SOLITONS AND STOCHASTIC NOISE DIAGNOSTICS

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A new approach is proposed to stochastic noise diagnostics. The approach is based on the use of nonlinear distributed filters (NDF's), which represent either a real spatiotemporal physical process or a mathematical construction, specially adjusted and implemented using software. Solitons, i.e., nonlinear solitary waves, are proposed as NDF. Mathematically, the NDF can be constructed using exactly solvable models of nonlinear physics. The above–said is illustrated in examples of stochastic Korteweg–de Vries (KDV), sine Gordon (SG), and nonlinear SchrЧЧ dinger (NS) equations.

INTRODUCTION

The common practice frequently requires decoding of signals of complicated nature, including irregular stochastic signals. The wide variety of such signals occur naturally. In particular, monitoring of the Earth's surface and its air and water basins deals with the physical characteristics such as temperature, humidity, pressure, concentration of a given impurity, intensity of sounding radiation, and so on, all of which are represented at a fixed spatial point as a time series of a complicated and irregular form, caused by the fluctuation nature of studied objects or stochastic interaction of the objects with their environment. This poses a variety of problems connected with decoding of randomly fluctuating signals (see, e.g., Refs. 1–4 as well as references therein). As far as the signal probabilistic properties are *a priori* unknown, one accepts statistical hypotheses to record them and then to extract the useful information. These hypotheses allow one to judge with some probability (depending on a hypothesis chosen) about the signal characteristics and useful information carried by them. The modern approach to the solution of this problem incorporates the estimation theory[5,6] and digital methods of signal processing based on spectral signal representation.[6]

Among the above–indicated problems, of particular importance are the problem of identification of Gaussian noise and that concerning the additive mixture of the deterministic signal $f(t)$ and Gaussian noise $\alpha\alpha(t)$. Their importance stems from the widespread conditions favoring the formation of Gaussian noise statistics (as dictated by the familiar mathematical central limit theorem) as well as from the current practice of signal shaping. In the present paper, we use conventional statistical approaches to show the possibility of *exact* identification of some widespread noise models, including Gaussian noise statistics, as well as the possibility, in principle, to solve *exactly* the problem of discriminating between arbitrary Gaussian noise and arbitrary deterministic signal in additive mixture.

Chosen as filtering and identifying elements are nonlinear distributed systems (see Refs. 7 and 8) such as solitary nonlinear waves, i.e., solitons. The above–said is illustrated in examples of single–soliton solutions to the stochastically perturbed KDV, SG, and NSE equations. It should be noted that many–soliton solutions can be used analogously.7 Preparatory to discussion, we emphasize that the practical implementation of nonlinear distributed filters (NDF's) based on nonlinear dynamic systems with distributed parameters can proceed in two different directions. In the first case, a filter is represented by a certain mathematical construction and is implemented as a special software product or subject–oriented processor. Exact single–soliton or N–soliton solutions for the above–enumerated stochastic equations can form the mathematical construction in this case.* In the second case, the filter is realized as a physical device whose particular module imitates dynamics described by the KDV, SG or NSE equations. The objective of the present paper is to investigate the first direction.

GENERAL EQUATION FOR AN AVERAGE SIGNAL ENVELOPE

The propagation of nonlinear solitary wave (soliton) in a stochastic medium, whose properties change in a random way over spatial and temporal coordinates, is accompanied by the soliton transformation. Its geometric and physical characteristics vary as functions of the specific features and properties of the stochastic medium itself. A soliton response to the medium impact thus serves as an indicator of the statistical properties of the medium.

Rigorous single– and many– soliton solutions to a number of nonlinear stochastic equations were found by Russian and foreign scientists.[9–11] In Ref. 12 it was emphasized that all these solutions represent a certain nonlinear function of a *linear* functional related to a random parameter that characterize the properties of the stochastic medium.

These circumstances provide the possibility to derive closed exact equations for averages of the form $\langle\Phi(z + \omega(t))\rangle$, with $\Phi(z)$ being a certain deterministic function of the variable z and $\omega(t) = \int_0^t \alpha(\tau)d\tau$ being the linear functional of the random process $\alpha(t)$, or to formulate a general method of calculating the averages $\langle\Phi\rangle$ for arbitrary random processes.

