SPECTRAL METHODS FOR NUMERICAL MODELING OF GEOPHYSICAL FIELDS

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Received December 28, 1995

Algorithms for modeling stationary processes and homogeneous isotropic fields with the prescribed 1-D *distribution and spectral power density (correlation function) are discussed in the paper. A random field with the* 1-D *lognormal distribution and power-law spectrum is used to imitate the* 2-D *field of optical thickness of marine stratocumulus clouds.*

1. INTRODUCTION

The solution of geophysical problems calls for modeling of random processes and fields such as those of temperature, humidity, wind velocity, cloud optical properties, rough sea surface, etc. As a rule, one knows only the one-dimensional (1-D) distribution and spectral power densities (correlation functions) and so can model realistic geophysical processes and fields only approximately.

This modeling relies upon the spectral representation[1]

$$w(x) = \int \cos(\lambda x)\ \xi(\mathrm{d}\lambda) + \int \sin(\lambda x)\ \eta(\mathrm{d}\lambda)\ , \qquad (1)$$

which is essentially an analog of the Fourier transform for a stationary random process $w(x)$. Here, ξ and η are the orthogonal stochastic measures. Methods for numerical simulation of stationary process and homogeneous random fields on the basis of spectral representation (1) are well studied in the literature[2–4] and are widely used to solve many applied problems.[4–9] Among them is the method of spectrum analysis and randomization proposed by Mikhailov[2,5,10] as well as its modification for modeling isotropic fields on a plane[4,11] (for its algorithmic implementation, also see Ref. 12).

In this paper, we outline algorithms for modeling of stationary processes and homogeneous isotropic fields with the prescribed 1-D distribution and spectral power density or correlation function (Sections 2–4). Nonisotropic fields can be modeled by changing a scale along a coordinate axis. In Section 5, an algorithm for modeling a field with the 1-D lognormal distribution and power-law spectrum is used to imitate 2-D fields of optical thickness of stratocumulus clouds. The structure functions and fractal dimension of this random field are also discussed in this section. For convenient reading, attached to the paper is Appendix A providing additional information about the structure functions and fractals.

2. SPECTRAL MODELS FOR GAUSSIAN STATIONARY PROCESSES

Let $w(x)$, $x \in R$, be a Gaussian stationary process with zero mean, unit variance, correlation function

$$K(x) = Ew(x + y)\ w(y)\ ,$$

and spectral power density $f(\lambda)$ with

$$K(x) = \int\limits_0^\infty \cos(\lambda x)\ f(\lambda)\mathrm{d}\lambda\ ,$$

$$f(\lambda) = \frac{2}{\pi} \int\limits_0^\infty \cos(\lambda x)\ K(x)\mathrm{d}x\ .$$

Here and below, E denotes the mathematical expectation over an ensemble of random process (field) realizations. The stochastic integral (1) is approximated by the expression of the form

$$w_n = \sum_{j=1}^{n} a_j \sqrt{-2\ \ln\alpha_j}\ \cos(\lambda_j x + 2\pi\beta_j)\ , \qquad (2)$$

where α_j and β_j are independent random variables distributed uniformly on [0, 1]. Here and below, n denotes the number of terms in the approximation of the stochastic integral. The independent random variables λ_j can be modeled in two ways.

Model A1: λ_j are distributed on the corresponding intervals $[b_{j-1}, b_j)$ with the probability densities

$$f_j(\lambda) = \frac{f(\lambda)}{a_j^2}\ ,\quad a_j^2 = \int\limits_{b_{j-1}}^{b_j} f(\lambda)\mathrm{d}\lambda\ ,$$

$$0 \le b_0 < b_1 < \ldots < b_n = +\infty\ .$$

0235-6880/96/07 629-07 $02.00

Model B1: $a_j^2 = 1/n$, λ_j are identically distributed on $[0, \infty)$ with probability density $f(\lambda)$.

