{ \hat{K}\Phi_{c1} = 0, \quad \Phi_{c1}\,|_{\Gamma_0} = 0, \quad \Phi_{c1}\,|_{\Gamma_H} = E_c(r_\perp, s); \right.$$

$$k \ge 2: \left\{ \hat{K}\Phi_{ck} = 0, \quad \Phi_{ck}\,|_{\Gamma_0} = 0, \quad \Phi_{ck}\,|_{\Gamma_H} = [\hat{R}_c\, \Phi_{ck-1}](H, r_\perp, s), \right.$$

then we obtain the following representations

$$\Phi_{c1}(z, r_\perp, s) = (\Theta, E_c) = \frac{1}{2\pi} \int\limits_{\Omega^-} d s_1^- \int\limits_{-\infty}^{\infty} \Theta(s_1^-; z, r_\perp - r_{\perp 1}, s)\, E_c(r_{\perp 1}, s_1^-)\, d r_{\perp 1};$$

$$\Phi_{ck}(z, r_\perp, s) = (\Theta, \hat{G}^{k-1}_c E_c) = \frac{1}{2\pi} \int\limits_{\Omega^-} d s_k^- \int\limits_{-\infty}^{\infty} \Theta(s_k^-; z, r_\perp - r_{\perp k}, s)\, d r_{\perp k} \frac{1}{2\pi} \int\limits_{\Omega^-} d s_{k-1}^- \times$$

$$\times \int\limits_{-\infty}^{\infty} d r_{\perp k-1} \int\limits_{\Omega^+} P_c(s_k^-, s_{k-1}^+) \Theta(s_{k-1}^-; H, r_{\perp k} - r_{\perp k-1}, s_{k-1}^+)\, d s_{k-1}^+ \frac{1}{2\pi} \int\limits_{\Omega^-} d s_{k-2}^- \times$$

$$\times \int_{-\infty}^{\infty} d\, r_{\perp k-2} \int_{\Omega^+} P_c(s_{k-1}^-, s_{k-2}^+)\, \Theta(s_{k-2}^-; H, r_{\perp k-1} - r_{\perp k-2}, s_{k-2}^+)\, d\, s_{k-2}^+ \dots \frac{1}{2\pi} \int_{\Omega^-} d\, s_2^- \times$$

$$\times \int_{-\infty}^{\infty} d\, r_{\perp 2} \int_{\Omega^+} P_c(s_3^-, s_2^+)\, \Theta(s_2^-; H, r_{\perp 3} - r_{\perp 2}, s_2^+)\, d\, s_2^+ \frac{1}{2\pi} \int_{\Omega^-} d\, s_1^- \int_{-\infty}^{\infty} E_c(r_{\perp 1}, s_1^-)\, d\, r_{\perp 1} \times$$

$$\times \int_{\Omega^+} P_c(s_2^-, s_1^+)\, \Theta(s_1^-; H, r_{\perp 2} - r_{\perp 1}, s_1^+)\, d\, s_1^+ = \frac{1}{2\pi} \int_{\Omega^-} d\, s_k^- \int_{-\infty}^{\infty} \Theta(s_k^-; z, r_\perp - r_{\perp k}, s)\, d\, r_{\perp k} \frac{1}{2\pi} \int_{\Omega^-} d\, s_{k-1}^- \times$$

$$\times \int_{-\infty}^{\infty} [\hat{R}_c\, \Theta](s_{k-1}^-; H, r_{\perp k} - r_{\perp k-1}, s_k^-)\, d\, r_{\perp k-1} \frac{1}{2p} \int_{X^-} d\, s_{k-2}^- \int_{-\infty}^{\infty} [\hat{R}_c\, \Theta](s_{k-2}^-; H, r_{\perp k-1} - r_{\perp k-2}, s_{k-1}^-)\, d\, r_{\perp k-2} \dots \frac{1}{2\pi} \int_{\Omega^-} d\, s_2^- \times$$

$$\times \int_{-\infty}^{\infty} [\hat{R}_c\, \Theta](s_2^-; H, r_{\perp 3} - r_{\perp 2}, s_3^-)\, d\, r_{\perp 2} \frac{1}{2\pi} \int_{\Omega^-} d\, s_1^- \int_{-\infty}^{\infty} E_c(r_{\perp 1}, s_1^-) [\hat{R}_c\, \Theta]\, (s_1^-; H, r_{\perp 2} - r_{\perp 1}, s_2^-)\, d\, r_{\perp 1}\, .$$

The sum of series (21) is an exact solution of the problem (8)

$$\Phi_c(z, r_\perp, s) = \sum_{k=1}^{\infty} (\Theta, \hat{G}_c^{k-1} E_c) = (\Theta, \hat{Y}_c\, E_c), \tag{22}$$

where

$$\hat{Y}_c\, E_c \equiv \sum_{k=1}^{\infty} \hat{G}_c^{k-1} E_c = \sum_{k=0}^{\infty} \hat{G}_c^k E_c [\hat{E} - \hat{G}_c]^{-1} E_c \tag{23}$$

is the sum of the Neumann series over the order of interaction of radiation with the reflecting boundary.

Presentation (10) in terms of FI Θ_c makes it possible to obtain a solution to the problem (8) for various preset sources E_c with the effect of homogeneous reflecting boundary calculated in advance. The terms from series (21) are expressed in terms of FI and for $k \ge 2$ these are nonlinear functionals which adequately describe the kth order process of interaction of radiation with the reflecting boundary for a preset irradiation of the boundary E_c. In terms of Fourier transforms we have

$$\hat{\Phi}_c(z, p, s) = \sum_{k=1}^{\infty} \varepsilon^k\, \hat{\Phi}_{ck}(z, p, s), \tag{24}$$

where the terms of the series are solutions to the recursive problems

$$k = 1: \ \left\{ \hat{L}(p)\, \check{\Phi}_{c1} = 0, \quad \check{\Phi}_{c1} \big|_{\Gamma_0} = 0, \quad \check{\Phi}_{c1} \big|_{\Gamma_H} = \check{E}_c(p, s); \right.$$

$$k \ge 2: \left\{ \hat{L}(p)\, \check{\Phi}_{ck} = 0, \quad \check{\Phi}_{ck} \big|_{\Gamma_0} = 0, \quad \check{\Phi}_{ck} \big|_{\Gamma_H} = [\hat{R}_c\, \check{\Phi}_{ck-1}](H, p, s). \right.$$

For $k \ge 2$ they are presented as linear functionals

$$\check{\Phi}_{c1}(z, p, s) = (\Psi, \check{E}_c) = \frac{1}{2\pi} \int_{\Omega^-} \Psi(s_1^-; z, p, s)\, \check{E}_c(p, s_1^-)\, d\, s_1^-;$$

$$\check{\Phi}_{ck}(z, p, s) = (\Psi, \hat{Q}_c^{k-1} \check{E}_c) = \frac{1}{2\pi} \int_{\Omega^-} \Psi(s_k^-; z, p, s)\, d\, s_k^- \frac{1}{2\pi} \int_{\Omega^-} [\hat{R}_c \Psi](s_{k-1}^-; H, p, s_k^-)\, d\, s_{k-1}^- \times$$

$$\times \frac{1}{2\pi} \int_{\Omega^-} [\hat{R}_c\, \Psi](s_{k-2}^-; H, p, s_{k-1}^-)\, d\, s_{k-2}^- \dots \frac{1}{2\pi} \int_{\Omega^-} [\hat{R}_c\, \Psi](s_2^-; H, p, s_3^-)\, d\, s_2^- \frac{1}{2\pi} \int_{\Omega^-} [\hat{R}_c\, \Psi](s_1^-; H, p, s_2^-) \times$$

$$\times \check{E}_c(p, s_1^-)\, d\, s_1^- = \frac{1}{2\pi} \int_{\Omega^-} \Psi(s_k^-; z, p, s)\, d\, s_k^- \ \frac{1}{2\pi} \int_{\Omega^-} d\, s_{k-1}^- \int_{\Omega^+} P_c(s_k^-, s_{k-1}^+)\, \Psi(s_{k-1}^-; H, p, s_{k-1}^+)\, d\, s_{k-1}^+ \times$$

$$\times \frac{1}{2\pi}\int_{\Omega^-} d\, s_{k-2}^- \int_{\Omega^+} P_c(s_{k-1}^-, s_{k-2}^+)\, \Psi(s_{k-2}^-; H, p, s_{k-2}^+)\, d\, s_{k-2}^+ \dots \frac{1}{2\pi}\int_{\Omega^-} d\, s_2^- \int_{\Omega^+} P_c(s_3^-, s_2^+)\, \Psi(s_2^-; H, p, s_2^+)\, d\, s_2^+ \times$$

$$\times \frac{1}{2\pi}\int_{\Omega^-} \check{E}_c(p, s_1^-)\, d\, s_1^- \int_{\Omega^+} P_c(s_2^-, s_1^+)\, \Psi(s_1^-; H, p, s_1^+)\, d\, s_1^+.$$

The sum of series (24) is the exact solution of the Fourier transform of the problem (8)

$$\check{\Phi}_c(z, p, s) = \sum_{k=1}^{\infty} (\Psi, \hat{Q}_c^{k-1}\, \check{E}_c) = (\Psi, \hat{Z}_c\, \check{E}_c), \tag{25}$$

where

$$\hat{Z}_c\, \check{E}_c \equiv \sum_{k=1}^{\infty} \hat{Q}_c^{k-1}\, \check{E}_c = \sum_{k=0}^{\infty} \hat{Q}_c^{k}\, \check{E}_c = [\hat{E} - \hat{Q}_c]^{-1}\, \check{E}_c \tag{26}$$

is the sum of the Neumann series in terms of Fourier transforms.