* This statement is in fact much more general, since the mathematical construction for NDF may be taken to be any rigorous solution of partial differential equation, provided it exhibits the required functional dependence, as well as simply a mathematically suitable construction of a desired form.

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In the case of Gaussian fluctuations $\alpha(t)$, the behavior of the average in the space of generalized variables obeys the diffusion equation with "time"–dependent diffusion coefficient[12,13]

$$\frac{\partial <\Phi>}{\partial t} = D(t) \frac{\partial^2 <\Phi>}{\partial z^2}, \quad (1)$$

where the variable diffusion coefficient $D(t)$ is closely related to the autocorrelation function of the process α

$$D(t) = \int_0^t <\alpha(t)\, \alpha(\tau)> \mathrm{d}\,\tau \equiv \int_0^t K(t, \tau)\, \mathrm{d}\,\tau. \quad (2)$$

Angular brackets denote the ensemble averages over $\alpha(t)$ realization. In the derivation of Eqs. (1) and (2) we set $<\alpha(t)> = 0$ for convenience. The initial conditions for $t = 0$ become $<\Phi>|_{t=0} = \Phi(z)$.

Expressions (1) and (2) indicate that the structure of $<\Phi(z + w(t))>$ serves as a test system to identify the noise with Gaussian statistics, by virtue of the fact that the evolution of the average $<\Phi>$ is governed by diffusion equation (1) solely for the Gaussian noise. Thus monitoring the behavior of the average envelope $<\Phi(z + w(t))>$ in t and z space, we can (1) provide an insight into whether or not the noise $\alpha(t)$ has Gaussian statistics, (2) reconstruct the autocorrelation function of the process (most readily for stationary noise case), and (3) elucidate whether or not the process $\alpha(t)$ is classified with the ergodic random processes. The third opportunity stems from the fact that expressions (1) and (2) are rigorous in the statistical sense provided that ensemble averaging over α realization is defined. However, in practice the noise statistics is analyzed using averages over characteristic times. In so doing, for comparison of theoretical results with experiment, extra assumptions must be made about the relations between the statistical averages (over the probability measure) and time averages. For ergodic random processes these averages are known to coincide.

Let us discuss briefly test systems in the form of soliton solutions to nonlinear stochastic equations.

As shown in Ref. 9, the rigorous solution to the stochastic KDV equation

$$u_t + 6\, u\, u_x + u_{x\,x\,x} = \beta(t) \quad (3)$$

has the form

$$u(x, t) = \nu(t) - 2\, k^2\, \mathrm{sech}^2 \left[k(x - x_0) - 4k^3\, t + 6\, k \int_0^t \nu(t)\, \mathrm{d}\, t \right], \quad (4)$$

where $\beta(t)$ is the Gaussian noise, $\nu(t) = \int_0^t \beta(\tau)\mathrm{d}\tau$, k is the parameter of spectral problem, and x_0 is the soliton location at initial time $t = 0$. Wadati[9] studied the evolution of the average soliton envelope $<u(x, t)>$ under the impact of Gaussian white noise $\beta(t)$, with $<\beta(t)> = 0$ and correlation function $<\beta(t + \tau)\, \beta(t)> = 2D\delta(\tau)$. He demonstrated that the evolution is governed by the diffusion equation of the form (1) with $D(t) \sim t^2$. The solution (4) is a particular case of the above–considered structure $\Phi(z + \omega(t))$ with $\Phi(z) = \mathrm{sech}^2 z$, $z = k(x - x_0) - 4k^3 t$, and $\alpha(t) \equiv \nu(t)$. In Ref. 10 the examples are given of exactly solvable stochastic SG and NS equations:

$$u_{x\,t} = (1 + \alpha(t)) \sin u; \quad (5)$$

$$i\, u_t + i\, \alpha(t)\, u_x + u_{x\,x} + 2u\, |u|^2 = \varepsilon(t)\, u, \quad (6)$$

where $\varepsilon(t)$ is a stochastic function. Analogously, the solutions to these equations are specified functions of either the linear functional $\omega(t) = \int_0^t \alpha(\tau)\mathrm{d}\tau$ for SG equation (5) or functionals $\omega(t)$ and $W(t) = \int_0^t \varepsilon(\tau)\mathrm{d}\tau$ for NS equation (6). Bass et al.[11] presented a review of some rigorously solvable models for the nonlinear wave theory. For all models, the solution for the wave amplitude typically has the structure $\Phi(z + \omega(t))$, where $\omega(t)$ is the definite integral of the random process, and z is a variable statistically independent of the process α. Therefore, all these solutions, after statistical averaging, for Gaussian fluctuations $\alpha(t)$ reduce to that of the linear diffusion equation with variable diffusion coefficient $D(t)$; thus they can be used as test systems to verify whether or not the investigated noise is Gaussian.