For both models A1 and B1, the random processes (2) have the 1-D Gaussian distribution functions, correlation function $K(x)$, and spectral power density $f(x)$. The conditions: 1) $\max\limits_{j \le n-1} |b_{j-1} - b_j| \underset{n\to\infty}{\to} 0$, $\int\limits_{b_0}^{b_{n-1}} f(\lambda)\mathrm{d}\lambda \to 1$ and 2) $n \to \infty$ ensure asymptotic Gaussian behavior of process (2) for models A1 and B1, respectively. Model B1 is simpler from the algorithmic viewpoint, whereas model A1 is more versatile and adequate for fewer harmonics.

As an example, we consider an algorithm for simulating a Gaussian stationary process with a power-law spectrum

$$f(\lambda) = \begin{cases} 0, & \lambda < \lambda_*, \\ \text{const } \lambda^{-k}, & \lambda \ge \lambda_*, \quad k > 1 \ . \end{cases} \tag{3}$$

It is easily shown that the independent random variable λ_j in Eq. (2) should be modeled for model A1 as

$$\lambda_j = \left[\gamma_j\, b_j^{1-k} + (1-\gamma_j)\, b_{j-1}^{1-k}\right]^{1/(1-k)},\ j = 1, \ldots, n-1,$$
$$\lambda_n = b_{n-1}\, \gamma_n^{1/(1-k)}\ ,\ \ b_0 = \lambda_* \tag{4}$$

and for model B1 as

$$\lambda_j = \lambda_*\, \gamma_j^{1/(1-k)},\ j = 1, \ldots, n, \tag{5}$$

where γ_j are independent random variables distributed uniformly on [0, 1].

Generally, the spectral model of a stationary process with mean μ and variance σ^2 is constructed using the formula

$$\mu + \sigma w_n(x),$$

where $w_n(x)$ is the spectral model defined by formula (2).

3. MODELING OF INHOMOGENEOUS ISOTROPIC GAUSSIAN RANDOM FIELDS ON A PLANE

We consider a homogeneous isotropic Gaussian field $w_\rho(x, y)$ with zero mean and correlation function

$$E\, w_\rho(x, y)\, w_\rho(0, 0) = J_0\left(\rho\sqrt{x^2 + y^2}\right),\ \ \rho > 0\ ,$$

where $J_0(\cdot)$ is the Bessel function of the first kind. The spectral representation of this field can be written as

$$w_\rho(x, y) = \int\limits_{R^2} \cos(\lambda x + \nu y)\ \xi(\mathrm{d}\lambda\, \mathrm{d}\nu) + \int\limits_{R^2} \sin(\lambda x + \nu y)\ \eta(\mathrm{d}\lambda\, \mathrm{d}\nu)\ ,$$

where ξ and η are orthogonal stochastic measures concentrated on the semicircle P in space R^2. As a numerical model of $w_\rho(x, y)$, we use the following approximation:

$$w_\rho^M(x, y) = M^{-1/2} \sum_{m=1}^{M} [\xi_m \cos(x\rho \cos\omega_m + y\rho \sin\omega_m) + \eta_m \sin(x\rho \cos\omega_m + y\rho \sin\omega_m)], \tag{6}$$

where ξ_m and η_m are independent standard normal variables, $\omega_m = \pi(m + \alpha')/M$, and α' is random variable distributed uniformly on [0, 1]. Geometrically, approximation (6) corresponds to the division of semicircle P into equal segments.