OPTICAL TRANSFER OPERATOR INVOLVING A HORIZONTALLY INHOMOGENEOUS OPERATOR OF REFLECTION WITH AND WITHOUT THE SEPARATION OF SPATIAL AND ANGULAR DEPENDENCES

Boundary–value problem (9) can be solved using linear functionals

$$\Phi_\nu z, r_\perp, s) = (\Theta_\nu, E_\nu)\ , \check{\Phi}_\nu(z, p, s) = (\Psi_\nu, \check{E}_\nu), \tag{27}$$

where FI $\Theta_v(s^-, z, r_\perp, s)$ is the solution to the general boundary–value problem

$$\left\{\hat{K}\, \Theta_\nu = 0,\ \Theta_\nu\,|_{\Gamma_0} = 0,\ \Theta_\nu\,|_{\Gamma_H} = \varepsilon\, \hat{R}_\nu \Theta_\nu + f_\delta(s^-; r_\perp, s), \right. \tag{28}$$

and the SFC $\Psi_\nu\,(s^-, z, p, s) = F[\Theta_\nu]$ is the solution of the complex equation of radiation transfer

$$\left\{\hat{L}(p)\, \Psi_\nu = 0,\ \Psi_\nu\,|_{\Gamma_0} = 0,\ \Psi_\nu\,|_{\Gamma_H} = \varepsilon\, \check{R}_\nu \Psi_\nu + \check{f}_\delta\,(s^-; s). \right. \tag{29}$$

Let us now introduce the operations of interaction of radiation with the boundary using FI Θ:

$$[\hat{G}_\nu\, f]\,(s^-; H, r_\perp, s) = \check{R}_\nu(\Theta, f\,) = \frac{1}{2\pi}\int_{\Omega^-} d\, s^{-\prime} \int_{-\infty}^{\infty} f(s^-; r_\perp', s^{-\prime})\, d\, r_\perp' \int_{\Omega^+} P_\nu(r_\perp, s, s^+)\Theta(s^{-\prime}; H, r_\perp - r_\perp', s^+)\, d\, s^+ \tag{30}$$

or the SFC Ψ (Ref. 15)

$$[\,\hat{Q}_\nu\, \check{f}\,](s^-; H, p, s) = F\,[\hat{G}_\nu\, f] = \check{R}_\nu\,(\Psi, \check{f}) = \frac{1}{2\pi}\int_{\Omega^-} d\, s^{-\prime} \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \check{f}\,(s^-; p'\, s^{-\prime})\, d\, p' \int_{\Omega^+} \check{P}_\nu(\, p - p', s, s^+)\Psi(s^{-\prime}; H, p', s^+)\, d\, s^+. \tag{31}$$

The components of perturbation series

$$\Theta_\nu(s^-; z, r_\perp, s) = \sum_{n=0}^{\infty} \varepsilon^n\, \Theta_{\nu n}(s^-; z, r_\perp, s) \tag{32}$$

satisfy the system of recursive problems

$$n = 0:\ \left\{\hat{K}\ Q_{\nu 0} = 0,\ \Theta_{\nu 0}\,|_{\Gamma_0} = 0,\ \Theta_{\nu 0}\,|_{\Gamma_H} = f_\delta(s^-; r_\perp, s);\right.$$

$$n \ge 1:\ \left\{\hat{K}\ Q_{\nu n} = 0,\ \Theta_{\nu n}\,|_{\Gamma_0} = 0,\ \Theta_{\nu n}\,|_{\Gamma_H} = [\hat{R}_\nu\, \Theta_{\nu n-1}](s^-; H, r_\perp, s)\right.$$

and are presented in the form of functionals ($\Theta_{\nu 0} = (\Theta, f_\delta) = \Theta$)

$$\Theta_{\nu 1}(s^-; z, r_\perp, s) = (\Theta, \hat{R}_\nu\, \Theta) = (\Theta, \hat{G}_\nu\, f_\delta) = \frac{1}{2\pi}\int_{\Omega^-} d\, s_1^- \int_{-\infty}^{\infty} \Theta(s_1^-; z, r_\perp - r_{\perp 1}, s) \times [\hat{R}_\nu\, \Theta]\,(s^-; H, r_{\perp 1}, s_1^-)\, d\, r_{\perp 1} =$$

$$= \frac{1}{2\pi} \int\limits_{\Omega^-} d\, s_1^- \int\limits_{-\infty}^{\infty} \Theta(s_1^-; z, r_\perp - r_{\perp 1}, s)\, d\, r_{\perp 1} \int\limits_{\Omega^+} P_\nu(r_{\perp 1}, s_1^-, s_0^+)\, \Theta\, (s^-; H, r_{\perp 1}, s_0^+)\, d\, s_0^+;$$

$$\Theta_{\nu n}(s^-; z, r_\perp, s) = (\Theta, \hat{R}_\nu\, \Theta_{\nu n-1}) = (\Theta, \hat{G}_\nu^n f_\delta) = (\Theta, \hat{G}_\nu^{n-1} [\hat{R}_\nu\, \Theta]) = \frac{1}{2\pi} \int\limits_{\Omega^-} d\, s_n^- \int\limits_{-\infty}^{\infty} \Theta(s_n^-; z, r_\perp - r_{\perp n}, s)\, d\, r_{\perp n} \times$$

$$\times \frac{1}{2\pi} \int\limits_{\Omega^-} d\, s_{n-1}^- \int\limits_{-\infty}^{\infty} d\, r_{\perp n-1} \int\limits_{\Omega^+} P_\nu(r_{\perp n}, s_n^-, s_{n-1}^+) \Theta(s_{n-1}^-; H, r_{\perp n} - r_{\perp n-1}, s_{n-1}^+)\, d\, s_{n-1}^+ \times$$

$$\times \ldots \frac{1}{2\pi} \int\limits_{\Omega^-} d\, s_2^- \int\limits_{-\infty}^{\infty} d\, r_{\perp 2} \int\limits_{\Omega^+} P_\nu(r_{\perp 3}, s_3^-, s_2^+) \Theta(s_2^-; H, r_{\perp 3} - r_{\perp 2}, s_2^+)\, d\, s_2^+ \frac{1}{2\pi} \int\limits_{\Omega^-} d\, s_1^- \int\limits_{-\infty}^{\infty} d\, r_{\perp 1} \int\limits_{\Omega^+} P_\nu(r_{\perp 2}, s_2^-, s_1^+)\, \Theta(s_1^-; H, r_{\perp 2} - r_{\perp 1}, s_1^+)\, ds_1^+ \times$$

$$\times \int\limits_{\Omega^+} P_\nu(r_{\perp 1}, s_1^-, s_0^+)\, \Theta\, (s^-; H, r_{\perp 1}, s_0^+)\, d\, s_0^+.$$

The sum of series (32) is an exact solution to the general boundary–value problem (28)

$$\Theta_\nu(s^-; z, r_\perp, s) = \sum_{n=0}^{\infty} (\Theta, \hat{G}_\nu^n f_\delta) = (\Theta, \hat{Y}_\nu f_\delta), \tag{33}$$

where

$$\hat{Y}_\nu f_\delta \equiv \sum_{n=0}^{\infty} \hat{G}_\nu^n f_\delta = [\hat{E} - \hat{G}_\nu]^{-1} f_\delta \tag{34}$$

is the Neumann series over the orders of interaction of radiation with the horizontally inhomogeneous anisotropically reflecting boundary.