Also, the studies mentioned above indicate that a soliton suffers transformation due to interaction with stochastic medium fluctuations. On the average, a soliton broadens but, being exactly integrable, preserves its area. At long times, the solution of diffusion equation (1) approaches asymptotically a self–similar mode, when the average $<\Phi>$ is described by a Gaussian curve independently of the form of the initial condition. It should be noted that the amplitude and width of the Gaussian profile are related to the autocorrelation function of noise $\alpha(t)$. For example, for stationary noise α its autocorrelation function and Gaussian pulse train width are related simply as[7]

$$K(t) = 2 \frac{\mathrm{d}}{\mathrm{d}\, t} \left(h(t) \frac{\mathrm{d}\, h(t)}{\mathrm{d}\, t} \right).$$

Thus, the shape and character of broadening of soliton average envelope serve as statistically exact identifiers of Gaussian noise statistics. In addition, with the Gaussian noise statistics, when the problem is to find the shape of the autocorrelation function, the character of soliton broadening may provide information about the salient features of the spectral function of noise (Fourier transform of the autocorrelation function). Whether the soliton broadening is slow or fast is directly related to the presence of "low–frequency" or resonance modes in the fluctuation spectrum.[12,14] While the formers accelerate the soliton broadening, the resonance fluctuation modes, in contrast, effectively decelerate the broadening (with the broadening rate being inversely proportional to the square of the fluctuation spectrum Q factor).

Let us now turn to the case of non–Gaussian process $\alpha(t)$ with memory (see, e.g., Ref. 15). This means that the probability distributions obey the generalized Fokker–Planck equations

$$\frac{\partial Q}{\partial t} = \hat{L}\, Q, \quad (7)$$

where $Q(\alpha, t\,|\Sigma, T) \equiv Q(\alpha, t\,|\alpha_1, t_1; ...; \alpha_n, t_n)$ is the conditional probability density, that is, the product $Q(\alpha, t\,|\Sigma, T)\mathrm{d}\Sigma$ $(\mathrm{d}\Sigma = \mathrm{d}\alpha_1, ..., \mathrm{d}\alpha_n)$ gives the probability that the random process $\alpha(t)$ takes the value α at time t provided that at times t_1, ..., t_n other than t the parameters of the process fell within the intervals

$(\alpha_1, \alpha_1 + d\alpha_1), ..., (\alpha_n, \alpha_n + d\alpha_n)$, respectively. The operator $\hat{L}$ (often called kinetic or generating operator) is given in differential form as

$$\hat{L} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \alpha^n} \Lambda_n(\alpha, t \mid \Sigma, T). \tag{8}$$

The kinetic coefficients $\Lambda_n(\alpha, t|\Sigma, T)$ are conditional averages defined as

$$\Lambda_n(\alpha, t \mid \Sigma, T) = \lim_{\tau \to 0} \int [(\alpha - \eta)^n Q(\alpha, t \mid \eta, t - \tau; \Sigma, T) / \tau] d\eta.$$

In the limiting case of Markovian processes the kinetic coefficients Λ_n depend only upon the states of the process α at time t, so that $\Lambda_n(\alpha, t \mid \Sigma, T) \equiv \Lambda_n(\alpha, t)$. Averaging of the function $\Phi(z + \omega(t))$ reduces in general to calculation of the characteristic function $\chi(k, t)$ of the process $w(t)$

$$<\Phi(x + \omega(t))> = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\, k\, \Phi(k)\, e^{i\kappa x} \chi(k, t)\ , \tag{9}$$

$$\chi(k, t) = <\exp (i\, k\, \omega(t))>,$$

where $\Phi(k)$ is the Fourier transform of the function $\Phi(z)$. Following the same argument as in Ref. 15 (p. 27), it is easy to show that the characteristic function χ can be determined from the solution of partial differential equation or integro–differential equation (the specific type of the equation depends upon the structure of the kinetic operator $\hat{L}$ of the stochastic process α) for a certain auxiliary function $R = R(t, k, \alpha)$

$$\frac{\partial R}{\partial t} = i\, k\, \alpha\, R + \hat{L}\, R \tag{10}$$

related to the characteristic function χ by the simple integral dependence

$$\chi(k, t) = \int_{(\alpha)} R(k, t, \alpha)\, d\, \alpha. \tag{11}$$

The initial conditions for the function R in Eq. (10) are as follows: $R(k, 0, \alpha) = \delta(\alpha)$.