Formula (6) can be written in more economical computational form as

$$w_\rho^M(x, y) = M^{-1/2} \sum_{m=1}^{M} \sqrt{-2\ln\alpha_m} \times \cos(x\rho\cos\omega_m + y\rho\sin\omega_m + 2\pi\beta_m), \tag{7}$$

where α_m and β_m are independent random variables distributed uniformly on [0, 1]. The simulation algorithm consists of forming arrays

$$A(m) = \sqrt{-2\ln\alpha_m/M}\ ,\ \ B(m) = \rho\cos\omega_m,$$
$$C(m) = \rho\sin\omega_m,\ D(m) = 2\pi\beta_m,\ m = 1, \ldots, M,$$

while the magnitudes of random field at a given point (x, y) are calculated by the formula

$$w_\rho^M(x, y) = \sum_{m=1}^{M} A(m) \cos[xB(m) + yC(m) + D(m)].$$

A homogeneous isotropic Gaussian field with mathematical expectation μ and correlation function $\sigma^2 J_0(\rho\sqrt{x^2 + y^2})$ is modeled by the formula

$$\mu + \sigma\, w_\rho^M(x, y).$$

Let $w_\rho(x, y)$, $\rho > 0$, be a family of independent homogeneous isotropic Gaussian fields on a plane, with zero mean and correlation function $J_0(\rho\sqrt{x^2+y^2})$. An arbitrary homogeneous isotropic Gaussian random field on the plane $w(x, y)$, with mean μ and correlation function

$$K(x, y) = E[w(x, y) - \mu][w(0, 0) - \mu] = B(\sqrt{x^2+y^2}),$$

can be represented as a superposition of fields $w_\rho(x, y)$:

$$w(x, y) = \mu + \int\limits_0^\infty w_\rho(x, y)\ z(\mathrm{d}\rho), \tag{8}$$

where $z(\mathrm{d}\rho)$ is the real orthogonal stochastic measure on $[0, +\infty)$, while the correlation function is given by the expression

$$K(x, y) = B(\sqrt{x^2+y^2}) = \int_0^\infty J_0(\rho\sqrt{x^2+y^2})G(\mathrm{d}\rho),$$

where $G(\mathrm{d}\rho) = Ez^2(\mathrm{d}\rho)$ is radial spectral measure. Henceforth we assume that for the radial spectral measure there exists the probability density $g(\rho)$:

$$G(\mathrm{d}\rho) = g(\rho)\mathrm{d}\rho \text{ and } g(\rho) = \rho \int_0^\infty rJ_0(r\rho)B(r)\mathrm{d}r.$$

The integral in Eq. (8) will be approximated by the sum

$$w^{(N)}(x, y) = \mu + \sum_{n=1}^{N} a_n\, w_{\rho_n}^{M_n}(x, y), \tag{9}$$

where the independent field realizations $w_{\rho_n}^{M_n}(x, y)$ are modeled by formula (7). As for random process, we adopt two models.

Model A2: ρ_n are random variables distributed on the corresponding intervals $[b_{n-1}, b_n)$ with probability density $g_n(\rho)$:

$$g_n(\rho) = a_n^2/g(\rho), \quad a_n^2 = \int_{b_{n-1}}^{b_n} g(\rho)\,\mathrm{d}\rho,$$

$$0 \le b_0 < b_1 < \ldots < b_N = +\infty .$$

Model B2: $a_n^2 = G(0, \infty)/N$, ρ_n are independent random variables distributed identically on $[0, \infty)$ with probability density $g(\rho)/G[0, \infty)$.

The algorithm for modeling Eq. (9) consists of forming arrays

$$A(n, m) = a_n\sqrt{-2\ln\alpha_{nm}/M_n}\,, \quad B(n, m) = \rho_n\cos\omega_{nm},$$

$$C(n, m) = \rho_n\sin\omega_{nm}\,, \; D(n, m) = 2\pi\beta_{nm}\,,$$

where $\omega_{nm} = \pi(m - \gamma_{nm})/M_n$ and the random variables α_{nm}, β_{nm}, and γ_{nm} are independent and uniformly distributed on [0, 1]. The magnitudes of the random field at the point (x, y) are calculated from the formula

$$w^{(N)}(x, y) = \mu + \sum_{n=1}^{N}\sum_{m=1}^{M_n} A(n, m) \times$$

$$\times \cos[B(n, m)x + C(n, m)y + D(n, m)]. \tag{10}$$

The algorithm is specified by the parameters N and M_n, while the sum $\sum_{n=1}^{N} M_n$ is taken over the number of harmonics in the random field approximation. The values of the parameters N and M_n are chosen empirically.