The terms of the parametric series

$$\Psi_\nu(s^-; z, p, s) = \sum_{n=0}^{\infty} \varepsilon^n\, \Psi_{\nu n}(s^-; z, p, s) \tag{35}$$

are solutions of the system of recursive problems

$$n = 0: \left\{ \hat{L}(p)\, \Upsilon_{\nu 0} = 0,\ \Psi_{\nu 0}\, |_{\Gamma_0} = 0,\ \Psi_{\nu 0}\, |_{\Gamma_H} = \check{f}_\delta(s^-; s); \right.$$

$$n \ge 1: \left\{ \hat{L}(p)\, \Upsilon_{\nu n} = 0,\ \Psi_{\nu n}\, |_{\Gamma_0} = 0,\ \Psi_{\nu n}\, |_{\Gamma_H} = [\check{R}_\nu\, \Psi_{\nu n-1}]\, (s^-; H, p, s); \right.$$

and for $n \ge 1$ are sought in terms of nonlinear functionals ($\Psi_{\nu 0} = (\Psi, \check{f}_\delta) = \Psi$)

$$\Psi_{\nu 1}(s^-; z, p, s) = (\Psi, \check{R}_\nu\, \Psi) = (\Psi, \hat{Q}_\nu \check{f}_\delta) = \frac{1}{2\pi} \int\limits_{\Omega^-} \Psi(s_1^-; z, p, s) \left[\check{R}_\nu\, \Psi\right] (s^-; H, p, s_1^-)\, d\, s_1^- =$$

$$= \frac{1}{2\pi} \int\limits_{\Omega^-} \Psi(s_1^-; z, p, s)\, d\, s_1^- \times \frac{1}{(2\pi)^2} \int\limits_{-\infty}^{\infty} d\, p_0 \int\limits_{\Omega^+} \check{P}_\nu(p - p_0, s_1^-, s_0^+) \Psi(s^-; H, p_0, s_0^+)\, d\, s_0^+;$$

$$\Psi_{\nu n}(s^-; z, p, s) = (\Psi, \check{R}_\nu\, \Psi_{\nu n-1}) = (\Psi, \hat{Q}_\nu^{n-1} \left[\check{R}_\nu\, \Psi\right]) = (\Psi, \hat{Q}_\nu^n \check{f}_\delta) = \frac{1}{2\pi} \int\limits_{\Omega^-} \Psi(s_n^-; z, p, s)\, d\, s_n^- \frac{1}{2\pi} \int\limits_{\Omega^-} d\, s_{n-1}^- \times$$

$$\times \frac{1}{(2\pi)^2} \int\limits_{-\infty}^{\infty} d\, p_{n-1} \times \int\limits_{\Omega^+} \check{P}_\nu(p - p_{n-1}, s_n^-, s_{n-1}^+) \Psi(s_{n-1}^-; H, p_{n-1}, s_{n-1}^+)\, d\, s_{n-1}^+ \ldots \frac{1}{2\pi} \int\limits_{\Omega^-} d\, s_2^- \frac{1}{(2\pi)^2} \int\limits_{-\infty}^{\infty} d\, p_2 \times$$

$$\times \int\limits_{\Omega^+} \check{P}_\nu(p_3 - p_2, s_3^-, s_2^+)\, \Psi\, (s_2^-; H, p_2, s_2^+)\, d\, s_2^+ \frac{1}{2\pi} \int\limits_{\Omega^-} d\, s_1^- \frac{1}{(2\pi)^2} \times \int\limits_{-\infty}^{\infty} d\, p_1 \int\limits_{\Omega^+} \check{P}_\nu(p_2 - p_1, s_2^-, s_1^+) \Psi(s_1^-; H, p_1, s_1^+)\, d\, s_1^+ \times$$

$$\times \frac{1}{(2\pi)^2} \int\limits_{-\infty}^{\infty} d\, p_0 \int\limits_{\Omega^+} \overset{\vee}{P}_\nu(\, p_1 - p_0,\, s_1^-,\, s_0^+) \Psi(s^-;\, H,\, p_0,\, s_0^+)\, d\, s_0^+.$$

The sum of series (35) is the exact solution to the problem (29)

$$\Psi_v(s^-;\, z,\, p,\, s) = \sum_{n=0}^{\infty} (\Psi, \hat{Q}_\nu^n \overset{\vee}{f}_\delta) = (\Psi, \hat{Z}_\nu \overset{\vee}{f}_\delta), \tag{36}$$

where

$$\hat{Z}_\nu \overset{\vee}{f}_\delta \equiv \sum_{n=0}^{\infty} \hat{Q}_\nu^n \overset{\vee}{f}_\mathrm{d} = \left[\hat{E} - \hat{Q}_\nu\right]^{-1} \overset{\vee}{f}_\delta \tag{37}$$

is the sum of the Neumann series over the orders of interaction of radiation with the boundary in terms of Fourier transforms.

If we introduce a series over the order of reflection from the boundary

$$\Phi_\nu(z,\, r_\perp,\, s) = \sum_{k=1}^{\infty} \varepsilon^k\, \Phi_{\nu k}(z,\, r_\perp,\, s) \tag{38}$$

with its terms found from the system of recursive problems

$$k = 1:\ \left\{\hat{K}\, \Phi_{\nu 1} = 0,\ \ \Phi_{\nu 1}\,|_{\Gamma_0} = 0,\ \ \Phi_{\nu 1}\,|_{\Gamma_H} = E_v(r_\perp,\, s);\right.$$

$$k \ge 2:\ \left\{\hat{K}\, \Phi_{\nu k} = 0,\ \Phi_{\nu k}\,|_{\Gamma_0} = 0,\ \Phi_{\nu k}\,|_{\Gamma_H} = \left[\hat{R}_v\, \Phi_{\nu k-1}\right](H,\, r_\perp,\, s);\right.$$

then

$$\Phi_{\nu 1} = (\Theta,\, E_\nu) = \frac{1}{2\pi} \int\limits_{\Omega^-} d\, s_1^- \int\limits_{-\infty}^{\infty} \Theta(s_1^-;\, z,\, r_\perp - r_{\perp 1},\, s)\, E_\nu\, (r_{\perp 1},\, s_1^-)\, d\, r_{\perp 1};$$

$$\Phi_{\nu k}(z,\, r_\perp,\, s) = (\Theta,\, \hat{R}_v\, \Phi_{\nu k-1}) = (\Theta,\, \hat{G}_\nu^{k-1}\, E_\nu) = \frac{1}{2\pi} \int\limits_{\Omega^-} d\, s_k^- \int\limits_{-\infty}^{\infty} \Theta(s_k^-;\, z,\, r_\perp - r_{\perp k},\, s)\, d\, r_{\perp k}\, \frac{1}{2\pi} \int\limits_{\Omega^-} d\, s_{k-1}^- \int\limits_{-\infty}^{\infty} d\, r_{\perp k-1} \times$$

$$\times \int\limits_{\Omega^+} P_\nu(r_{\perp k},\, s_k^-,\, s_{k-1}^+)\, \Theta(s_{k-1}^-;\, H,\, r_{\perp k} - r_{\perp k-1},\, s_{k-1}^+)\, d\, s_{k-1}^+ \dots \frac{1}{2\pi} \int\limits_{\Omega^-} d\, s_2^- \int\limits_{-\infty}^{\infty} d\, r_{\perp 2} \int\limits_{\Omega^+} P_\nu(r_{\perp 3},\, s_3^-,\, s_2^+)\, \Theta(s_2^-;\, H,\, r_{\perp 3} - r_{\perp 2},\, s_2^+)\, d\, s_2^+ \times$$

$$\times \frac{1}{2\pi} \int\limits_{\Omega^-} d\, s_1^- \int\limits_{-\infty}^{\infty} E_\nu\, (r_{\perp 1},\, s_1^-)\, d\, r_{\perp 1} \int\limits_{\Omega^+} P_\nu(r_{\perp 2},\, s_2^-,\, s_1^+)\, \Theta\, (s_1^-;\, H,\, r_{\perp 2} - r_{\perp 1},\, s_1^+)\, d\, s_1^+.$$

The sum of series (38) is the exact solution of the general boundary problem (9)

$$\Phi_\nu(z,\, r_\perp,\, s) = \sum_{k=1}^{\infty} (\Theta,\, \hat{G}_\nu^{k-1}\, E_\nu) = (\Theta,\, \hat{Y}_\nu\, E_\nu), \tag{39}$$

where

$$\hat{Y}_\nu\, E_\nu \equiv \sum_{k=1}^{\infty} \hat{G}_\nu^{k-1}\, E_v = \sum_{k=0}^{\infty} \hat{G}_\nu^k\, E_\nu = \left[\hat{E} - \hat{G}_\nu\right]^{-1} E_\nu \tag{40}$$

is the sum of the Neumann series over the orders of reflections from the boundary.