Thus based on general relations (9)–(11) derived for a wide class of random noise models, including non–Markovian ones, it is possible to develop mathematical constructions for testing that noise. In particular, proposed are systems (including those constructed based on soliton solutions) for testing noise often encountered in applications, such as Pearson noise, Poisson noise, and the like (detailed results will be given in a separate paper).

ADDITIVE MIXTURES OF DETERMINISTIC SIGNALS AND STOCHASTIC NOISE

It is a common practice (or a customary assumption) that a receiver stochastic input signal is a sum of a certain deterministic function of time, $f(t)$, and random noise, $\alpha(t)$. As shown in Ref. 7, once the process $\alpha(t)$ obeys the Gaussian statistics, the indicated mixture can be completely discriminated. This is done using nonlinear distributed filters as a discriminating system. Filters based on exact single– and many–soliton solutions of stochastically perturbed KDV equation (3) were thoroughly investigated. The idea of complete filtration relies on the fact that the noise and deterministic signal determine a pattern of the transport of a fixed substance in the space of generalized variables. Important is the fact that the deterministic component of the mixture is responsible for convective transport of the substance, while the stochastic component – for the transport by diffusion. As a result, the convective transport rate retrieves the shape and characteristics of deterministic signal, while the dynamics of diffusion is used to retrieve the noise characteristics. It should be also noted that the deterministic component contributes only to the convective transport.

Mathematically, the distributed filter for discrimination of the additive mixture $\xi(t) = f(t) + \alpha(t)$ rests upon averaging of the functional $\Phi(z + \int_0^t f(\tau)d\tau + \int_0^t \alpha(\tau)d\tau)$ over the trajectories of the stochastic process $\alpha(t)$. For Gaussian fluctuations α, the average is determined as a solution of the transfer equation of the form[7]

$$\frac{\partial <\Phi>}{\partial t} + f(t) \frac{\partial <\Phi>}{\partial z} = D(t) \frac{\partial^2 <\Phi>}{\partial z^2} \tag{12}$$

with the initial condition $<\Phi>|_{t=0} = \Phi(z)$. Structurally, most simple NDF's are obtained from single–soliton solution of SG equation (5) and NS equation (6) (they can be found in Ref. 10 in explicit form). When the function $\Phi(z)$ is represented by single–soliton solutions of Eqs. (5) and (6), analysis of Eq. (12) shows that the presence of the deterministic process $f(t)$ in this mixture determines the trajectory of maximum soliton amplitude. The dynamics of soliton broadening is still characterized by the stochastic component $\alpha(t)$.

Thus movement of soliton maximum amplitude along a certain trajectory indicates the presence of a regular process in the mixture (if the variable t is interpreted as time, as in KDV equation (3)) or some regular structure within a stochastic medium (when t is understood as a spatial coordinate as in Eqs. (5) and (6)). Hence, the soliton geometric characteristics: location of its maximum, width, and/or amplitude are highly vivid indicators of additive mixture of signal and noise components.

CONCLUSION

The general approach has been proposed to mathematical construction of nonlinear distributed filters to test various models of random noise. In the context of results (9)–(11), one can develop NDF's specialized for particular models of noise, like the diffusion equation (1) for the Gaussian noise. Analysis has shown that the NDF's on the basis of the soliton solutions provide highly convenient and vivid means for studying the probabilistic properties and characteristics of fluctuating media. It seems promising to use "soliton NDF" to segregate between additive mixtures of noise and a deterministic process in view of the fact that the latter contributes solely to the convective transport of the average $<\Phi>$.

The approach being developed is believed to provide the basis for future development of subject–oriented software and hardware intended for practical noise spectroscopy.

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