The limiting behavior of the spectral models was studied in Refs. 2–5 and 13. Here, we note only that the A2 and B2 models, describing adequately the correlation function and the spectrum of the approximated field, obey the one-dimensional Gaussian distribution and become asymptotically Gaussian as $N \to \infty$ and $\max_{n<N}|b_{n-1} - b_n| \to 0$, $G[b_0, b_{N-1}) \to G[0, +\infty)$ for the A2 model.

If the correlation function of the field is exponential, that is, $B(r) = \exp(-ar)$, $g(\rho) = a\rho(\rho^2 + a^2)^{-3/2}$ and for the spectral model B2 the random variables ρ_n can be calculated as

$$\rho_n = a\sqrt{\delta_n^{-2} - 1},$$

where δ_n are independent random variables distributed uniformly on [0, 1].

For homogeneous isotropic Gaussian field on a plane, having power-law radial spectrum

$$g(\rho) = \begin{cases} 0, & \rho \in [0, \rho_*), \\ \mathrm{const}\cdot\rho^{-k}, & \rho > \rho_*, \; \rho_* > 0, \quad k > 1 \end{cases} \tag{11}$$

formula analogous to Eqs. (4) and (5) can be used to calculate ρ_n for the A2 model, that is,

$$\rho_n = \left[\delta_n\, b_n^{1-k} + (1 - \delta_n)\, b_{n-1}^{1-k}\right]^{1/(1-k)}, \; n = 1, \ldots, N-1,$$

$$\rho_n = b_{N-1}\,\delta_N^{1/(1-k)}\,, \quad b_0 = \rho_*,$$

and for the model B2

$$\rho_n = \rho_*\,\delta_n^{1/(1-k)}, \; n = 1, \ldots, N,$$

where δ_n are independent random variables uniformly distributed on [0, 1].

Now we give relationships between the spectral measure $F(\mathrm{d}\lambda) = f(\lambda)\mathrm{d}\lambda$ of the stationary random process obtained as a trace of an isotropic homogeneous field on a certain straight line, that is,

$$f(\lambda) = \frac{2}{\pi}\int_0^\infty B(r)\cos(\lambda r)\,\mathrm{d}r\,, \; \lambda \in [0, +\infty),$$

and the radial spectral measure $G(\mathrm{d}\rho) = g(\rho)\mathrm{d}\rho$ of the isotropic random field (see Refs. 3, 4, and 14–6):

$$f(\lambda) = \frac{2}{\pi}\int_{|\lambda|}^{+\infty} \frac{g(\rho)}{\sqrt{\rho^2 - \lambda^2}}\,\mathrm{d}\rho\,, \quad \lambda \in [0, +),$$

$$F(\lambda) = \frac{2}{\pi}\int_0^{\pi/2} G\left(\frac{\lambda}{\sin\theta}\right)\mathrm{d}\theta =$$

$$= G(\lambda) + \frac{2}{\pi}\int_{|\lambda|}^{+\infty} \arcsin(\lambda/\rho)g(\rho)\mathrm{d}\rho\,, \tag{12}$$

$$1 - G(\rho) = \rho \int_0^{\pi/2} \frac{f(\rho/\sin\theta)}{\sin^2\theta}\, d\theta ,$$

$$g(\rho) = \frac{\rho\, f(R)}{\sqrt{R^2 - \rho^2}} - \rho \int_{|\rho|}^{R} \frac{f'(\lambda)}{\sqrt{\lambda^2 - \rho^2}}\, d\lambda .$$