In terms of Fourier transforms

$$\overset{\vee}{\Phi}_\nu(z,\, p,\, s) = \sum_{k=1}^{\infty} \varepsilon^k\, \overset{\vee}{\Phi}_{\nu k}(z,\, p,\, s), \tag{41}$$

where the components are solutions of the system of recursive problems

$$k = 1:\ \left\{\hat{L}(\, p)\, \overset{\vee}{\Phi}_{\nu 1} = 0,\ \overset{\vee}{\Phi}_{\nu 1}\,|_{\Gamma_0} = 0,\ \overset{\vee}{\Phi}_{\nu 1}\,|_{\Gamma_H} = \overset{\vee}{E}_\nu(\, p,\, s);\right.$$

$k \geq 2$: $\left\{\hat{L}(p)\check{\Phi}_{\nu k} = 0,\ \check{\Phi}_{\nu k}\,|_{\Gamma_0} = 0,\ \check{\Phi}_{\nu k}\,|_{\Gamma_H} = \left[\check{R}_\nu \check{\Phi}_{\nu k-1}\right](H, p, s);\right.$

and are defined as functionals

$$\check{\Phi}_{\nu 1}(z, p, s) = (\Psi, \check{E}_\nu) = \frac{1}{2\pi}\int\limits_{\Omega^-} \Psi(s_1^-; z, p, s)\,\check{E}_\nu(p, s_1^-)\, d s_1^-;$$

$$\check{\Phi}_{\nu k}(z, p, s) = (\Psi, \check{R}_\nu \check{\Phi}_{\nu k-1}) = (\Psi, \hat{Q}_\nu^{k-1}\check{E}_\nu) = \frac{1}{2\pi}\int\limits_{\Omega^-} \Psi(s_k^-; z, p, s)\, d s_k^- \frac{1}{2\pi}\int\limits_{\Omega^-} d s_{k-1}^- \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty} d p_{k-1} \times$$

$$\times \int\limits_{\Omega^+} \check{P}_\nu(p - p_{k-1}, s_k^-, s_{k-1}^+)\Psi(s_{k-1}^-; H, p_{k-1}, s_{k-1}^+)\, d s_{k-1}^+ \ldots \frac{1}{2\pi}\int\limits_{\Omega^-} d s_2^- \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty} d p_2 \int\limits_{\Omega^+} \check{P}_\nu(p_3 - p_2, s_3^-, s_2^+)\times$$

$$\times\, \Psi(s_2^-; H, p_2, s_2^+)\, d s_2^+ \frac{1}{2\pi}\int\limits_{\Omega^-} d s_1^- \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty} \check{E}_\nu(p_1, s_1^-)\, d p_1 \int\limits_{\Omega^+} \check{P}_\nu(p_2 - p_1, s_2^-, s_1^+)\Psi(s_1^-; H, p_1, s_1^+)\, d s_1^+ .$$

The sum of series (41) is the exact solution of the problem (9) expressed in terms of Fourier transforms

$$\check{\Phi}_\nu(z, p, s) = \sum_{k=1}^{\infty} (\Psi, \hat{Q}_\nu^{k-1}\check{E}_\nu) = (\Psi, \hat{Z}_\nu \check{E}_\nu), \tag{42}$$

where

$$\hat{Z}_\nu \check{E}_\nu \equiv \sum_{k=1}^{\infty} \hat{Q}_\nu^{k-1}\check{E}_\nu = \sum_{k=0}^{\infty} \hat{Q}_\nu^{k}\check{E}_\nu = \left[\hat{E} - \hat{Q}_\nu\right]^{-1}\check{E}_\nu \tag{43}$$

is the sum of Neumann series over the orders of reflection from the boundary in terms of Fourier transforms.

If spatial and angular dependences may be separated in the kernel of the operator of reflection (as in Ref. 2)

$$P_\nu(r_\perp, s, s^+) = q(r_\perp)\, P_c(s, s^+),$$

$$\left[\hat{R}_\nu \Phi\right](H, r_\perp, s) = q(r_\perp)\left[\hat{R}_H \Phi\right](H, r_\perp, s),$$

$$\left[\hat{R}_H \Phi\right](H, r_\perp, s) \equiv \int\limits_{\Omega^+} \Phi(H, r_\perp, s^+)\, P_c(s, s^+)\, d s^+,$$

then we have a particular case for the presentation (30)

$$\left[\hat{G}_\nu f\right](s^-; H, r_\perp, s) = \hat{R}_\nu(\Theta, f) = q(r_\perp)\left[\hat{R}_H(\Theta, f)\right] = q(r_\perp)(\left[\hat{R}_H \Theta\right], f) = q(r_\perp)\frac{1}{2\pi}\int\limits_{\Omega^-} d s^{-\prime}\int\limits_{-\infty}^{\infty} f(s^-; r'_\perp, s^{-\prime})\, d r'_\perp \times$$

$$\times \int\limits_{\Omega^+} P_c(s, s^+)\Theta(s^{-\prime}; H, r_\perp - r'_\perp, s^+)\, d s^+ = q(r_\perp)\frac{1}{2\pi}\int\limits_{\Omega^-} d s^{-\prime}\int\limits_{-\infty}^{\infty} f(s^-; r'_\perp, s^{-\prime})\left[\hat{R}_H \Theta\right](s^{-\prime}; H, r_\perp - r'_\perp, s)\, d r'_\perp,$$

$$\left[\hat{R}_H \Theta\right](s^-; H, r_\perp, s) = \int\limits_{\Omega^+} P_c(s, s^+)\Theta(s^-; H, r_\perp, s^+)\, d s^+.$$

In terms of the Fourier transforms we obtain a particular form of the presentation (31)

$$\left[\hat{R}_\nu \check{\Phi}\right](H, p, s) = F\left[\hat{R}_\nu \Phi\right](H, p, s) = \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty} \check{q}(p - p')\, d p' \int\limits_{\Omega^+} \check{\Phi}(H, p', s^+)\, P_c(s, s^+)\, d s^+ = \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty} \check{q}(p - p')\, d p' \left[\hat{R}_H \check{\Phi}\right](H, p', s)\, d p',$$

$$\left[\hat{R}_H \check{\Phi}\right](H, p, s) = \int\limits_{\Omega^+} \check{\Phi}(H, p, s^+)\, P_c(s, s^+)\, d s^+;$$

$$[\hat{Q}_\nu \check{f}](s^-; H, p, s) = F[\hat{G}_\nu f] = \check{R}_v(\Psi, \check{f}) = \frac{1}{2\pi}\int_{\Omega^-} d s^{-\prime}\, \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \check{f}(s^-; p', s^{-\prime})\, \check{q}(p-p')\, d p' \int_{\Omega^+} P_c(s, s^+)\Psi(s^{-\prime}; H, p', s^+)\, d s^+ =$$

$$= \frac{1}{2\pi}\int_{\Omega^-} d s^{-\prime}\, \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \check{q}(p-p')\, \check{f}(s^-; p', s^{-\prime})\, [\hat{R}_H\Psi](s^{-\prime}; H, p', s)\, d p'.$$

$$[\hat{R}_H\Psi](s^-; H, p, s) = \int_{X^+} P_c(s, s^+)\Psi(s^-; H, p, s^+)\, d s^+ .$$