The last expression assumed that $f(\lambda) > 0$ for $\lambda \in [0, R)$ and $f(\lambda) = 0$ for $\lambda \in [R, +\infty)$. From formula (12) it follows in particular that if the radial spectrum is described by the power-law dependence, that is, Eq. (11) is fulfilled, the spectral density $f(\lambda)$ will be also the power-law function for $\lambda \geq p_*$:

$$G(\rho) = C_1[1 - (\rho/\rho_*)^{-k+1}] , \quad \rho > \rho_*, \quad (k > 1),$$

$$F(\lambda) = \frac{2}{\pi} \int_0^{\pi/2} C_1 \left\{ 1 - \left[\frac{\lambda/\sin\theta}{\rho_*} \right]^{-k+1} \right\} d\theta = C_2 + C_3\, \lambda^{-k+1},$$

$\lambda > \rho_*$,

where C_1, C_2, and C_3 are constants.

For particular examples of correlation functions of homogeneous isotropic random fields on a plane and their associated radial spectral densities, see Ref. 14.

4. MODELING OF HOMOGENEOUS RANDOM FIELD WITH THE LOGNORMAL 1-D DISTRIBUTION AND POWER-LAW SPECTRUM

The random variable ν obeying the lognormal distribution with the parameters μ and σ can be represented as $\nu = \exp(w)$, where w is the Gaussian variable with mean μ and variance σ^2. As is well known,

$$E\nu = \exp\left(\frac{\sigma^2}{2} + \mu\right) ,$$

$$D\nu = E\nu^2 - (E\nu)^2 = e^{\sigma^2 + 2\mu}\, [e^{\sigma^2} - 1] .$$

We assume $w(t)$ to be a Gaussian random function with mathematical expectation μ, variance σ^2, and correlation function $K_w(t, s) = Ew(t, s)\, w(s)$. Then the random function

$$\nu(t) = \exp(w(t)) \tag{13}$$

obeys the 1-D lognormal distribution with the parameters μ and σ and the correlation function

$$K_\nu(t, s) = R_{\mu\sigma}(K_w(t, s)),$$

where $R_{\mu\sigma}$ is the function determining distortion of correlations due to nonlinear transformations (13) (see, for example, Refs. 17 and 18). The function $R_{\mu\sigma}$ can be found as follows. We denote by $\hat{K}_\nu(t, s)$ the normalized correlation function of the $\nu(t)$ field

$$\hat{K}_\nu(t, s) = \frac{K_\nu(t, s) - (E\nu)^2}{D\nu} , \quad D_\nu = K_\nu(t, t) - (E\nu)^2$$

and by $\hat{K}_w(t, s)$ the normalized correlation function of Gaussian field $w(t)$

$$\hat{K}_w(t, s) = \frac{K_w(t, s) - \mu^2}{\sigma^2} .$$

The normalized correlation functions are related by the formula[18]

$$\hat{K}_\nu(t, s) = \frac{e^{\sigma^2 \hat{K}_w(t, s)} - 1}{e^{\sigma^2} - 1} , \tag{14}$$

from which the function $R_{\mu\sigma}$ is readily found.

Generally, modeling of random function with lognormal 1-D distribution and correlation function $K_\nu(t, s)$ according to formula (13) calls for the precalculated correlation function

$$K_w(t, s) = R^{-1}_{\mu\sigma}(K_\nu(t, s)).$$

Distortion of the correlation function due to nonlinear transformation (13) is given by the parameter

$$\Delta = \max_\rho \left\{ \rho - \frac{e^{\sigma^2 \rho} - 1}{e^{\sigma^2} - 1} \right\} ,$$

where ρ runs through all possible values of the normalized correlation function $\hat{K}_w(t, s)$.