In the case when spatial and angular variables are split, one may find that the most general expressions for the n approximations of FI Θ_ν and SFC Ψ_ν, of the terms of series (38) and (41), obtained above, are reduced to the following form:

$$\Theta_{\nu n}(s^-; z, r_\perp, s) = \frac{1}{2\pi}\int_{\Omega^-} ds_n^- \int_{-\infty}^{\infty}\Theta(s_n^-; z, r_\perp - r_{\perp n}, s)\, q(r_{\perp n})\, dr_{\perp n}\, \frac{1}{2\pi}\int_{\Omega^-} ds_{n-1}^- \int_{-\infty}^{\infty} q(r_{\perp n-1})\, [\hat{R}_H\Theta](s_{n-1}^-; H, r_{\perp n} - r_{\perp n-1}, s_n^-)\, dr_{\perp n-1} \times$$

$$\times \dots \frac{1}{2\pi}\int_{\Omega^-} ds_2^- \int_{-\infty}^{\infty} q(r_{\perp 2})\, [\hat{R}_H\Theta](s_2^-; H, r_{\perp 3} - r_{\perp 2}, s_3^-)\, dr_{\perp 2}\, \frac{1}{2\pi}\int_{\Omega^-} ds_1^- \int_{-\infty}^{\infty} q(r_{\perp 1})\, [\hat{R}_H\Theta](s_1^-; H, r_{\perp 2} - r_{\perp 1}, s_2^-)[\hat{R}_H\Theta](s^-; H, r_{\perp 1}, s_1^-)\, d r_{\perp 1} ;$$

$$\Psi_{\nu n}(s^-; z, p, s) = \frac{1}{2\pi}\int_{\Omega^-}\Psi(s_n^-; z, p, s)\, ds_n^-\, \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\check{q}(p - p_{n-1})\, d p_{n-1}\, \frac{1}{2\pi}\int_{\Omega^-}[\hat{R}_H\Psi](s_{n-1}^-; H, p_{n-1}, s_n^-)\, ds_{n-1}^- \dots \frac{1}{(2p)^2}\int_{-\infty}^{\infty}\check{q}(p_2 - p_1)\, d p_1 \times$$

$$\times \frac{1}{2\pi}\int_{\Omega^-}[\hat{R}_H\Psi](s_1^-; H, p_1, s_2^-)\, ds_1^-\, \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\check{q}(p_1 - p_0)\, [\hat{R}_H\Psi](s^-; H, p_0, s_1^-)\, d p_0 = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\check{q}(p - p_{n-1})\, d p_{n-1} \times$$

$$\times \dots \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\check{q}(p_1 - p_0)\, d p_0\, \frac{1}{2\pi}\int_{\Omega^-}\Psi(s_n^-; z, p, s)\, ds_n^-\, \frac{1}{2\pi}\int_{\Omega^-}[\hat{R}_H\Psi](s_{n-1}^-; H, p_{n-1}, s_n^-)\, ds_{n-1}^- \times$$

$$\times .. \frac{1}{2\pi}\int_{\Omega^-}[\hat{R}_H\Psi](s_1^-; H, p_1, s_2^-)\, [\hat{R}_H\Psi](s^-; H, p_0, s_1^-)\, ds_1^- ;$$

$$\Phi_{\nu k}(z, r_\perp, s) = \frac{1}{2\pi}\int_{\Omega^-} ds_k^- \int_{-\infty}^{\infty}\Theta(s_k^-; z, r_\perp - r_{\perp k}, s)\, q(r_{\perp k})\, dr_{\perp k}\, \frac{1}{2\pi}\int_{\Omega^-} ds_{k-1}^- \int_{-\infty}^{\infty} q(r_{\perp k-1})\, [\hat{R}_H\Theta](s_{k-1}^-; H, r_{\perp k} - r_{\perp k-1}, s_k^-)\, dr_{\perp k-1} \times$$

$$\times \dots \frac{1}{2\pi}\int_{\Omega^-} ds_2^- \int_{-\infty}^{\infty} q(r_{\perp 2})\, [\hat{R}_H\Theta](s_2^-; H, r_{\perp 3} - r_{\perp 2}, s_3^-)\, dr_{\perp 2}\, \frac{1}{2\pi}\int_{\Omega^-} ds_1^- \int_{-\infty}^{\infty}[\hat{R}_H\Theta](s_1^-; H, r_{\perp 2} - r_{\perp 1}, s_2^-)\, E_v(r_{\perp 1}, s_1^-)\, dr_{\perp 1} ;$$

$$\check{\Phi}_{\nu k}(z, p, s) = \frac{1}{2\pi}\int_{\Omega^-}\Psi(s_k^-; z, p, s)\, ds_k^-\, \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\check{q}(p - p_{k-1})\, d p_{k-1}\, \frac{1}{2\pi}\int_{\Omega^-}[\hat{R}_H\Psi](s_{k-1}^-; H, p_{k-1}, s_k^-)\, ds_{k-1}^- \dots \frac{1}{2\pi}\int_{\Omega^-} ds_2^- \times$$

$$\times \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\check{q}(p_3 - p_2)\, [\hat{R}_H\Psi](s_2^-; H, p_2, s_3^-)\, d p_2\, \frac{1}{2\pi}\int_{\Omega^-} ds_1^-\, \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\check{q}(p_2 - p_1)\, \check{E}_\nu(p_1, s_1^-)\, [\hat{R}_H\Psi](s_1^-; H, p_1, s_2^-)\, d p_1 .$$

OPTICAL TRANSFER OPERATOR INVOLVING A HORIZONTALLY INHOMOGENEOUS REFLECTION OPERATOR WITH SEPARATED OFF HORIZONTALLY HOMOGENEOUS COMPONENT

Problem (1) may be solved in several ways.

Technique 1. Presentation in the form of a linear functional

$$\Phi(z, r_\perp, s) = (\Theta_R, E) \qquad (44)$$

in terms of the FI $\Theta_R(s^-; z, r_\perp, s)$ which is a solution of the problem

$$\{\hat{K}Q_R = 0\ ,\ Q_R|_{\Gamma_0} = 0\ ,\ Q_R|_{\Gamma_H} = \varepsilon\hat{R}\, Q_R + f_\delta(s^-; r_\perp, s)\ , \qquad (45)$$

or, in terms of Fourier transforms

$$\check{\Phi}(z, p, s) = F[(\Theta_R, E)] = (\Psi_R, \check{E}) \qquad (46)$$

using SFC $\Psi_R(s^-; z, p, s) = F[\Theta_R]$ which is a solution of the problem

$$\{\hat{L}(p)\Psi_R=0,\ \Psi_R|_{\Gamma_0}=0,\ \Psi_R|_{\Gamma_H}=\varepsilon\check{R}\Psi_R+\check{f}_\delta(s^-; s). \tag{47}$$

Let us define the operations of interaction of radiation with the boundary using the FI Θ

$$[\hat{G}_R f](s^-; H, r_\perp, s) = [\hat{G}_c f](s^-; H, r_\perp, s) + [\hat{G}_\nu f](s^-; H, r_\perp, s) = \hat{R}(\Theta, f) = ([\hat{R}_c Q], f) + \hat{R}_\nu(\Theta, f) \tag{48}$$

and the SFC Ψ

$$[\hat{Q}_R \check{f}](s^-; H, p, s)=F[\hat{G}_R f]=\left[\hat{Q}_c\check{f}\right](s^-; H, p, s)+[\hat{G}_\nu f](s^-; H, p, s)=\check{R}(\Psi, \check{f})=\check{R}_\nu(\Psi, \check{f})+([\hat{R}_c\Psi], \check{f}). \tag{49}$$

Now let us introduce a parametric series

$$\Theta_R(s^-; z, r_\perp, s) = \sum_{n=0}^{\infty}\varepsilon^n\Theta_{Rn}(s^-; z, r_\perp, s)$$

with the components satisfying the system of recursive problems

$$n = 0: \{\hat{K}Q_{R0}=0,\ Q_{R0}|_{\Gamma_0}=0,\ Q_{R0}|_{\Gamma_H}=f_\delta(s^-; r_\perp, s);$$

$$n \ge 1: \{\hat{K}Q_{Rn}=0,\ Q_{Rn}|_{\Gamma_0}=0,\ Q_{Rn}|_{\Gamma_H}=\left[\hat{R}Q_{Rn-1}\right](s^-; H, r_\perp, s)$$

and explicitly expressed in terms of the FI $\Theta(s^-; z, r_\perp, s)$

$$\Theta_{R0}(s^-; z, r_\perp, s) = (\Theta, f_\delta) = \Theta(s^-; z, r_\perp, s);$$

$$\Theta_{Rn}(s^-; z, r_\perp, s) = (\Theta, \hat{R}\Theta_{Rn-1})=(\Theta, \hat{G}_R^n f_\delta) = (\Theta, \hat{G}_R^{n-1}\left[\hat{R}Q\right]) = (\Theta, (\hat{G}_c + \hat{G}_\nu)^{n-1}\left[\hat{R}Q\right]);$$

$$\Theta_R = \sum_{n=0}^{\infty}(\Theta, \hat{G}_R^n f_\delta) = (\Theta, \hat{Y}_R f_\delta), \tag{50}$$

where

$$\hat{Y}_R f_\delta \equiv \sum_{n=0}^{\infty}\hat{G}_R^n f_\delta = [\hat{E} - \hat{G}_R]^{-1} f_\delta. \tag{51}$$

In terms of Fourier transforms the terms of the series

$$\Psi_R(s^-; z, p, s) = \sum_{n=0}^{\infty}\varepsilon^n\Psi_{Rn}(s^-; z, p, s)$$

are solutions of the system of recursive problems

$$n=0: \{\hat{L}(p)\Psi_{R0}=0,\ \Psi_{R0}|_{\Gamma_0}=0,\ \Psi_{R0}|_{\Gamma_H}=\check{f}_\delta(s^-; s);$$

$n\ge 1$:

$$\{\hat{L}(p)\Psi_{Rn}=0,\ \Psi_{Rn}|_{\Gamma_0}=0,\ \Psi_{Rn}|_{\Gamma_H}=\left[\check{R}\Psi_{Rn-1}\right](s^-; H, p, s),$$

which can be presented as functionals in terms of the SFC $\Psi(s^-; z, p, s)$

$$\Psi_{R0}(s^-; z, p, s) = (\Psi, f_\delta) = \Psi(s^-; z, p, s);$$

$$\Psi_{Rn}(s^-; z, p, s) = (\Psi, \check{R}\Psi_{Rn-1}) = (\Psi, \hat{Q}_R^n\check{f}_\delta) = (\Psi, \hat{Q}_R^{n-1}\left[\check{R}\Psi\right]) = (\Psi, (\hat{Q}_c + \hat{Q}_\nu)^{n-1}\left[\check{R}\Psi\right]);$$

$$\Psi_R = \sum_{n=0}^{\infty}(\Psi, \hat{Q}_R^n\check{f}_\delta) = (\Psi, \hat{Z}_R\check{f}_\delta), \tag{52}$$

$$\hat{Z}_R\check{f}_\delta \equiv \sum_{n=0}^{\infty}\hat{Q}_R^n\check{f}_\delta = \left[\hat{E} - \hat{Q}_R\right]^{-1}\check{f}_\delta. \tag{53}$$

Let us now define the operations

$$\left[\hat{G}_{\nu c} f\right](s^-; H, r_\perp, s) = \hat{R}_\nu(\Theta_c, f), \tag{54}$$

$$[\hat{Q}_{\nu c}\check{f}](s^-; H, p, s)=F\left[\hat{G}_{\nu c} f\right] = \check{R}_\nu(\Psi_c, \check{f}), \tag{55}$$

which are similar to $\hat{G}_\nu$(30) and $\hat{Q}_\nu$(31), respectively, and only differ from the latter by the functions of influence (Θ instead of Θ_c) and by SFCs (Ψ insread of Ψ_c), which allow for the contribution of horizontally homogeneous component of the reflection coefficient.

One may introduce the series

$$\Theta_R(s^-; z, r_\perp, s) = \sum_{n=0}^{\infty}\varepsilon^n\Theta_{Rcn}(s^-; z, r_\perp, s)$$

with its components being solutions of the system of recursive problems

$$n=0: \{\hat{K}Q_{Rc0}=0,\ Q_{Rc0}|_{\Gamma_0}=0,\ Q_{Rc0}|_{\Gamma_H}=\hat{R}_c Q_{Rc0}+f_\delta(s^-; r_\perp, s),$$

$$n\ge 1: \{(\hat{K}Q_{Rcn}=0,\ Q_{Rcn}|_{\Gamma_0}=0,\ Q_{Rcn}|_{\Gamma_H}=\hat{R}_c Q_{Rcn}+\hat{R}_\nu Q_{Rcn-1}),$$

which are expressed in terms of the functionals with FI $\Theta_c(s^-; z, r_\perp, s)$

$$\Theta_{Rc0}(s^-; z, r_\perp, s) = (\Theta_c, f_\delta) = \Theta_c(s^-; z, r_\perp, s);$$

$$\Theta_{Rcn}(s^-; z, r_\perp, s) = (\Theta_c, \hat{R}_\nu\Theta_{Rcn-1}) = (\Theta_c, \hat{G}_{\nu c}^n f_\delta) = (\Theta_c, \hat{G}_{\nu c}^{n-1}\left[\hat{R}_\nu Q_c\right]);$$

$$\Theta_R = \sum_{n=0}^{\infty}(\Theta_c, \hat{G}_{\nu c}^n f_\delta) = (\Theta_c, \hat{Y}_{\nu c} f_\delta), \tag{56}$$

$$\hat{Y}_{\nu c} f_\delta \equiv \sum_{n=0}^{\infty}\hat{G}_{\nu c}^n f_\delta = \left[\hat{E} - \hat{G}_{\nu c}\right]^{-1} f_\delta. \tag{57}$$

In terms of Fourier transforms

$$\Psi_R(s^-; z, p, s) = \sum_{n=0}^{\infty}\varepsilon^n\Psi_{Rcn}(s^-; z, p, s);$$

$$n=0: \{\hat{L}(p)\Psi_{Rc0}=0,\ \Psi_{Rc0}|_{\Gamma_0}=0,\ \Psi_{Rc0}|_{\Gamma_H}=\hat{R}_c\Psi_{Rc0}+\check{f}_\delta(s^-; s);$$

$$n\ge 1: \{\hat{L}(p)\Psi_{Rcn}=0,\ \Psi_{Rcn}|_{\Gamma_0}=0,\ \Psi_{Rcn}|_{\Gamma_H}=\hat{R}_c\Psi_{Rcn}+\check{R}_\nu\Psi_{Rcn-1};$$

$$\Psi_{Rc0}(s^-; z, p, s) = (\Psi_c, \check{f}_\delta) = \Psi_c(s^-; z, p, s);$$

$$\Psi_{Rcn}(s^-;\, z,\, p,\, s) = (\Psi_c\, ,\, \check{R}_\nu \Psi_{Rcn-1}) = (\Psi_c\, ,\, \hat{Q}^n_{\nu c}\, \check{f}_\delta) =$$

$$= (\Psi_c\, ,\, \hat{Q}^{n-1}_{mc} \Big[\check{R}_m Y_c \Big])\, ;$$

$$\Psi_R = \sum_{n=0}^{\infty} (\Psi_c\, ,\, \hat{Q}^n_{\nu c}\, \check{f}_\delta) = (\Psi_c\, ,\, \hat{Z}_{\nu c}\, \check{f}_\delta)\, , \tag{58}$$

$$\hat{Z}_{\nu c}\, \check{f}_\delta = \sum_{n=0}^{\infty} \hat{Q}^n_{\nu c}\, \check{f}_\delta = \Big[\hat{E} - \hat{Q}_{\nu c} \Big]^{-1} \check{f}_\delta\, . \tag{59}$$

Let us now introduce the operations

$$\Big[\hat{G}_{c\nu} f \Big] (s^-;\, H,\, r_\perp,\, s) = \hat{R}_c(\Theta_\nu,\, f)\, ; \tag{60}$$

$$\Big[\hat{Q}_{c\nu}\, \check{f} \Big] (s^-;\, H,\, p,\, s) = F \Big[\hat{G}_{c\nu} f \Big] = \hat{R}_c\, (\Psi_\nu,\, \check{f})\, , \tag{61}$$

similar to $\hat{G}_\nu$(30) and $\hat{Q}_\nu$(31), in which their FI Θ and SFC Ψ are replaced by FI Θ_ν, and SFC Ψ_ν, respectively, accounting for the contribution from horizontally inhomogeneous component of the reflection coefficient.