Given $\hat{K}_\nu(t, s) \geq 0$ and σ is small (e.g., $\sigma \leq 1$), the distortion of the correlations can be neglected (see Ref. 17). Then, as an approximate model of homogeneous isotropic random field $\nu(x, y)$ obeying the lognormal 1-D distribution with the parameters μ and σ and power-law spectrum (11), we can take

$$\nu^{(N)}(x, y) = \exp(w^{(N)}(x, y)),$$

where $w^{(N)}$ is spectral model (10) of homogeneous isotropic Gaussian field with mathematical expectation μ and power-law spectrum (11) with $\int_0^\infty g(\rho) d\rho = \sigma^2$.

The approach based on the transformation of Gaussian functions is frequently used to model non-Gaussian random processes and fields (see Refs. 3, 4, 17–19 for details).

5. MATHEMATICAL MODELS FOR OPTICAL THICKNESS OF STRATOCUMULUS CLOUDS

Two-parametric fractal models generated by multiplicative cascade processes provide adequate description of the distribution of liquid water in stratocumulus clouds.[20,21] In this case, the vertical optical thickness τ_ν and the liquid water path W are related by the expression

$$\tau_\nu = 3W/2r_{\rm eff} , \tag{15}$$

where W is in g/m^2 and the effective droplet radius r_{eff} is in μm (Ref. 22). Typically, stratocumulus have $W = 90$ g/m^2, $r_{eff} = 10$ μm, $\tau_v = 13$, and cloud thickness of about 300 m. According to Eq. (15), the distribution of the liquid water path is well approximated by the lognormal distribution and can be transformed into the optical thickness distribution necessary for discussion of radiative transfer.

Of these model, we consider the simpler one reproducing fractal in only one direction with a scaling factor of 2. Extension to a higher dimension and other scaling factors is straightforward. Model starts with a plane-parallel homogeneous cloud slab of finite extent in the vertical direction and in one horizontal direction and infinite in the other horizontal direction. This slab is divided into two parts of equal lengths and a part of liquid water f_0 is transferred from one half to the other, with the direction of transfer chosen randomly with equal probabilities. At the next stage, each of the two parts is divided in the same way in halves, and a part of liquid water f_1 is transferred from one quarter of the slab to the neighboring one, with the transfer directions chosen randomly and independently. This procedure is applied to each quarter-slab, etc.

Usually, $n = 10–12$ cascade steps are used to generate a cloud and the initial plane-parallel cloud slab is divided into cells (pixels) of equal extent and different optical thicknesses. It is shown (e.g., see Ref. 20) that when

$$f_n = fc^n , \tag{16}$$

the distribution of W (or optical thickness τ) will have a power-law spectrum $f(\lambda) \sim \lambda^{-k}$ with the spectral exponent $k = 1 - \log_2 c^2$ independent of the value of f for $c < 1$. Analysis of data obtained as part of the program FIRE for marine stratocumulus clouds over California revealed a $\lambda^{-5/3}$ wavelength spectrum[23] corresponding to the Kolmogorov-Obukhov law and hence the spectral parameter c is

$$c = 2^{-1/3} \approx 0.8 . \tag{17}$$

The sole free parameter f is specified empirically from the standard deviation of log W distribution and for marine Sc it is approximately 0.5 (Refs. 20 and 23).

Instead of cascade processes, an algorithm for simulating random processes (fields) with the 1-D lognormal distribution and power-law spectrum described in Sec. 4 can be used to model numerical realizations of the distribution of optical thickness of marine Sc. The model input parameters are now the mean $\langle\tau\rangle$ and the variance D_τ of optical thickness (liquid water path) as well as the exponent k characterizing the slope of the power-law energy spectrum. Figure 1 shows sampling realization of the 2-D field of optical thickness obtained with $\langle\tau\rangle = 13$, $D_\tau = 29$, and $k = 5/3$ typical of marine Sc (Ref. 20).