For the terms of parametric series

$$\Theta_R(s^-;\, z,\, r_\perp,\, s) = \sum_{n=0}^{\infty} \varepsilon^n\, \Theta_{R\nu n}(s^-;\, z,\, r_\perp,\, s)\, ,$$

which are solutions of the system of recursive problems

$$n=0 : \{\, \hat{K} Q_{R\nu 0} = 0\, ,\; Q_{R\nu 0}|_{\Gamma_0} = 0\, ,\; Q_{R\nu 0}|_{\Gamma_H} = \hat{R}_\nu Q_{R\nu 0} + f_\delta(s^-;\, r_\perp,\, s)\, ;$$

$$n \ge 1 : \{\, \hat{K} Q_{R\nu n} = 0\, ,\; Q_{R\nu n}|_{\Gamma_0} = 0\, ,\; Q_{R\nu n}|_{\Gamma_H} = \hat{R}_\nu Q_{R\nu n} + \hat{R}_c Q_{R\nu n-1}\, ;$$

there exist presentations in the form of functionals with $\Theta_\nu(s^-;\, z,\, r_\perp,\, s)$

$$\Theta_{R\nu 0}(s^-;\, z,\, r_\perp,\, s) = (\Theta_\nu\, ,\, f_\delta) = \Theta_\nu(s^-;\, z,\, r_\perp,\, s)\, ;$$

$$\Theta_{R\nu n}(s^-;\, z,\, r_\perp,\, s) = (\Theta_\nu\, ,\, \hat{R}_c\, \Theta_{R\nu n-1}) = (\Theta_\nu\, ,\, \hat{G}^n_{c\nu} f_\delta) =$$
$$= (\Theta_\nu\, ,\, \hat{G}^{n-1}_{c\nu} \Big[\hat{R}_c Q_\nu \Big])\, ;$$

$$\Theta_R = \sum_{n=0}^{\infty} (\Theta_\nu\, ,\, \hat{G}^n_{c\nu} f_\delta) = (\Theta_\nu\, ,\, \hat{Y}_{c\nu} f_\delta)\, , \tag{62}$$

$$\hat{Y}_{c\nu} f_\delta \equiv \sum_{n=0}^{\infty} \hat{G}^n_{c\nu} f_\delta = \Big[\hat{E} - \hat{G}_{c\nu} \Big]^{-1} f_\delta\, . \tag{63}$$

And for the Fourier transforms

$$\Psi_R(s^-;\, z,\, p,\, s) = \sum_{n=0}^{\infty} \varepsilon^n \Psi_{R\nu n}(s^-;\, z,\, p,\, s)\, ;$$

$$n=0 : \{\, \hat{L}\,(p) \Psi_{R\nu 0} = 0\, ,\; \Psi_{R\nu 0}|_{\Gamma_0} = 0\, ,\; \Psi_{R\nu 0}|_{\Gamma_H} = \check{R}_m \Psi_{R\nu 0} + \check{f}_\delta(s^-;\, s)\, ;$$

$$n \ge 1 : \{\, \hat{L}\,(p) \Psi_{R\nu n} = 0\, ,\; \Psi_{R\nu n}|_{\Gamma_0} = 0\, ,\; \Psi_{R\nu n}|_{\Gamma_H} = \check{R}_\nu \Psi_{R\nu n} + \check{R}_c \Psi_{R\nu n-1}\, ;$$

$$\Psi_{R\nu 0}(s^-;\, z,\, p,\, s) = (\Psi_\nu\, ,\, \check{f}_\delta) = \Psi_\nu(s^-;\, z,\, p,\, s)\, ;$$

$$\Psi_{R\nu n}(s^-;\, z,\, p,\, s) = (\Psi_\nu\, ,\, \hat{R}_c \Psi_{R\nu n-1}) = (\Psi_\nu\, ,\, \hat{Q}^n_{c\nu}\, \check{f}_\delta) =$$
$$= (\Psi_\nu\, ,\, \hat{Q}^{n-1}_{c\nu} \Big[\hat{R}_c \Psi_\nu \Big])\, ;$$

$$\Psi_R = \sum_{n=0}^{\infty} (\Psi_\nu\, ,\, \hat{Q}^n_{c\nu}\, \check{f}_\delta) = (\Psi_\nu\, ,\, \hat{Z}_{c\nu}\, \check{f}_\delta)\, , \tag{64}$$

$$\hat{Z}_{c\nu}\, \check{f}_\delta \equiv \sum_{n=0}^{\infty} \hat{Q}^n_{c\nu}\, \check{f}_\delta = \Big[\hat{E} - \hat{Q}_{c\nu} \Big]^{-1} \check{f}_\delta\, . \tag{65}$$

Technique 2. This is a presentation in the form of a series

$$\Phi_R\,(z,\, r_\perp,\, s) = \sum_{k=1}^{\infty} \varepsilon^k\, \Phi_{Rck}(z,\, r_\perp,\, s)\, ,$$

with its terms being solutions of the system of recursive problems

$$k=1 : \{\, \hat{K} \Phi_{Rc1} = 0\, ,\; \Phi_{Rc1}|_{\Gamma_0} = 0\, ,\; \Phi_{Rc1}|_{\Gamma_H} = \hat{R}_c \Phi_{Rc1} + E(r_\perp,\, s)\, ;$$

$$k \ge 2 : \{\, \hat{K} \Phi_{Rck} = 0\, ,\; \Phi_{Rck}|_{\Gamma_0} = 0\, ,\; \Phi_{Rck}|_{\Gamma_H} = \hat{R}_c \Phi_{Rck} + \hat{R}_m \Phi_{Rck-1}$$

and expressed either in terms of the FI $\Theta_c(s^-;\, z,\, r_\perp,\, s)$

$$\Phi_{Rc1}(z,\, r_\perp,\, s) = (\Theta_c\, ,\, E)\, ; \quad \Phi_{Rck}(z,\, r_\perp,\, s) = (\Theta_c\, ,\, \hat{R}_\nu \Phi_{Rck-1}) =$$
$$= (\Theta_c\, ,\, \hat{G}^{k-1}_{\nu c} E)\, ,$$

or the SFC $\Psi_c(s^-;\, z,\, p,\, s)$

$$\check{\Phi}_{Rc1}(z,\, p,\, s) = (\Psi_c\, ,\, \check{E})\, ;$$

$$\check{\Phi}_{Rck}(z,\, p,\, s) = (\Psi_c\, ,\, \check{R}_\nu\, \check{\Phi}_{Rck-1}) = (\Psi_c\, ,\, \hat{Q}^{k-1}_{\nu c}\, \check{E})\, ;$$

$$\Phi_R = \sum_{k=1}^{\infty} (\Theta_c\, ,\, \hat{G}^{k-1}_{\nu c}\, E) = (\Theta_c\, ,\, \hat{Y}_{\nu c}\, E)\, , \tag{66}$$

$$\hat{Y}_{\nu c}\, E \equiv \sum_{k=1}^{\infty} \hat{G}^{k-1}_{\nu c}\, E = \sum_{k=0}^{\infty} \hat{G}^k_{\nu c}\, E = \Big[\hat{E} - \hat{G}_{\nu c} \Big]^{-1} E\, ; \tag{67}$$

$$\check{\Phi}_R(z,\, p,\, s) = \sum_{k=1}^{\infty} (\Psi_c\, ,\, \hat{Q}^{k-1}_{\nu c}\, \check{E}) = (\Psi_c\, ,\, \hat{Z}_{\nu c}\, \check{E})\, , \tag{68}$$

$$\hat{Z}_{\nu c}\, \check{E} \equiv \sum_{k=1}^{\infty} \hat{Q}^{k-1}_{\nu c}\, \check{E} = \sum_{k=0}^{\infty} \hat{Q}^k_{\nu c}\, \check{E} = \Big[\hat{E} - \hat{Q}_{\nu c} \Big]^{-1} \check{E}\, . \tag{69}$$

Technique 3. This is a presentation in the form of a series

$$\Phi_R(z,\, r_\perp,\, s) = \sum_{k=1}^{\infty} \varepsilon^k \Phi_{R\nu k}(z,\, r_\perp,\, s)$$

with its components satisfying the systems of recursive problems

$$k=1 : \{\, \hat{K} \Phi_{R\nu 1} = 0\, ,\; \Phi_{R\nu 1}|_{\Gamma_0} = 0\, ,\; \Phi_{R\nu 1}|_{\Gamma_H} = \hat{R}_\nu \Phi_{R\nu 1} + E(r_\perp,\, s)\, ;$$

$$k \ge 2 : \{\, \hat{K} \Phi_{R\nu k} = 0\, ,\; \Phi_{R\nu k}|_{\Gamma_0} = 0\, ,\; \Phi_{R\nu k}|_{\Gamma_H} = \hat{R}_\nu \Phi_{R\nu k} + \hat{R}_c \Phi_{R\nu k-1}$$

and written either in terms of FI $\Theta_\nu(s^-;\, z,\, r_\perp,\, s)$

$$\Phi_{R\nu 1}(z,\, r_\perp,\, s) = (\Theta_\nu\, ,\, E)\, ;$$

$$\Phi_{R\nu k}(z,\, r_\perp,\, s) = (\Theta_\nu\, ,\, \hat{R}_c \Phi_{R\nu k-1}) = (\Theta_\nu\, ,\, \hat{G}^{k-1}_{c\nu} E)\, ,$$