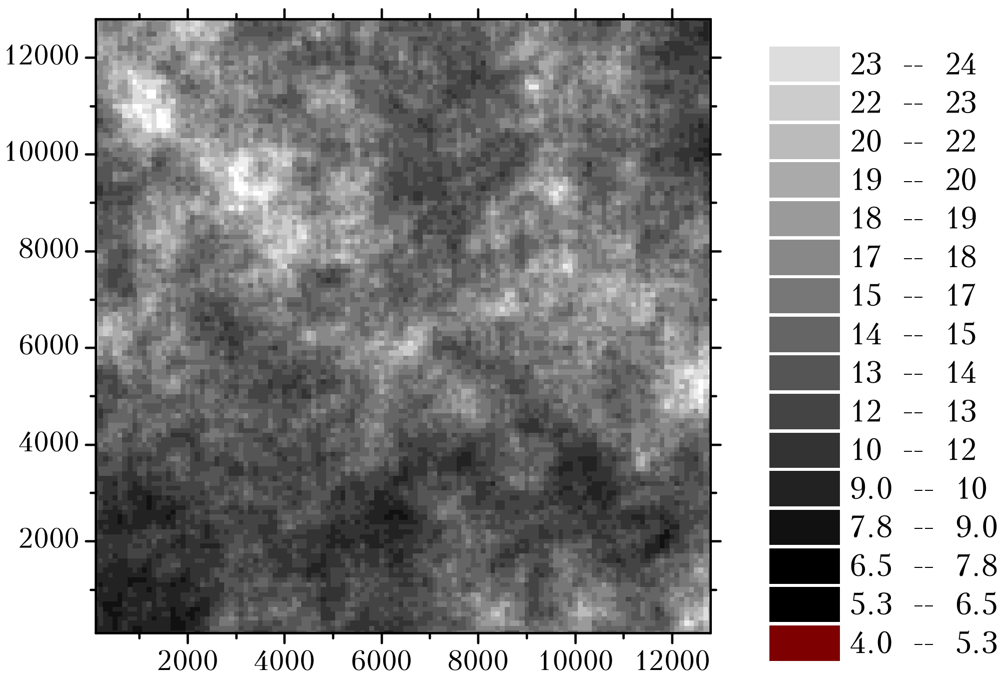

FIG. 1. Computer realization of the 2-D *field of optical thickness of marine stratocumulus clouds for* $\langle\tau\rangle = 13$, $D_\tau = 29$, *and* $k = 5/3$.

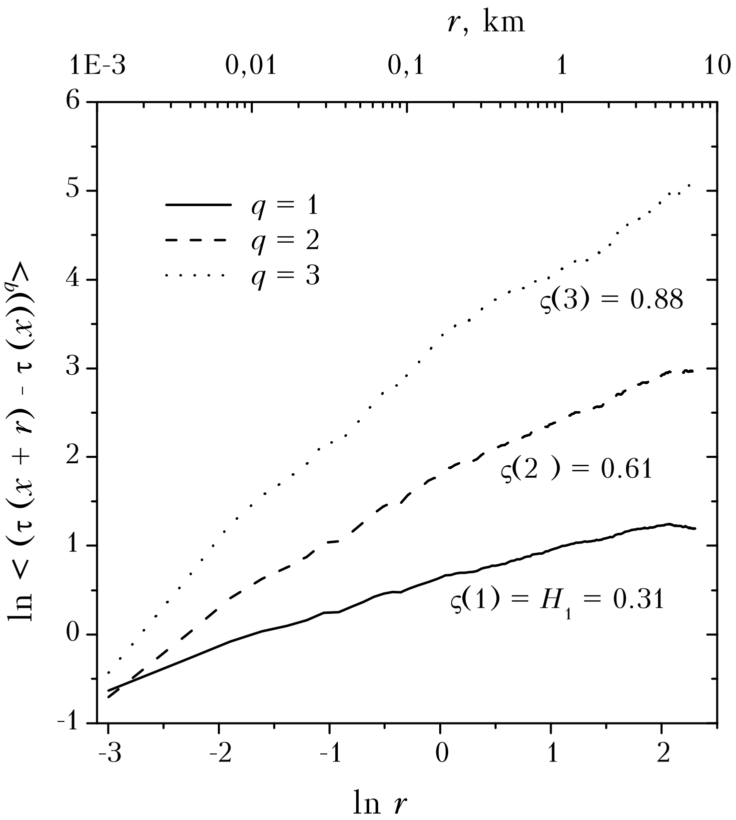

FIG. 2. Structure function of optical thickness of stratocumulus clouds.