or the SFC $\Psi_v(s^-; z, p, s)$ when in terms of the Fourier transforms

$$\check{\Phi}_{Rv1}(z, p, s) = (\Psi_v, \check{E})\ ; \quad \check{\Phi}_{Rvk}(z, p, s) = (\Psi_v, \hat{R}_c \check{\Phi}_{Rvk-1}) =$$

$$= (\Psi_v, \hat{Q}^{k-1}_{cv} \check{E})\ ;$$

$$\Phi_R = \sum_{k=1}^{\infty} (\Theta_v, \hat{G}^{k-1}_{cv} E) = (\Theta_v, \hat{Y}_{cv} E)\ , \tag{70}$$

$$\hat{Y}_{cv} E \equiv \sum_{k=1}^{\infty} \hat{G}^{k-1}_{cv} E = \sum_{k=0}^{\infty} \hat{G}^{k}_{cv} E = [\hat{E} - \hat{G}_{cv}]^{-1} E\ ; \tag{71}$$

$$\check{\Phi}_R(z, p, s) = \sum_{k=1}^{\infty} (\Psi_v, \hat{Q}^{k-1}_{cv} \check{E}) = (\Psi_v, \hat{Z}_{cv} \check{E})\ , \tag{72}$$

$$\hat{Z}_{cv} \check{E} \equiv \sum_{k=1}^{\infty} \hat{Q}^{k-1}_{cv} \check{E} = \sum_{k=0}^{\infty} \hat{Q}^{k}_{cv} \check{E} = \left[\hat{E} - \hat{Q}_{cv}\right]^{-1} \check{E}\ . \tag{73}$$

The function of influence $\Theta(s^-; z, r_\perp, s)$ and the spatial frequency characteristic $\Psi(s^-; z, p, s)$, in fact, describe the field of radiation in the layer produced due to the processes of multiple scattering of a laser beam propagating along the direction s^- at its boundary $z = H$ at the center of the system of horizontal coordinates x, y. This fundamental solution is the kernel of OTO for the problems with the following set of source and reflection characteristic pairs: 1) $E(r_\perp, s)$, $P(r_\perp, s, s')$; 2) $E(r_\perp, s)$, $P(s, s')$; 3) $E(s)$, $P(r_\perp, s, s')$; 4) $E(r_\perp)$, $P(r_\perp, s, s')$; 5) $E(r_\perp)$, $P(s, s')$; 6) E, $P(r_\perp, s, s')$.

The cases should be noted when other fundamental solutions are used, which are particular representations of FI Θ and SFC Ψ.

The function of influence

$$\Theta_r(z, r_\perp, s) = \frac{1}{2\pi} \int_{\Omega^-} \Theta(s^-; z, r_\perp, s)\, d s^-$$

and the spatial frequency characteristic

$$\Psi_r(z, p, s) = F[\Theta_r] = \frac{1}{2\pi} \int_{\Omega^-} \Psi(s^-; z, p, s)\, d s^-$$

determined from the boundary problems

$$\left\{\hat{K}\, \Theta_r = 0,\ \ \Theta_r \mid_{\Gamma_0} = 0,\ \ \Theta_r \mid_{\Gamma_H} = \delta(r_\perp)\ ;\right.$$

$$\left\{\hat{L}(p)\, \Psi_r = 0,\ \ \Psi_r \mid_{\Gamma_0} = 0,\ \ \Psi_r \mid_{\Gamma_H} = 1\ ,\right.$$

are kernels of the functionals for the cases when the source parameters and the reflection coefficient make the following pairs: 7) $E(r_\perp, s)$, $P(r_\perp, s')$; 8) $E(r_\perp, s)$, $P(s')$; 9) $E(s)$, $P(r_\perp, s')$; 10) $E(r_\perp)$, $P(r_\perp, s')$; 11) $E(r_\perp)$, $P(s')$; 12) E, $P(r_\perp, s')$.

Using the function of influence

$$\Theta_z(s^-; z, s) = \int_{-\infty}^{\infty} \Theta(s^-; z, r_\perp, s)\, d r_\perp$$

which is a solution to the problem $(\hat{K}_z = \hat{D}_z - \hat{S})$ for a monodirectional wide beam

$$\left\{\hat{K}_z \Theta_z = 0,\ \ \Theta_z \mid_{\Gamma_0} = 0,\ \ \Theta_z \mid_{\Gamma_H} = \delta(s - s^-)\ ,\right.$$

one can derive the functionals for the case of horizontally homogeneous sources and reflection 13) $E(s)$, $P(s,s')$; 14) E, $P(s,s')$.

Using the transmission function, which is not corrected for multiple scattering effects

$$W(z, s) = \frac{1}{2\pi} \int_{\Omega^-} \Theta_z(s^-; z, s)\, d s^- =$$

$$= \frac{1}{2\pi} \int_{\Omega^-} d s^- \int_{-\infty}^{\infty} \Theta(s^-; z, r_\perp, s)\, d r_\perp\ ,$$

which satisfied the problem for a single isotropic source

$$\left\{\hat{K}_z W = 0,\ \ W \mid_{\Gamma_0} = 0,\ \ W \mid_{\Gamma_H} = 1\ ,\right.$$

one finds the solution for the pair 15) E, $P(s')$.

The function of influence $\Theta(s^-; z, r_\perp, s)$ is a solution to the first boundary–value problem (4), while FI $\Theta_R(s^-; z, r_\perp, s)$ is the solution to the general boundary–value problem (45) for a horizontally inhomogeneous source of the type of a laser beam.

Functions of influence $\Theta_c(s^-; z, r_\perp, s)$ and $\Theta_v(s^-; z, r_\perp, s)$ are particular cases of the FI Θ_R, these are the solutions to problems (11) and (28), respectively, with their operators of reflection $\hat{R} = \hat{R}_c$ and $\hat{R} = \hat{R}_v$ Functions of influence Θ_R, Θ_v, and Θ_c describe the radiation field produced by a stationary narrow beam with its coordinates $x = 0$, $y = 0$, $z = H$, $s = s^-$, they account for the contribution from multiple scattering in the medium and contribution from multiple reflections from the underlying surface, these reflections described by corresponding reflection coefficients of $P_v + P_c$, P_v, and P_c. The spatial frequency characteristic is defined as the Fourier transform of the function of influence $\Psi(s^-; z, p, s) = F[\Theta]$, which is a solution to the first boundary–value problem for the complex equation of radiation transfer (5) and $\Psi_R(s^-; z, p, s) = F[\Theta_R]$, $\Psi_v(s^-; z, p, s) = F[\Theta_v]$, $\Psi_c(s^-; z, p, s) = F[\Theta_c]$, which are solutions to the complex problems (47), (29), and (12), respectively.

The exact solutions to the general boundary–value problems (1), (8) and (9), i.e. the functionals (10), (22), (27), (39), (44), (66) and (70) are essentially different presentations of the optical transfer operator in terms of the functions of influence, while the functionals (10), (25), (27), (42), (46), (68), and (72) are essentially different representations of the OTO in Fourier transforms in terms of spatial frequency characteristics.

Functionals (16), (33), (50), (56), and (62) yield different representations of the function of influence for the general boundary–value problem, while functionals (19), (36), (52), (58), and (64) describe corresponding spatial frequency characteristics. Meanwhile functions of influence (16), (33), and (50), being the exact solutions to the problem for the case with reflecting bottom, are defined in terms of FI

$\Theta(s^{-}, z, r_{\perp}, s)$, which is a solution to the problem for the nonreflecting boundaries. Similarly, spatial frequency characteristics (19), (36), and (52) are the exact solutions to the complex equation of radiation transfer in a layer with reflecting bottom and are explicitly expressed in terms of the SFC $\Psi(s^{-}, z, r_{\perp}, s)$, which is a solution to the problem with nonreflecting boundaries.

The Neumann series (23) and(40) describe the "scenario" at the underlying surface in terms of FI, while (26) and (42) yield its Fourier transform in terms of the SFC. The optical transfer operators in the terms of linear functionals (10), (22), (25), (27), (39), (42), (44), (46), (66), (68), (70), and (72) describe the transfer of the "scenario" through a turbid layer and may be employed to solve the problems on radiative correction at remote sensing of the underlying surface from any height (both within the layer and outside it) and along any direction.

In this paper we have omitted cumbersome nontrivial transformations and only present original final results within the framework of the theory of the optical transfer operator which is based on the kinetic equations of radiation transfer in turbid media. Instead of solving the initial problems (1), (8), and (9) it is sufficient to define the FI Θ or the SFC Ψ, and then to compute the functionals using relevant approximation.

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