Structure functions of the order $q = 1$, 2, and 3 calculated for the 1-D realization of the optical thickness are depicted in Fig. 2. The exponent H_1 for the first moment of the absolute increment of τ is 0.31 and hence the fractal dimension $D = 2 - H_1 = 1.69$ (see Appendix A). The exponent $\zeta(q)$ of the structure function is well approximated by the linear function $\zeta(q) = qH_1$, $q = 1, 2, 3$. This is in agreement with the results for the cascade cloud field model. The exponent k of the power-law energy spectrum calculated as $k = \zeta(2) + 1 = 1.61$ is close to a desired value of 5/3.

The spectral model of Sc optical thickness field has some advantages over the cascade one. The input parameters for the spectral model have more habitual statistical meaning: mean, variance, and exponent of the power-law energy spectrum. For the cascade model, the *piecewise-constant* cloud field is constructed in the fixed horizontally *bounded* volume, whereas spectrum-based algorithms allow one to model *continuous* cloud fields in horizontally *infinite* volume.

ACKNOWLEDGMENTS

The work was supported by the DOE's ARM Program (under Contract No. 350114–A–Q1) and by International Science Foundation and Russian Government Grant No. JI 7100.

APPENDIX A. STRUCTURE FUNCTIONS AND FRACTAL DIMENSION OF RANDOM PROCESSES

Smoothness of any continuous function is well characterized by the Hölder exponent α:

$$|f(x + \Delta x) - f(x)| \le \text{const}|\Delta x|^{\alpha}, \quad 0 < \alpha \le 1, \qquad \text{(A1)}$$

whose larger value implies smoother function f. The limiting case $\alpha = 1$ corresponds to the class of differentiable functions.

For a stochastic process ϕ, it is possible to find a statistical analog α as the exponent H_1 for the first moment of its absolute increment $|\Delta\phi(x, r)| = |\phi(x + r) - \phi(x)|$, $r > 0$:

$$\langle|\Delta\phi(x, r)|\rangle \propto r^{-H_1}, \quad 0 < H_1 \le 1,$$

where $\langle\cdot\rangle$ denotes an ensemble average. The exponent H_1 is related to the fractal dimension of the plot of $\phi(x)$, considered as a random geometric object in 2-D space, by the expression[24]

$$D = 2 - H_1, \quad 1 \le D \le 2. \qquad \text{(A2)}$$

Then the codimension of the plot $H_1 = 2 - D$ changes from 0 (discontinuous process at each point) to 1 (differentiable process) and hence is a direct and natural measure of the smoothness.

For structure functions of the q order we have

$$\langle|\Delta\phi(x, r)|^{q}\rangle \propto r^{-\zeta(q)} , \qquad (A3)$$

where $\zeta(1) = H_1$. "Simple" scaling, or monoscaling, means that $\zeta(q)$ is a linear function, $\zeta(q) = q\zeta(1) = qH_1$; in this case, H_1 is the only quantity required for the two-point statistical description of the stochastic process. A classical example of monoscaling is Brownian motion.[24] If $\zeta(q)$ is a nonlinear function, the stochastic process will possess multiscaling, or multifractality,[25] or multiaffinity[26] and a variety of exponents $\zeta(q)$ is required to describe this process statistically.

In their discussion of the relation between the structure function of the second order ($q = 2$) and energy spectrum $f(\lambda)$, Monin and Yaglom[27] have shown that

$$k = \zeta(2) + 1 > 1. \qquad (A4)$$

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